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SOME RESULTS ON KIRK'S ASYMPTOTIC CONTRACTIONS

E.M. BRISEID

Department of Mathematics
Darmstadt University of Technology
Schlossgartenstrasse 7, 64289 Darmstadt, Germany
E-mail: briseid@mathematik.tu-darmstadt.de
eyvindbriseid@gmail.com

Abstract. In [W.A. Kirk, Fixed points of asymptotic contractions, J. Math. Anal. Appl. 277 (2003) 645-650], W.A. Kirk proved a fixed point theorem for asymptotic contractions $f : X \to X$ on complete metric spaces (X, d). In the theorem it is assumed that some Picard iteration sequence $(f^n(x_0))$ is bounded. Here we prove that for an asymptotic contraction $f : X \to X$ on a metric space (X, d) any Picard iteration sequence is bounded, thus making the above mentioned assumption of the theorem superfluous. We also provide, given an asymptotic contraction $f : X \to X$ on a complete metric space (X, d), an explicit rate of convergence for a Picard iteration sequence $(f^n(x_0))$ which does not depend on a bound on the iteration sequence, but which instead depends on (strictly positive) upper and lower bounds on $d(x_0, f(x_0))$. This is thus in a sense an improvement on the results in [E.M. Briseid, A rate of convergence for asymptotic contractions, J. Math. Anal. Appl., 330 (2007) 364-376], where the rate of convergence depends on a bound on the iteration sequence. In both cases the rate of convergence also depends on some moduli for the mapping appearing as parameters, but is again in both cases otherwise fully uniform. We can also easily adapt to the situation where the space is not complete.
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1. Introduction

In [6], W.A. Kirk introduced asymptotic contractions and proved a fixed point theorem for these. We give for reference the definition and the theorem.

Definition 1.1. ([6]) *A function $f : X \to X$ on a metric space (X, d) is called an* asymptotic contraction in the sense of Kirk *with moduli $\phi, \phi_n : [0, \infty) \to [0, \infty)$ if ϕ, ϕ_n are continuous, $\phi(s) < s$ for all $s > 0$ and for all $x, y \in X$*

$$d(f^n(x), f^n(y)) \leq \phi_n(d(x, y)),$$

and moreover $\phi_n \to \phi$ uniformly on the range of d.

Theorem 1.2. ([6]) *Let (X, d) be a complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk. If for some $x \in X$ the Picard iteration sequence $(f^n(x))$ is bounded, then f has a unique fixed point $z \in X$ and for every starting point $x \in X$ the iteration sequence $(f^n(x))$ converges to z.*

(Note that, as remarked in e.g. [1], [5], in the statement of the theorem in [6] the assumption that the mapping must be continuous was inadvertently left out. Note also that in [6] the mappings in Definition 1.1 are called just asymptotic contractions.) We prove here that the assumption that one iteration sequence is bounded can be dropped. To this end we consider the modified definition of an asymptotic contraction given in [2]. Our work makes use of the analysis of Kirk's theorem on asymptotic contractions given by P. Gerhardy in [4]. In [4], Gerhardy uses techniques from the program of proof mining as developed by U. Kohlenbach (see e.g. [7], [8]) to develop a quantitative version of Kirk's theorem and give an elementary proof of the theorem.

See also I.D. Arandelović ([1]), Y.-Z. Chen ([3]), J. Jachymski and I. Jóźwik ([5]), and K. Włodarczyk, D. Klim and R. Plebaniak ([9]) for further work concerning versions of asymptotic contractions. See [2] for some comments on these (except [9], which appeared quite recently). In [3] and [5] conditions are given which allow one to remove the requirement that some iteration sequence is bounded from the corresponding theorems, but this removal is obtained only by introducing further limit requirements on the relevant moduli. We need here no such extra conditions.

The following definition is a modification of Gerhardy's definition of asymptotic contractions, i.e. Definition 2 in [4].

Definition 1.3. ([2]) *A function $f : X \to X$ on a metric space (X, d) is called a* (generalized) asymptotic contraction *if for each $b > 0$ there exist moduli $\eta^b : (0, b] \to (0, 1)$ and $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ and the following hold:*

(1) *There exists a sequence of functions* $\phi_n^b : (0,\infty) \to (0,\infty)$ *such that for each* $0 < l \leq b$ *the function* $\beta_l^b := \beta^b(l,\cdot)$ *is a modulus of uniform convergence for* $(\phi_n^b)_{n\in\mathbb{N}}$ *on* $[l,b]$, *i.e.*

$$\forall \varepsilon > 0 \forall s \in [l,b] \forall m,n \geq \beta_l^b(\varepsilon)(|\phi_m^b(s) - \phi_n^b(s)| \leq \varepsilon).$$

Furthermore, if $\varepsilon < \varepsilon'$ *then* $\beta_l^b(\varepsilon) \geq \beta_l^b(\varepsilon')$.

