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A NEW FAMILY OF MODIFIED NEWTON METHODS WITH CUBIC CONVERGENCE

YANJIAN LU AND XIUBIN XU

Department of Mathematics
Zhejiang Normal University
Jinhua 321004, China
E-mails: xxu@zjnu.cn luyj25@163.com

Abstract. In this paper, we present a new family of modified Newton methods, which includes, as particular cases, some known results. It is proved that each method in the family is cubically convergent. A general error analysis is given, and the computational efficiency in term of function evaluations is provided. Numerical illustrations are given to compare the proposed methods with some other methods of the same kind.

Key Words and Phrases: Newton's method, order of convergence, iterative method, nonlinear equations, computational efficiency.
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1. Introduction

One of the most important problems in numerical analysis is the solution of a nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function defined on an open interval D. To solve this equation we can use iterative methods. The best known and the most widely used example of these types of methods is the classical Newton method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically.

There exists an extension due to Potra and Pták called the "two-step method" (see [1]) that may be rewritten as the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n) + f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)} \tag{2}$$

This method converges cubically. In general, it is cheaper than any other third-order methods requiring the evaluation of the second derivative.

For quite a long time, this was the only known method converging cubically apart from methods that involve higher-order derivatives (for a recent review of the latter methods, see [2])

Recently, some new modified Newton methods with cubic convergence have been developed, for example, see [3]-[9].

Weerakoon and Fernando [3] derived the Newton method from the Newton-Leibniz formula

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(\mu)d\mu \tag{3}$$

by approximating the integral by the rectangular rule

$$\int_{x_n}^{x} f'(\mu)d\mu \approx (x - x_n)f'(x_n)$$

and using $f(x) = 0$. When they used the trapezoidal rule, they arrived at the modified Newton-type iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})}, \tag{4}$$

and proved that this scheme converges cubically.

Since Weerakoon and Fernando's work, the deduction of Newton method from the integral formulation and the modified Newton methods from quadrature approach have received more and more attention. The midpoint rule for the integral of (3) gives the following cubically convergent scheme [4, 5]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2f'(x_n)})}, \tag{5}$$

which has also been derived by Homeier in [6] via some other approach.

In [7], Kou, Li and Wang considered the computation of the definite integral on a new interval of integration arising from Newton-Leibniz formula

$$f(x) = f(y_n) + \int_{y_n}^{x} f'(\mu)d\mu, \tag{6}$$

where $y_n = x_n + \frac{f(x_n)}{f'(x_n)}$. They used the midpoint rule to approximate the integral $\int_{y_n}^{x} f'(\mu)d\mu$ and gave the scheme

$$x_{n+1} = x_n - \frac{f(x_n + \frac{f(x_n)}{f'(x_n)}) - f(x_n)}{f'(x_n)}. \tag{7}$$

They proved that it also converges cubically.

Frontini and Sormani [8] generalized the approach of Weerakoon and Fernando by using general interpolatory quadrature rules of order at least one of the type

$$\int_{x_n}^{x} g(\mu)d\mu \approx (x - x_n) \sum_{i=1}^{m} \omega_i g(\zeta_i) \tag{8}$$

with $\zeta_i = x_n + \tau_i(x - x_n)$, knots τ_i in $[0, 1]$ and weights ω_i satisfying $\sum_{i=1}^{m} \omega_i = 1, \sum_{i=1}^{m} \omega_i \tau_i = 1/2$, and they obtained a family of cubically convergent iterative methods

$$x_{n+1} = x_n - \frac{f(x_n)}{\sum_{i=1}^{m} \omega_i f'(\hat{\zeta}_i)}$$

with $\hat{\zeta}_i = x_n - \tau_i \frac{f(x_n)}{f'(x_n)}$.

In [9], instead of using the Newton-Leibniz formula for $y = f(x)$, Homeier used it for the inverse function $x(y)$

$$x(y) = x(y_n) + \int_{y_n}^{y} x'(\mu)d\mu.$$

By approximating $\int_{y_n}^{y} x'(\mu)d\mu$ via an interpolatory quadrature similar to (8), they introduced such a class of cubically convergent modified Newton methods

$$x_{n+1} = x_n - f(x_n) \sum_{i=1}^{m} \omega_i \frac{1}{f'(\hat{\zeta})}$$

with $\hat{\zeta}_i = x_n - \tau_i \frac{f(x_n)}{f'(x_n)}$. The instance, corresponding to $m = 2, \omega_1 = \omega_2 = \frac{1}{2}, \tau_1 = 1 - \tau_2 = 0$, is

$$x_{n+1} = x_n - \frac{f(x_n)}{2} \left[\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - \frac{f(x_n)}{f'(x_n)})} \right], \tag{9}$$

which has also been derived by Özban in [5] independently.

