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ON SOME VOLTERRA INTEGRAL INEQUALITIES

NICOLAIE LUNGU

Technical University of Cluj-Napoca
Department of Mathematics
C. Daicoviciu 15
E-mail: nlungu@math.utcluj.ro

Abstract. This paper presents a certain Volterra integral inequalities which is based on operatorial inequalities for Picard and weakly Picard operators.
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1. Introduction

The Gronwall inequality was generalized in various directions. In [12]-[14], I.A. Rus established some new operatorial inequalities for Picard and weakly Picard operators. The theory of Picard operators (PO, for short) is very useful in studying the properties of the solutions of Volterra integral equations.

In this paper we presented a generalization of the results from [5] and [10]. The notions and notations from [13], [14] are used. Two important results established in [13], [14] are Lemma 1.1 and Lemma 1.2.

Lemma 1.1. (I.A. Rus [13]) (Abstract Gronwall lemma). *Let* $(X, \to, \leq)$ *be an ordered L-space and* $A : X \to X$ *an operator. We suppose that*

(i) A is PO;

(ii) A is increasing.

If we denote by x_A^* *the unique fixed point of A, then*

(a) $x \leq A(x) \;\Rightarrow\; x \leq x_A^*$;

(b) $x \geq A(x) \;\Rightarrow\; x \geq x_A^*$.

Lemma 1.2. (I.A. Rus [14]) (Abstract Gronwall-comparison lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that:*

(i) A and B are POs;

(ii) A is increasing;

(iii) $A \leq B$.

Then

$$x \leq A(x) \;\Rightarrow\; x \leq x_B^*,$$

x_B^* is the unique fixed point of B.

2. Main results

2.1. Volterra integral equations

In what follows we consider the integral equation

$$u(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_n) + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) u(s_1, x_2, \ldots, x_n) ds_1$$
$$+ \int_0^{x_1} \int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_n) u(s_1, s_2, x_3, \ldots, x_n) ds_1 ds_2 + \cdots +$$
$$+ \int_0^{x_1} \int_0^{x_2} \ldots \int_0^{x_n} K_n(s_1, s_2, \ldots, s_n) u(s_1, s_2, \ldots, s_n) ds_1 ds_2 \ldots ds_n, \quad (2.1)$$

where we suppose that $a_i > 0$, $i = \overline{1, n}$, $D = \prod_{i=1}^{n} [0, a_i]$, $h \in C(D, \mathbb{R})$, $K_i \in C(D)$ for $i \in \{1, 2, \ldots, n\}$. There exists for all $i \in \{1, \ldots, n\}$, $M_{K_i} > 0$ such that $|K_i(x)| \leq M_{K_i}$, $\forall\, x \in D$. Equation (2.1) is a generalization of the corresponding equation from [5] and [10].

Theorem 2.1. *If $K_i \in C(D)$, $h \in C(D)$, $i \in \{1, 2, \ldots, n\}$, then equation (2.1) has in $C(D)$ a unique solution u_A^*, and $A^n(u)$ converge uniformly to u_A^* as $n \to \infty$, for all $u \in C(D)$.*

Proof. Let $A : C(D) \to C(D)$ be the operator defined by

$$A(u)(x_1, \ldots, x_n) := \text{second part of } (2.1).$$

If we consider the Banach space $(C(D), \|\cdot\|_B)$ where $\|\cdot\|_B$ is the Bielecki norm

$$\|u\|_B := \max_{x \in D} (|u(x)| e^{-\tau x_1}), \text{ for } \tau > 0, \quad (2.2)$$

then the operator A is Lipschitz with the constant

$$L_A = \frac{1}{\tau}(M_{K_1} + M_{K_2}a_2 + \cdots + M_{K_n}a_2 \dots a_n). \tag{2.3}$$

Thus A is a contraction with respect to $\|\cdot\|_B$, for $\tau > 0$ suitable chosen. Therefore A is PO in the L-space $(CD, \xrightarrow{\|\cdot\|_B})$ and the conclusion follows from the Banach fixed point theorem.

Theorem 2.2. *We suppose that* $h \in C(D, \mathbb{R}_+)$, $K_i \in C(D, \mathbb{R}_+)$, *for* $i \in \{1, 2, \dots, n\}$. *Then*

(a) $u_A^*(x) > 0, \ \forall\, x \in D$;

(b) If $h(x_1, \dots, x_n)$, $K_i(x_1, \dots, x_n)$ *are increasing with respect to* $x_{i+1}, \dots, x_n$, $i \in \{1, \dots, n-1\}$, *then* u_A^* *is increasing.*

Proof. (a) Let $u_0 \in C(D, \mathbb{R}_+)$ and $u_n := A^n(u_0)$, $n \in \mathbb{N}^*$. Then from (2.1) we have that $u_n(x) > 0$, for all $x \in D$ and u_n is increasing. By Theorem 2.1, $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u_A^* ($u_n \xrightarrow{unif} u_A^*$). So, we have (a).

