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LOCALIZATION RESULTS VIA KRASNOSELSKII'S FIXED POINT THEOREM IN CONES

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Abstract. The purpose of this paper is to give an existence result for the nonlinear fourth-order boundary value problem

$$u^{(4)}(t) = f(u(t)), \quad t \in [0,1]$$
$$u(0) = u(1) = A,$$
$$u''(0) = u''(1) = B$$

where $f : [0,\infty) \to \mathbb{R}$ is continuous and A, B are positive real numbers. We use a result related to the existence of positive solutions for nonlinear integral equations in Banach spaces, presented in [7].

Key Words and Phrases: ordered Banach space, fourth-order boundary value problem, Krasnoselskii's compression-expansion fixed point theorem.
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1. Introduction and preliminary results

Localization of solutions for nonlinear operator equations can be obtained by using variational methods [14, 15, 9], upper and lower solution method [1, 4, 6] or existence results related to ordered Banach spaces [6, 12, 2, 8]. In this paper we seek positive solutions of a fourth-order boundary value problem using the results related to the existence of positive solutions for nonlinear integral equation in ordered Banach spaces, presented in [7].

Boundary value problems for m-order differential equations describe physical, biological and chemical phenomena. Fourth-order boundary value problems were studied by many authors [3, 10, 11]. A special attention received the Lidstone boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u''(t)), \;\; 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

For example, in [3] using the Five Functionals Fixed Point Theorem there are established some growth conditions on f to obtain three symmetric positive solutions, and the Avery-Henderson fixed point theorem is applied in [10] together with some growth conditions on f to prove the existence of at least two positive solutions for three point boundary value problems.

The main tool of our approach is the Krasnoselskii's compression-expansion fixed point theorem.

Theorem 1.1. *Let $(E, |\cdot|)$ be a Banach space, and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are two open subsets of E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let the operator $T : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be completely continuous and either*

$$|T(x)| \leq |x|, \;\; x \in C \cap \partial\Omega_1 \;\; \text{and} \;\; |T(x)| \geq |x|, \;\; x \in C \cap \partial\Omega_2$$

or

$$|T(x)| \geq |x|, \;\; x \in C \cap \partial\Omega_1 \;\; \text{and} \;\; |T(x)| \leq |x|, \;\; x \in C \cap \partial\Omega_2$$

is true. Then T has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

In [7] the Krasnoselskii's compression-expansion fixed point theorem is used for the nonlinear integral equation

$$U(t) = \int_0^T k(t,s) F(U)(s)\, ds, \;\; t \in [0,T],$$

where $k : [0,T] \times [0,T] \to \mathbb{R}_+$ and $F : C([0,T]; K) \to C([0,T]; K)$ is an operator. In that follows, we extend this result to the nonlinear integral equation

$$U(t) = g(t) + \int_0^T k(t,s) F(U)(s)\, ds, \;\; t \in [0,T] \tag{1.1}$$

where $g : [0,T] \to K$, $k : [0,T] \times [0,T] \to \mathbb{R}_+$ and $F : C([0,T]; K) \to C([0,T]; K)$ is an operator.

Theorem 1.2. *Let $(X, |\cdot|)$ be a real Banach space and let $K \subset X$ be a cone of X. Assume that the norm $|\cdot|$ is monotone with respect to K and that the following conditions are satisfied:*

(H_1) *For each* $t \in [0,T]$, $k_t = k(t,\cdot) \in L^1(0,T;\mathbb{R}_+)$ *and the map* $t \mapsto k_t$ *is continuous from* $[0,T]$ *to* $L^1(0,T)$*;*

(H_2) *There exists* $\mu \in (0,1)$, $\kappa \in L^1(0,T)$ *and an interval* $[a,b] \subset [0,T]$, $a < b$, *such that*

$$\begin{aligned} k(t,s) &\leq \kappa(s), \ \ t \in [0,T], a.e \ \ s \in [0,T] \\ &\textit{and} \\ \mu\kappa(s) &\leq k(t,s) \ \ t \in [a,b], a.e \ \ s \in [0,T]; \end{aligned}$$