(2) *For all* $x,y \in X$, *for all* $b \geq \varepsilon > 0$ *and for all* $n \in \mathbb{N}$ *such that* $\beta_\varepsilon^b(1) \leq n$, *we have:*

$$b \geq d(x,y) \geq \varepsilon \;\; \textit{gives} \;\; d(f^n(x), f^n(y)) \leq \phi_n^b(\varepsilon) d(x,y).$$

(3) *Define* $\phi^b := \lim_{n\to\infty} \phi_n^b$. *Then for each* $0 < \varepsilon \leq b$ *we have*

$$\phi^b(s) + \eta^b(\varepsilon) \leq 1$$

for each $s \in [\varepsilon, b]$.

If f is an asymptotic contraction in the sense of Definition 1.1, then f is also an asymptotic contraction in the sense of Definition 1.3. (See Proposition 9 in [4] and the comment following Definition 2.2 in [2].) We will for short refer to the mappings $f : X \to X$ fulfilling Definition 1.3 as just asymptotic contractions. When there is no risk of ambiguity we will often drop the superscripts from η^b and β^b. Unless otherwise specified we will, given a metric space (X,d), a mapping $f : X \to X$ and an $x_0 \in X$, let (x_n) be the sequence defined by $x_{n+1} := f(x_n)$.

The above definition is in fact, as the following simple example shows, strictly more general than Definition 2 in [4], and therefore also strictly more general than Definition 1.1. Consider $f : [0,\infty) \to [0,\infty)$ defined by

$$f(x) := \begin{cases} 1/x & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is an asymptotic contraction in our extended sense, with for all $b > 0$ $\phi_n^b(t) := 0$, $\phi^b(t) := 0$ and $\beta_l^b(t) := 2$ for $t \in (0,\infty)$, and with $\eta^b(t) := 1/2$ for $t \in (0,b]$. But f is not an asymptotic contraction in the sense of Gerhardy, i.e. in the sense of Definition 2 in [4]. For, since $1/2 \geq d(\varepsilon, 1/2+\varepsilon) \geq 1/2$ for $0 < \varepsilon < 1/2$, we should then have

$$d(f(\varepsilon), f(1/2+\varepsilon)) \leq \phi_1^{1/2}(1/2) \cdot d(\varepsilon, 1/2+\varepsilon) = \frac{\phi_1^{1/2}(1/2)}{2}$$

for all such ε, where $\phi_1^{1/2}$ is as in Definition 2 in [4]. But

$$\{d(f(\varepsilon), f(1/2+\varepsilon)) : 0 < \varepsilon < 1/2\}$$

is unbounded.

We will make use of the following two results from [4], which we in [2] note hold also for asymptotic contractions in our sense (i.e. in the sense of Definition 1.3).

Proposition 1.4. ([4]) *Let (X, d) be a metric space, let $f : X \to X$ be an asymptotic contraction and let $b > 0$ and η^b, β^b be given. For $b \geq \varepsilon > 0$ we let $K(\eta^b, \beta^b, \varepsilon) := \beta^b_\varepsilon(\eta^b(\varepsilon)/2)$. With η^b, β^b fixed we write K_ε for $K(\eta^b, \beta^b, \varepsilon)$. Then for all $b \geq \varepsilon > 0$ we have for all $k \geq K_\varepsilon$ that if $b \geq d(x, y) \geq \varepsilon$, then*

$$d(f^k(x), f^k(y)) \leq \left(1 - \frac{\eta^b(\varepsilon)}{2}\right) \cdot d(x, y).$$

Lemma 1.5. ([4]) *Let (X, d) be a metric space, let $f : X \to X$ be an asymptotic contraction and let $b > 0$ and η, β be given. For $b \geq \varepsilon > 0$ let $K_\varepsilon := \beta_\varepsilon(\eta(\varepsilon)/2)$ and $\delta := \eta(\varepsilon) \cdot \varepsilon/4$. Then for every $b \geq \varepsilon > 0$, for all $n \geq K_\varepsilon$ and all $x, y \in X$ with $d(x, y) \leq b$, if*

$$d(x, f^n(x)), d(y, f^n(y)) \leq \delta,$$

then $d(x, y) \leq \varepsilon$.

2. Main results

We need the following lemma, which draws heavily on Lemma 11 in [4].