These modifications of Newton method are interesting and very important because per iteration they only require the evaluations of the function f and its first derivative f', not requiring the second derivation f'', but they also converge cubically.

In this paper, instead of using interpolation for an integrand, we use it for the function in Newton's method directly. In Section 2, we introduce a new family of modified Newton methods, which includes (2) and (7) as special cases, and show that these methods are cubically convergent. In Section 3, we compare one of our new methods with the above mentioned various methods by numerical test results. The results show that our proposed method behaves equal or better performance in many cases.

2. New methods and their convergence

In order to obtain the solutions of nonlinear equation $f(x) = 0$, we may combine Newton method and Lagrange polynomial interpolation for $f(x)$. Consider the following interpolation

$$f(x) \approx \sum_{i=0}^{m} f(x_i) l_i(x),$$

where $x_i = x_n - \beta_i \frac{f(x_n)}{f'(x_n)}$, β_i is a set of disposable parameters, and $l_i(x)$ is the set of basis functions.

By replacing $f(x_n)$ in Newton method (1) with $\sum_{i=0}^{m} f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)}) l_i(x_n)$, we have

$$x_{n+1} = x_n - \frac{\sum_{i=0}^{m} f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)}) l_i(x_n)}{f'(x_n)}$$

Setting $l_i(x_n) = \alpha_i$, we obtain a new family of modified Newton methods

$$x_{n+1} = x_n - \frac{\sum_{i=0}^{m} \alpha_i f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)})}{f'(x_n)} \tag{10}$$

where α_i is a set of parameters just depend on β_i.

Noting that for every non-zero values of the disposable parameters, per iteration of (10), the method only requires the evaluations of the function f and its first derivative f'. Now we shall prove the following theorem.

Theorem 1. *Let $x^* \in D$ be a simple zero of a function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D. Assume that f has derivatives up to the third order in D*

and x_0 is sufficiently close to x^*. Then the iterative method (10) is cubically convergent to x^* if and only if the parameters satisfy

$$\sum_{i=0}^{m} \alpha_i - \sum_{i=0}^{m} \alpha_i \beta_i = 1 \quad \textit{and} \quad \sum_{i=0}^{m} \alpha_i \beta_i^2 = 1.$$

Furthermore, in such cases, the errors satisfy the following difference equation:

$$e_{n+1} = [2A_2^2 + (\sum_{i=0}^{m} \alpha_i \beta_i^3 - 1)A_3]e_n^3 + O(e_n^4),$$

where $e_n = x_n - x^*$ and $A_j = \frac{f^{(j)}(x^*)}{j! f'(x^*)}$, $j = 2, 3$.

Proof. Since f has derivatives up to the third order, by expanding $f(x_n)$ and $f'(x_n)$ at x^*, we get

$$f(x_n) = f'(x^*)(e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4))$$

and

$$f'(x_n) = f'(x^*)(1 + 2A_2 e_n + 3A_3 e_n^2 + O(e_n^3)).$$

Since the terms in the parentheses are polynomials in terms of e_n, direct division gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - A_2 e_n^2 + 2(A_2^2 - A_3)e_n^3 + O(e_n^4),$$

and hence

$$x_n - \beta_i \frac{f(x_n)}{f'(x_n)} = x^* + (1 - \beta_i)e_n + A_2 \beta_i e_n^2 - 2(A_2^2 - A_3)\beta_i e_n^3 + O(e_n^4).$$

This, together with Taylor expansion of $f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)})$ at x^*, gives

$$f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)}) = f'(x^*)[(1 - \beta_i)e_n + A_2(1 - \beta_i + \beta_i^2)e_n^2$$
$$-(2A_2^2 \beta_i^2 - A_3(1 - \beta_i + 3\beta_i^2 - \beta_i^3))e_n^3 + O(e_n^4)].$$

Therefore we have

$$\sum_{i=0}^{m} \alpha_i f(x_n - \beta_i \frac{f(x_n)}{f'(x_n)}) = f'(x^*)\Big[(\sum_{i=0}^{m} \alpha_i - \sum_{i=0}^{m} \alpha_i \beta_i)e_n$$
$$+A_2(\sum_{i=0}^{m} \alpha_i - \sum_{i=0}^{m} \alpha_i \beta_i + \sum_{i=0}^{m} \alpha_i \beta_i^2)e_n^2 - (2A_2^2 \sum_{i=0}^{m} \alpha_i \beta_i^2$$

$$-A_3(\sum_{i=0}^{m}\alpha_i-\sum_{i=0}^{m}\alpha_i\beta_i+3\sum_{i=0}^{m}\alpha_i\beta_i^2-\sum_{i=0}^{m}\alpha_i\beta_i^3)e_n^3+O(e_n^4)\Big]. \tag{11}$$