(H_3) *The map* $g : [0,T] \to K$ *is continuous and satisfies the inequality*

$$\mu g(t) \leq g(t'), \ \ t \in [0,T], t' \in [a,b] \tag{1.2}$$

(H_4) *There exists* $\Phi : K \to K$ *such that*

$$\Phi(x) \leq F(U)(t), \ \ t \in [a,b]; \tag{1.3}$$

whenever $U \in C([0,T];K), x \in K$ *and* $x \leq U(t)$ *for all* $t \in [a,b]$*;*

(H_5) *There exists* $\alpha > 0$ *such that*

$$|F(U)(t)| \leq \frac{\alpha - |g|_\infty}{\sup\limits_{t\in[0,T]} \int_0^T k(t,s)\,ds} \tag{1.4}$$

for all $t \in [0,T]$ *and* $U \in C([0,T];K)$ *with* $|U| = \alpha$*;*

(H_6) *There exists* $\beta > 0, \beta \neq \alpha$ *and* $t^* \in [0,T]$ *such that*

$$\inf\{|\Phi(x)| : x \in K, |x| = \mu\beta\} \cdot \int_a^b k(t^*,s)\,ds \geq \beta + |g(t^*)|; \tag{1.5}$$

(H_7) *The operator* N_0 *defined by*

$$N_0(U)(t) = \int_0^T k(t,s)\,F(U)(s)\,ds$$

is completely continuous from $C([0,T];K)$ *to* $C([0,T];X)$.

Then (1.1) *has at least one solution* $U \in C([0,T];K)$ *such that*

$$\mu U(t) \leq U(t') \ \ \textit{for } t \in [0,T], t' \in [a,b] \tag{1.6}$$

and $0 < \min\{\alpha,\beta\} \leq |U|_\infty \leq \max\{\alpha,\beta\}$.

Proof. To apply Krasnoselskii's theorem, let $E = C([0,T];X)$ be endowed with the norm $|U|_\infty = \max_{t\in[0,T]} |U(t)|$ and and let us consider the cone

$$C = \left\{U \in C([0,T];K) : \mu U(t) \le U(t') \ \text{ for } t \in [0,T], t' \in [a,b]\right\}.$$

We make the notations $r_1 = \min\{\alpha,\beta\}$ and $r_2 = \max\{\alpha,\beta\}$ and we denote

$$\Omega_1 = \{U \in C([0,T];X) : |U|_\infty < r_1\}$$

$$\Omega_2 = \{U \in C([0,T];X) : |U|_\infty < r_2\}.$$

Consider the operator

$$N(U)(t) = g(t) + \int_0^T k(t,s) F(U)(s)\, ds, \ \ t \in [0,T].$$

From $(H_1)-(H_3)$ and (H_7) we have that $N : C \to C$ and N is completely continuous.

Let $U \in C$ and $|U|_\infty = \alpha$. Using (H_5) we deduce that

$$|N(U)(t)| \le |g(t)| + \int_0^T k(t,s) |F(U)(s)|\, ds \le \alpha$$

for all $t \in [0,T]$. Hence $|N(U)|_\infty \le |U|_\infty$.

Let $U \in C$, $|U|_\infty = \beta$ and we can consider that $\mu U(t_0) \le U(t')$ for all $t' \in [a,b]$ and $t_0 \in [0,T]$ with $|U(t_0)| = \beta$. Now (H_4) implies that $\Phi(\mu U(t_0)) \le F(U)(s)$ for $s \in [a,b]$. Then

$$\begin{aligned} N(U)(t^*) &= g(t^*) + \int_0^T k(t^*,s) F(U)(s)\, ds \\ &\ge g(t^*) + \Phi(\mu U(t_0)) \int_0^T k(t^*,s)\, ds. \end{aligned}$$

So, $|N(U)(t^*)| \ge |\Phi(\mu U(t_0))| \int_0^T k(t^*,s)\, ds - |g(t^*)|$ for $t^* \in [0,T]$ and from (H_6) it follows that $|N(U)|_\infty \ge |U|_\infty$.