Lemma 2.1. *Let (X, d) be a metric space, let $f : X \to X$ be an asymptotic contraction and let $b > 0$ and η, β be given. For $b \geq \delta > 0$ let $K_\delta := \beta_\delta(\eta(\delta)/2)$ and*

$$M_\delta := K_\delta \cdot \left\lceil \frac{\lg(\delta) - \lg(b)}{\lg(1 - \frac{\eta(\delta)}{2})} \right\rceil.$$

Then for all $x_0, y_0 \in X$ such that for all $n \geq 0$ we have $d(x_n, y_n) \leq b$ there exists an $m \leq M_\delta$ such that

$$d(x_m, y_m) \leq \delta.$$

Proof. Let $K_\delta := \beta_\delta(\eta(\delta)/2)$. Assume for some M and all $m < M$ we have $d(x_{mK_\delta}, y_{mK_\delta}) \geq \delta$. Then repeatedly using Proposition 1.4 we have

$$d(x_{MK_\delta}, y_{MK_\delta}) \leq \left(1 - \frac{\eta(\delta)}{2}\right)^M \cdot d(x_0, y_0) \leq \left(1 - \frac{\eta(\delta)}{2}\right)^M \cdot b.$$

Solving the inequality $(1 - \eta(\delta)/2)^M \cdot b \leq \delta$ with respect to M gives the described upper bound $M_\delta = K_\delta \cdot M$ on an m such that $d(x_m, y_m) \leq \delta$. $\square$

Now we can prove our theorem.

Theorem 2.2. *Let (X, d) be a metric space and let $f : X \to X$ be an asymptotic contraction. Let $x_0 \in X$. Then the Picard iteration sequence (x_n) is bounded.*

Proof. Assume $d(x_0, f(x_0)) > 0$, for else there is nothing to prove. Let $b := d(x_0, f(x_0))$, and let η, β be the associated moduli of f from Definition 1.3. We first prove that $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$. So let $b \geq \varepsilon > 0$. Since $b \geq d(x_0, f(x_0)) \geq b$ we can conclude by Proposition 1.4 that

$$d(x_n, f(x_n)) \leq \left(1 - \frac{\eta^b(b)}{2}\right) \cdot b$$

for $n \geq K_b$, where $K_b = \beta^b_b(\eta^b(b)/2)$. Then by considering x_{K_b} and x_{K_b+1} as the starting points x'_0 and y'_0 of two Picard iteration sequences (x'_n) and (y'_n) with the property that $d(x'_n, y'_n) < b$ for $n \geq 0$, we know by Lemma 2.1 that there exists $m \leq K_b + M_\varepsilon$ such that $d(x_m, f(x_m)) \leq \varepsilon$. Here M_ε is as in Lemma 2.1. Let $c := d(x_m, f(x_m))$ for some particular such m. If $c = 0$, then x_m is a fixed point, and (x_n) is bounded. So assume $c > 0$. Then Proposition 1.4 gives that

$$d(x_n, f(x_n)) \leq \left(1 - \frac{\eta^b(c)}{2}\right) \cdot c \leq \left(1 - \frac{\eta^b(c)}{2}\right) \cdot \varepsilon < \varepsilon,$$

for $n \geq K_b + M_\varepsilon + K_c$, with $K_c = \beta^b_c(\eta^b(c)/2)$. So $\lim_{n\to\infty} d(x_n, f(x_n)) = 0$.

Let now $N := \beta^1_{1/2}(\eta^1(1/2)/2)$ and $\delta := \eta^1(1/2) \cdot 1/8$. Let furthermore M be so large that for $n \geq M$ we have

$$d(x_n, f(x_n)) < 1/2$$

and

$$d(x_n, f^N(x_n)) \leq \delta.$$

Then Lemma 1.5 yields that for $m, n \geq M$ we have

$$d(x_m, x_n) > 1$$

or

$$d(x_m, x_n) \leq 1/2.$$

So in particular, for $n \geq M$ we have $d(x_M, x_n) > 1$ or $d(x_M, x_n) \leq 1/2$. If for all $n \geq M$ we have $d(x_M, x_n) \leq 1/2$, then (x_n) is bounded. So suppose there exists $n \geq M$ such that $d(x_M, x_n) > 1$. Let $n' > M$ be the first such $n \in \mathbb{N}$. Then

$$d(x_{n'-1}, x_{n'}) + d(x_{n'-1}, x_M) \geq d(x_{n'}, x_M) > 1,$$

so $d(x_{n'-1}, x_M) \leq 1/2$ gives $d(x_{n'-1}, x_{n'}) > 1/2$. But

$$d(x_{n'-1}, x_{n'}) = d(x_{n'-1}, f(x_{n'-1})) < 1/2,$$

which thus contradicts our choice of M and n'. Thus $d(x_M, x_n) \leq 1/2$ for all $n \geq M$, and hence (x_n) is bounded. □