It follows from (10) that

$$e_{n+1}=e_n-\frac{\sum_{i=0}^{m}\alpha_i f(x_n-\beta_i\frac{f(x_n)}{f'(x_n)})}{f'(x_n)}. \tag{12}$$

Substituting (11) into (12) yields

$$\begin{aligned}e_{n+1}&=(1-\sum_{i=0}^{m}\alpha_i+\sum_{i=0}^{m}\alpha_i\beta_i)e_n+(\sum_{i=0}^{m}\alpha_i-\sum_{i=0}^{m}\alpha_i\beta_i-\sum_{i=0}^{m}\alpha_i\beta_i^2)A_2e_n^2\\&-\Big[(2\sum_{i=0}^{m}\alpha_i-2\sum_{i=0}^{m}\alpha_i\beta_i-4\sum_{i=0}^{m}\alpha_i\beta_i^2)A_2^2-(2\sum_{i=0}^{m}\alpha_i-2\sum_{i=0}^{m}\alpha_i\beta_i\\&-3\sum_{i=0}^{m}\alpha_i\beta_i^2+\sum_{i=0}^{m}\alpha_i\beta_i^3)A_3\Big]e_n^3+O(e_n^4).\end{aligned} \tag{13}$$

Thus $e_{n+1}=O(e_n^3)$ if and only if $\sum_{i=0}^{m}\alpha_i-\sum_{i=0}^{m}\alpha_i\beta_i=1$ and $\sum_{i=0}^{m}\alpha_i\beta_i^2=1$. In such cases, we obtain the order of e_n

$$e_{n+1}=[2A_2^2+(\sum_{i=0}^{m}\alpha_i\beta_i^3-1)A_3]e_n^3+O(e_n^4). \tag{14}$$

The proof is complete.

Remark 1. (i). *We can see from (13) that the family (10) is quadratically convergent if we only require* $\sum_{i=0}^{m}\alpha_i-\sum_{i=0}^{m}\alpha_i\beta_i=1$. *Thus the well-known Newton method (1) can be obtained from (10) with*

$$m=0,\qquad \alpha_0=1,\qquad \beta_0=0.$$

(ii). *From the previous theorem, it is not difficult to see that the method (7) may be obtained from (10) with*

$$m=1,\qquad \alpha_0=-1,\qquad \beta_0=0,\qquad \alpha_1=1,\qquad \beta_1=-1;$$

while, for the method (2), we have

$$m=1,\qquad \alpha_0=1,\qquad \beta_0=0,\qquad \alpha_1=1,\qquad \beta_1=1.$$

(iii). *Specializing to*

$$m=1,\qquad \alpha_0=\alpha_1=1/4,\qquad \beta_0=0,\qquad \beta_1=-2,$$

we obtain a new cubically convergent scheme

$$x_{n+1}=x_n-\frac{f(x_n)+f(x_n+2\frac{f(x_n)}{f'(x_n)})}{4f'(x_n)} \tag{15}$$

with the order relation

$$e_{n+1} = (2A_2^2 - 3A_3)e_n^3 + O(e_n^4).$$

We define a computational efficiency for an iterative method as $p^{\frac{1}{\omega}}$ [10], where p is the order of the method and ω is the number of function evaluations required by the method. The method (15) has the computational efficiency equal to $\sqrt[3]{3} \approx 1.442$, which is better than the one of Newton method ($\sqrt{2} \approx 1.414$).

3. Numerical examples

In this section, we present some numerical test results for various cubically convergent schemes in Table 1. All tests were done using Matlab 6.1 on a PC using 32 digit floating point arithmetics. Compared are Newton method (1), the various third-order modifications of Newton method (mNm) mentioned in Section 1, and our new method (15).