Therefore, Krasnoselskii's Theorem applies. □

In many applications we are interested in multiple solutions. Under similar conditions to (H_5) and (H_6) we obtain the following result:

Theorem 1.3. *Assume that* $(H_1)-(H_4), (H_7)$ *hold and*

(M_1) *for some* $n \in \mathbb{N}\setminus\{0\}$ *there exist* $\alpha_i > 0, i = \overline{1,n}$ *such that* (H_5) *is satisfied with* $\alpha = \alpha_i$ *for every* $i \in \{1,\ldots,n\}$

(M_2) *for some* $m \in \mathbb{N}\setminus\{0\}$ *there exist* $\beta_j > 0$ *and* $t_j^* \in [0,T]\,, j = \overline{1,m}$ *such that* (H_6) *is satisfied with* $\beta = \beta_j$ *and* $t^* = t_j^*$ *for every* $j \in \{1,\ldots,m\}$

Then

(I) *If* $m = n+1$ *and* $0 < \beta_1 < \alpha_1 < \ldots < \beta_n < \alpha_n < \beta_{n+1}$, *then* (1.1) *has at least* $2n$ *nonnegative solutions* $U_1,\ldots,U_{2n} \in C([0,T];K)$ *such that*

$$0 < \beta_1 < |U_1|_\infty < \alpha_1 < \ldots < \alpha_n < |U_{2n}|_\infty < \beta_{n+1}$$

(II) *If* $m = n$ *and* $0 < \beta_1 < \alpha_1 < \ldots < \beta_n < \alpha_n$, *then* (1.1) *has at least* $2n-1$ *nonnegative solutions* $U_1,\ldots,U_{2n-1} \in C([0,T];K)$ *such that*

$$0 < \beta_1 < |U_1|_\infty < \alpha_1 < \ldots < \beta_n < |U_{2n-1}|_\infty < \alpha_n$$

(III) *If* $n = m+1$ *and* $0 < \alpha_1 < \beta_1 < \ldots < \alpha_m < \beta_m < \alpha_{m+1}$, *then* (1.1) *has at least* $2m$ *nonnegative solutions* $U_0,\ldots,U_{2m} \in C([0,T];K)$ *such that*

$$0 < \alpha_1 < |U_1|_\infty < \beta_1 < \ldots < \beta_m < |U_{2m}|_\infty < \alpha_{m+1}$$

(IV) *If* $n = m$ *and* $0 < \alpha_1 < \beta_1 < \ldots < \beta_n < \alpha_m < \beta_m$, *then* (1.1) *has at least* $2m-1$ *nonnegative solutions* $U_0,\ldots,U_{2m-1} \in C([0,T];K)$ *such that*

$$0 < \alpha_1 < |U_1|_\infty < \beta_1 < \ldots < \alpha_m < |U_{2m-1}|_\infty < \beta_m.$$

2. Application

In this section, we apply Theorem 1.2 to localize positive solutions for the nonlinear fourth-order boundary value problem

$$u^{(4)}(t) = f(u(t)),\ \ t \in [0,1] \tag{2.7}$$
$$u(0) = u(1) = A,$$
$$u''(0) = u''(1) = -B$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and A, B are nonnegative real numbers.

Theorem 2.4. *Let* $\varepsilon \in \left(0,\frac{1}{2}\right)$, $\mu \in \left(0,\frac{1}{2}-\varepsilon\right]$, $\sigma_m = \min\{A,B\}$ *and* $\sigma_M = \max\{A,B\}$. *Suppose that the function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous and*

(T_1) *there is a map* $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+^2$, $\Phi = (\Phi_1,\Phi_2)$ *such that*

$$(0,0) \le (x_1,x_2) \le (y_1,y_2) \ \textit{implies}\ \Phi(x_1,x_2) \le (y_2, f(y_1)) \tag{2.8}$$

(T_2) *there exists* $\alpha > 0$ *such that*

$$\max\{f(y_1), y_2\} < 8(\alpha - \sigma_M), \tag{2.9}$$

for any $(y_1, y_2) \in \mathbb{R}_+^2$ *with* $\max\{y_1, y_2\} = \alpha$*;*

(T_3) *there exists* $\beta > 0, \beta \neq \alpha$ *and* $t^* \in [0,1]$ *such that*

$$\Sigma \cdot g(t^*) < \beta + \sigma_m, \tag{2.10}$$

where

$$\Sigma = \inf\{\max\{\Phi_1(x_1, x_2), \Phi_2(x_1, x_2)\} : x_1, x_2 \in \mathbb{R}_+, \max\{x_1, x_2\} = \mu\beta\}$$

and

$$g(t) = \begin{cases} \varepsilon t & if \quad t \in \left[0, \frac{1}{2} - \varepsilon\right] \\ \frac{t-t^2}{2} - \frac{1}{8} + \frac{\varepsilon - \varepsilon^2}{2} & if \quad t \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \\ \varepsilon(1-t) & if \quad t \in \left[\frac{1}{2} + \varepsilon, 1\right] \end{cases}.$$