(b) The conclusion follows immediately from (a) and from (2.1).

2.2. Volterra integral inequations

Consider now the inequation

$$u \le A(u). \tag{2.4}$$

By definition a solution of (2.4) is a **lower** solution of (2.1).

Theorem 2.3. *We suppose that conditions of Theorem 2.2 are satisfied. If* $u \in C(D, \mathbb{R}_+)$ *is a lower solution of (2.1), then*

$$u(x) \le \overline{u}^*(x), \tag{2.5}$$

where

$$\overline{u}^*(x) = h(x) + \int_0^{x_1} K_1(s_1, x_2, \dots, x_n) h(s_1, x_2, \dots, x_n)$$
$$\times \exp\left(\int_{s_1}^{x_1} K_1(t_1, x_2, \dots, x_n)dt_1 + \cdots + \right.$$
$$\left. + \int_{s_1}^{x_1}\int_0^{x_2}\dots\int_0^{x_n} K_n(t_1, s_2, \dots, s_n)dt_1 ds_2 \dots ds_n\right) ds_1 + \cdots +$$
$$+ \int_0^{x_1}\dots\int_0^{x_n} K_n(s)h(s)\exp\left(\int_{s_1}^{x_1} K_1(t_1, x_2, \dots, x_n)dt_1\right.$$

$$+\int_{s_1}^{x_1}\int_{s_2}^{x_2} K_2(t_1,t_2,x_3,\dots,x_n)dt_1dt_2+\dots+\int_{s_1}^{x_1}\dots\int_{s_n}^{x_n} K_n(t)dt\Bigg)\,ds, \tag{2.6}$$

$t=(t_1,\dots,t_n)$, $dt=dt_1\dots dt_n$, $s=(s_1,\dots,s_n)$, $ds=ds_1\dots ds_n$, $x=(x_1,\dots,x_n)$.

Proof. If we consider equation (2.1), and u_A^* the unique fixed point of A, then, by Lemma 1.1 we have

$$u(x)\le u_A^*(x).$$

In what follows we shall prove that $u_A^*(x)\le \overline{u}^*(x)$.

If we denote

$$g(x)=\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)u_A^*(s_1,x_2,\dots,x_n)ds_1+\dots+$$

$$+\int_0^{x_1}\dots\int_0^{x_n} K_n(s_1,\dots,s_n)u_A^*(s_1,\dots,s_n)ds_1\dots ds_n,\quad g(0,x_2,\dots,x_n)=0,$$

we have

$$u_A^*(x)=h(x)+g(x). \tag{2.7}$$

Using (2.7) and the fact that $g(x)$ is increasing, we obtain

$$\frac{\partial g}{\partial x_1}\le\Bigg[K_1(x)+\int_0^{x_2}K_2(x_1,s_2,x_3,\dots,x_n)ds_2+\dots+$$

$$+\int_0^{x_2}\dots\int_0^{x_n}K_n(x_1,s_2,\dots,s_n)ds_2\dots ds_n\Bigg]g(x)$$

$$+\Big[K_1(x_1,\dots,x_n)h(x_1,\dots,x_n)$$

$$+\int_0^{x_2}K_2(x_1,s_2,x_3,\dots,x_n)h(x_1,s_2,x_3,\dots,x_n)ds_2+\dots+$$

$$+\int_0^{x_2}\dots\int_0^{x_n}K_n(x_1,s_2,\dots,s_n)h(x_1,s_2,\dots,s_n)ds_2\dots ds_n\Big]. \tag{2.8}$$

Let

$$f(x)=g(x)\exp\Bigg[-\int_0^{x_1}K_1(s_1,x_2,\dots,x_n)ds_1$$

$$-\int_0^{x_1}\int_0^{x_2}K_2(s_1,s_2,x_3,\dots,x_n)ds_1ds_2-\dots-\int_0^{x_1}\dots\int_0^{x_n}K_n(s)ds\Bigg]. \tag{2.9}$$

A simple calculation gives

$$\frac{\partial f}{\partial x_1}=\Bigg[\frac{\partial g}{\partial x_1}-g(x)\Bigg(K_1(x)+\int_0^{x_2}K_2(x_1,s_2,x_3,\dots,x_n)ds_2+\dots+$$

$$+\int_0^{x_2}\dots\int_0^{x_n} K_n(x_1,s_2,\dots,s_n)ds_2\dots ds_n\Big)\Big]$$

$$\times\exp\left[-\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)ds_1-\dots-\int_0^{x_1}\dots\int_0^{x_n} K_n(s)ds\right]. \quad (2.10)$$