As test functions, we use the functions

$f_1(x) = x^3 + 4x^2 - 10,$

$f_2(x) = e^{x^2+7x-30} - 1,$

$f_3(x) = e^{1-x} - 1,$

$f_4(x) = \frac{1}{x} - \sin x + 1,$

$f_5(x) = x - 3\ln x,$

$f_6(x) = x^2 + \sin\frac{x}{5} - \frac{1}{4},$

$f_7(x) = \frac{1}{x} - 1,$

$f_8(x) = \arctan x.$

For every function, we try to seek an approximation x_n of the root x^* of equation $f(x) = 0$ after n times iterations. The corresponding function absolute values $|f(x_n)|$ are also given. The number of function evaluations (NFE) is counted as the sum of the number of evaluations of the function f itself plus the number of evaluations of the first derivative f'.

Table 1: Comparison of various iterative methods

	n	NFE	x_n	$\|f(x_n)\|$
	$f_1, x_0 = -0.8$			

	n	NFE	x_n	$\lvert f(x_n)\rvert$
Newton	36	72	1.3652	0
mNm (2)	11	33	1.3652	3.55271e-15
mNm (4)	56	168	1.3652	0
mNm (5)	24	72	1.3562	0
mNm (7)	14	42	1.3652	0
mNm (9)	11	33	1.3562	0
mNm (15)	5	15	1.3562	0
	$f_2, x_0 = 5$			
Newton	36	72	3	0
mNm (2)	26	78	3	3.55271e-14
mNm (4)	25	75	3	0
mNm (5)	22	66	3	9.23706e-14
mNm (7)	22	66	3	0
mNm (9)	20	60	3	0
mNm (15)	18	54	3	0
	$f_3, x_0 = 4$			
Newton	23	46	1	0
mNm (2)	Failure (Divided by zero)			
mNm (4)	Divergence			
mNm (5)	105	315	1	0
mNm (7)	7	21	1	0
mNm (9)	9	27	1	0
mNm (15)	8	24	1	0
	$f_4, x_0 = -1.3$			
Newton	27	54	-0.6294	6.43929e-14
mNm (2)	57	171	-1.7389e+3	1.55431e-15
mNm (4)	Divergence			
mNm (5)	6	18	-0.6294	2.22045e-16
mNm (7)	7	21	-3.9886	1.11022e-16
mNm (9)	4	12	-0.6294	4.64073e-14

	n	NFE	x_n	$\|f(x_n)\|$
mNm (15)	5	15	-0.6294	4.44089e-16
	$f_5, x_0 = 2$			
Newton	5	10	1.8572	2.22045e-16
mNm (2)	4	12	1.8572	0
mNm (4)	4	12	1.8572	0
mNm (5)	4	12	1.8572	2.22045e-16
mNm (7)	4	12	1.8572	2.22045e-16
mNm (9)	4	12	1.8572	2.22045e-16
mNm (15)	4	12	1.8572	0
	$f_6, x_0 = 1$			
Newton	6	12	0.4100	1.13243e-14
mNm (2)	5	15	0.4100	0
mNm (4)	5	15	0.4100	0
mNm (5)	5	15	0.4100	0
mNm (7)	5	15	0.4100	2.77556e-17
mNm (9)	4	12	0.4100	2.77556e-17
mNm (15)	5	15	0.4100	0
	$f_7, x_0 = 2$			
Newton	Failure (Divided by zero)			
mNm (2)	Failure (Divided by zero)			
mNm (4)	Failure (Divided by zero)			
mNm (5)	6	18	1	0
mNm (7)	2	6	1	0
mNm (9)	Failure (Divided by zero)			
mNm (15)	6	18	1	0
	$f_8, x_0 = 2$			
Newton	Failure (Divided by zero)			
mNm (2)	Divergence			
mNm (4)	Failure (Divided by zero)			

	n	NFE	x_n	$\lvert f(x_n)\rvert$
mNm (5)	5	15	0	0
mNm (7)	5	15	0	1.15797e-16
mNm (9)	Failure (Divided by zero)			
mNm (15)	6	18	0	1.46643e-19

The results show that our new method (15) performs better than the other methods do in the cases f_1, f_2, f_3 and f_4 (only one method beats (15) in the cases f_3 and f_4, respectively). The cubically convergent schemes, especially our new method (15), can compete with the Newton method and seem to be superior in difficult cases where Newton method fails in convergence, as functions f_7 and f_8.

4. Conclusion

In this work, we presented a new family of iterative methods (15) based on interpolatory approach. We have proved that, if we set the values $\sum_{i=0}^{m} \alpha_i - \sum_{i=0}^{m} \alpha_i\beta_i = 1$ and $\sum_{i=0}^{m} \alpha_i\beta_i^2 = 1$, then the convergence orders of the iterative methods are always three. Among this family of methods, we can obtain efficient iterative methods different from any known schemes. Some examples were tested, showing equal or better performance than some other methods of the same kind.

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