Then (2.7) *has at least one solution* $u \in C([0,1]; \mathbb{R}_+)$ *such that*

$$\mu u(t) \leq u(t') \text{ and } \mu u''(t) \geq u''(t')$$

for $t \in [0,1]$, $t' \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ *and*

$$0 < \min\{\alpha, \beta\} \leq \max_{t \in [0,1]} \{u(t), -u''(t)\} \leq \max\{\alpha, \beta\}.$$

Proof. Let $C([0,1]; \mathbb{R}_+)$ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}_+ = [0, \infty)$. We make the notation $\mathbb{K} = C([0,1]; \mathbb{R}_+) \times C([0,1]; \mathbb{R}_+)$. For any $U = (u_1, u_2) \in \mathbb{K}$ we consider

$$\|U\| = \max_{t \in [0,1]} \max\{u_1(t), u_2(t)\}.$$

Letting $u_1 = u$ and $u_2 = -u''$ we see that problem (2.7) is equivalent to the system

$$\begin{cases} u_1'' = -u_2 \\ u_1(0) = u_1(1) = A \\ u_2'' = -f(u_1(t)) \\ u_2(0) = u_2(1) = B. \end{cases} \tag{2.11}$$

The Green function associated to the operator $-u''$ and the boundary conditions $u(0) = u(1) = 0$ is $G : [0,1] \times [0,1] \to \mathbb{R}$,

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1 \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.12}$$

Hence, (2.7) is equivalent to the nonlinear integral equation

$$U(t) = g(t) + \int_0^1 G(t,s) F(U(s))\, ds, \quad t \in [0,1], \tag{2.13}$$

where $U = (u_1, u_2) \in C([0,1]; \mathbb{R}_+^2) \cap C^2([0,1]; \mathbb{R}_+^2)$, the function $g : [0,1] \to \mathbb{R}_+^2$ is given by

$$g(t) = (A, B), t \in [0,1]$$

and the operator $F : \mathbb{K} \to \mathbb{K}$ is defined by

$$F(U) = F(u_1, u_2) = (u_2, f(u_1)).$$

In what it follows, we show that all conditions of Theorem 1.2 are satisfied. Consider $\kappa : [0,1] \to \mathbb{R}_+$, $\kappa(s) = s(1-s)$ and $0 < \varepsilon < \frac{1}{2}$. We have $\mu\kappa(s) \le G(t,s)$ for $t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$ and all $s \in [0,1]$. Here $\mu \in \left(0, \frac{1}{2} - \varepsilon\right]$. On the other hand, $G(t,s) \le \kappa(s)$, for all $t, s \in [0,1]$. The proof of these inequalities may be found in [8]. Hence, condition (H_2) is satisfied.

Let $X = \mathbb{R}^2$ and $K = \mathbb{R}_+^2$ be the positive cone of $\mathbb{R}^2$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we have $(x_1, x_2) \le (y_1, y_2)$ if and only if $x_1 \le y_1$ and $x_2 \le y_2$.

Inequality (1.2) is equivalent to $\mu \cdot (A, B) \le (A, B)$ and is satisfied for any $\mu \le 1$.

Let $(x_1, x_2) \in \mathbb{R}_+^2$ and $U = (u_1, u_2) \in C([0,1]; \mathbb{R}_+^2) \cap C^2([0,1]; \mathbb{R}_+^2)$ with $(x_1, x_2) \le U(t)$, for every $t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$. From hypothesis (T_1) we obtain

$$\Phi(x_1, x_2) \le (u_2(t), f(u_1(t))) = F(U)(t), t \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

Hence, (T_1) implies (H_4).

We have (see [3, 8])

$$\sup_{t \in [0,1]} \int_0^1 G(t,s)\, ds = \frac{1}{8}.$$

Therefore, (1.4) is equivalent to (2.9).