Using now (2.8) and (2.10) we obtain

$$\frac{\partial f}{\partial x_1}\le\Big[K_1(x)h(x)+\int_0^{x_2} K_2(x_1,s_2,x_3,\dots,x_n)h(x_1,s_2,x_3,\dots,x_n)ds_2+\dots+$$

$$+\int_0^{x_2}\dots\int_0^{x_n} K_n(x_1,s_2,\dots,s_n)h(x_1,s_2,\dots,s_n)ds_2\dots ds_n\Big]$$

$$\times\exp\left[-\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)ds_1-\dots-\int_0^{x_1}\dots\int_0^{x_n} K_n(s)ds\right]. \quad (2.11)$$

Taking into account (2.9) it follows that

$$g(x)=f(x)\exp\Big[\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)ds_1$$

$$+\int_0^{x_1}\int_0^{x_2} K_2(s_1,s_2,x_3,\dots,x_n)ds_1ds_2+\dots+\int_0^{x_1}\dots\int_0^{x_n} K_n(s)ds\Big].$$

Hence

$$g(x)\le\int_0^{x_1} K_1(s_1,x_2,\dots,x_n)h(s_1,x_2,\dots,x_n)$$

$$\times\exp\Big(\int_{s_1}^{x_1} K_1(t_1,x_2,\dots,x_n)dt_1+\dots+$$

$$+\int_{s_1}^{x_1}\int_0^{x_2}\dots\int_0^{x_n} K_n(t_1,s_1,\dots,s_n)dt_1ds_2\dots ds_n\Big)ds_1+\dots+$$

$$+\int_0^{x_1}\dots\int_0^{x_n} K_n(s)h(s)\exp\Big(\int_{s_1}^{x_1} K_1(t_1,x_2,\dots,x_n)dt_1$$

$$+\int_{s_1}^{x_1}\int_{s_2}^{x_2} K_2(t_1,t_2,x_3,\dots,x_n)dt_1dt_2+\dots+\int_{s_1}^{x_1}\dots\int_{s_n}^{x_n} K_n(t)dt\Big)ds, \quad (2.12)$$

and therefore

$$u_A^*(x)\le h(x)+\alpha(x)$$

where $\alpha(x)$ is the right hand side of (2.12). If we denote $\overline{u}^*(x)=h(x)+\alpha(x)$, then

$$u_A^*(x)\le\overline{u}^*(x).$$

On the other hand $u(x)\le u_A^*(x)$ implies $u(x)\le\overline{u}^*(x)$.

Remark 1. If $h(x)$ is increasing and strictly positive, then setting $v(x) = u(x)/h(x)$, from (2.1) we obtain

$$v(x) \leq 1 + \int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) v(s_1, x_2, \ldots, x_n) ds$$

$$+ \int_0^{x_1} \int_0^{x_2} K_2(s_1, s_2, x_3, \ldots, x_n) v(s_1, s_2, x_3, \ldots, x_n) ds_1 ds_2 + \cdots +$$

$$+ \int_0^{x_1} \ldots \int_0^{x_n} K_n(s) v(s) ds. \tag{2.13}$$

This is a particular case of equation (4.1) from [10], for $\alpha = 1$. Moreover, we have

$$v(x) \leq \exp\left(\int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) ds_1 + \cdots + \int_0^{x_1} \ldots \int_0^{x_n} K_n(s) ds \right)$$

and

$$u(x) \leq h(x) \exp\left(\int_0^{x_1} K_1(s_1, x_2, \ldots, x_n) ds_1 + \cdots + \int_0^{x_1} \ldots \int_0^{x_n} K_n(s) ds \right). \tag{2.14}$$

Remark 2. If we consider the equation

$$u(x_1, \ldots, x_n) = h(x_1, \ldots, x_n) + \int_0^{x_1} F_1(s_1, x_2, \ldots, x_n, u(s_1, x_2, \ldots, x_n)) ds_1$$

$$+ \int_0^{x_1} \int_0^{x_2} F_2(s_1, s_2, x_3, \ldots, x_n, u(s_1, s_2, x_3, \ldots, x_n)) ds_1 ds_2 + \cdots +$$

$$+ \int_0^{x_1} \ldots \int_0^{x_n} F_n(s_1, \ldots, s_n, u(s_1, \ldots, s_n)) ds_1 \ldots ds_n, \tag{2.15}$$

and we take

$$F_i(x, u) \leq K_i(x) u(x),$$

where $|K_i(x)| \leq M_{K_i}$, for $i \in \{1, 2, \ldots, n\}$, then the operator A is right hand side of (2.15). By Lemma 1.2 we consider the operator B second part of (2.1) and we obtain an analogous result.

Other applications were studied by R.P. Agarwal [1], [2], A. Buică [3], C. Corduneanu [4], V. Lakshmikantham, S. Leela, A. A. Martynyuk [6], D. Popa, N. Lungu [11], V. Ya. Stetsenko, M. Shaban [15] and M. Zima [16].

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