A simple computation shows

$$\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} G(t,s)\, ds = g(t), \quad t \in [0,1].$$

Then (T_3) is necessary for (H_6). Hypothesis (H_7) is implied by the continuity of g, G and f. □

Remark 2.1. *For $f:\mathbb{R}_+\to\mathbb{R}_+$ a nondecreasing map, we can take $\Phi:\mathbb{R}_+^2\to\mathbb{R}_+^2$ given by*

$$\Phi(x_1,x_2)=(x_2,f(x_1)).$$

Remark 2.2. *For $t^*=\frac{1}{2}$, inequality* (2.10) *can be replaced by*

$$\Sigma>\frac{2(\beta+\sigma_m)}{\varepsilon(1-\varepsilon)},$$

see [3, 8] *for details.*

Remark 2.3. *Theorem* 2.4 *can be extended to the nonlinear $2n$-order boundary value problem*

$$\begin{aligned}&u^{(2n)}(t)=f(u(t)),\ \ t\in[a,b]\\&u(a)=u(b)=A_1\\&u''(a)=u''(b)=A_2\\&u^{(4)}(a)=u^{(4)}(b)=A_3\\&\ldots\\&u^{(2n-2)}(a)=u^{(2n-2)}(b)=A_n\end{aligned}$$

and some similar results can be established for non autonomous equations.

3. Multiple solutions result

This part was inspired by [3] where a three symmetric positive solution result is established.

Consider the fourth order Lidstone boundary value problem

$$\begin{cases}u^{(4)}(t)=f(u(t)),\ \ 0<t<1\\u(0)=u(1)=u''(0)=u''(1)=0.\end{cases}\tag{3.14}$$

where $f:\mathbb{R}_+\to\mathbb{R}_+$ is continuous. Here $A=B=0$. To obtaining three positive solutions we have the next results:

Theorem 3.5. *Let $\varepsilon\in\left(0,\frac{1}{2}\right)$, $\mu\in\left(0,\frac{1}{2}-\varepsilon\right]$ and $f:\mathbb{R}_+\to\mathbb{R}_+$ is continuous. Suppose that there exist $0<\beta_1<\alpha_1<\beta_2<\alpha_2$ such that*

(M_1) *there exists a map $\Phi:\mathbb{R}_+^2\to\mathbb{R}_+^2$ such that*

$$(x_1,x_2)\le(y_1,y_2)\ \textit{ implies }\Phi(x_1,x_2)\le(y_2,f(y_1))$$

(M_2) *for all $y_i=(y_{i1},y_{i2})\in\mathbb{R}_+^2$ with $\max\{y_{i1},y_{i2}\}=\alpha_i$, $i\in\{1,2\}$, we have*

$$\max\{f(y_{i1}),y_{i2}\}<8\alpha_i,\ \ i\in\{1,2\}$$

(M_3) *there exist* $t_1^*, t_2^* \in [0,1]$ *such that*

$$\Sigma_i \cdot g(t_i^*) < \beta_i, \;\; i \in \{1,2\}.$$

where $\Sigma_i = \inf\{|\Phi(x_1,x_2)| : x_1, x_2 \in \mathbb{R}_+, \max\{x_1,x_2\} = \mu\beta_i\}$ *and*

$$g(t) = \begin{cases} \varepsilon t & if \quad t \in \left[0, \frac{1}{2} - \varepsilon\right] \\ \frac{t-t^2}{2} - \frac{1}{8} + \frac{\varepsilon - \varepsilon^2}{2} & if \quad t \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right) \\ \varepsilon(1-t) & if \quad t \in \left[\frac{1}{2} + \varepsilon, 1\right] \end{cases}.$$

Then the problem (3.14) *has at least three positive solutions* u_1, u_2, u_3 *such that*

$$0 < \beta_1 < \max_{t\in[0,1]} |u_1(t)| < \alpha_1 < \max_{t\in[0,1]} |u_2(t)| < \beta_2 < \max_{t\in[0,1]} |u_3(t)| < \alpha_2$$

with

$$\mu u_i(t) \le u_i(t_i') \;\; and \; \mu u_i''(t) \ge u_i''(t_i'), i \in \{1,2,3\}$$

for $t \in [0,1]$, $t_i' \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right]$, $i \in \{1,2,3\}$.

Proof. The conclusions follow from Theorem 1.3 for $n = m = 2$. □

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