Corollary 2.3. *Let (X, d) be a nonempty, complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk. Then f has a unique fixed point $z \in X$, and for every starting point $x \in X$ the iteration sequence $(f^n(x))$ converges to z.*

Proof. Since $f : X \to X$ is an asymptotic contraction in the sense of Kirk f is also an asymptotic contraction in the sense of Definition 1.3. Hence Theorem 2.2 yields that all Picard iteration sequences are bounded. The rest follows from Theorem 1.2. □

Similarly we can improve the results in [2]. As an instance of this we give the following improvement of Theorem 3.3 in [2].

Corollary 2.4. *Let (X, d) be a metric space, and let $f : X \to X$ be an asymptotic contraction with moduli η^b and β^b for each $b > 0$. Then all Picard iteration sequences are Cauchy. Assume that for some $x_0 \in X$ the limit $z := \lim_{n\to\infty} x_n$ exists. Then for any $x_0 \in X$ the iteration sequence (x_n) converges to z, irrespective of whether z is a fixed point or not. If (x_n) is bounded by $b > 0$ then (x_n) converges to z with the rate of convergence specified in Theorem* 3.1 *in* [2].

Proof. Immediate from Theorem 2.2 and from Theorem 3.3 in [2]. □

In [9], K. Włodarczyk, D. Klim and R. Plebaniak introduce set-valued asymptotic contractions and prove a theorem for these which is reminiscent of Kirk's theorem for asymptotic contractions. This theorem contains a condition of boundedness much like in Kirk's original theorem. Our results here lead us to conjecture that the boundedness condition can be removed also from Theorem 2.1 in [9].

Conjecture 2.5. *Let (X, d) be a complete metric space and let the mapping $T : X \to 2^X$ be closed and a set-valued asymptotic contraction. Then T has a unique endpoint v in X. Furthermore, for each $w_1 \in X$ and for each sequence (w_n) which is a generalized sequence of iterations with respect to T and w_1, we have that (w_n) converges to v.*

For details, see [9].

The rate of convergence in Corollary 2.4 depends on a bound b on the iteration sequence in question, and also on the moduli η^b, β^b. We will in the following denote this rate of convergence by Φ, so that given a metric space (X, d), an asymptotic contraction $f : X \to X$ with moduli η and β, an $x_0 \in X$ such that $\lim_{n\to\infty} x_n = z$ and such that (x_n) is bounded by $b > 0$, and also a real number $b \geq \varepsilon > 0$, then $n \geq \Phi(b, \eta^b, \beta^b, \varepsilon)$ gives $d(x_n, z) \leq \varepsilon$.

We now give a rate of convergence which does not depend on a bound on the iteration sequence, but which instead depends on (strictly positive) upper and lower bounds on $d(x_0, f(x_0))$, and also on the moduli η, β (for some specific values b). Specifically, we have the following.

Proposition 2.6. *Let (X, d) be a metric space, and let $f : X \to X$ be an asymptotic contraction with moduli η^b and β^b for each $b > 0$. Assume that for some $x_0 \in X$ the limit $z := \lim_{n\to\infty} x_n$ exists. Let $b \geq c > 0$ and let $x_0 \in X$ be such that $b \geq d(x_0, f(x_0)) \geq c$. Then (x_n) has the following rate of convergence. Let $\varepsilon > 0$. Let Φ be the rate of convergence in Corollary 2.4, and let*

$$K_c := \beta^b_c(\eta^b(c)/2),$$
$$N := \beta^b_{b/2}(\eta^b(b/2)/2),$$
$$\alpha := \eta^b(b/2) \cdot b/8,$$

$$K' := \left\lceil \frac{\lg(\alpha/N) - \lg(b)}{\lg(1-\eta^b(\alpha/N)/2)} \right\rceil,$$
$$K_{\alpha/N} := \beta^b_{\alpha/N}(\eta^b(\alpha/N)/2),$$
$$b' := \max\{\varepsilon, K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b\}.$$

Let $n \in \mathbb{N}$ satisfy $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon) + K_c$. Then

$$d(x_n, z) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$. Let $x_0 \in X$, and let $b, c > 0$ be such that $b \geq d(x_0, f(x_0)) \geq c$. (If $d(x_0, f(x_0)) = 0$ then $x_0 = z$.) Let $K_c := \beta^b_c(\eta^b(c)/2)$. Then Proposition 1.4 gives that

$$d(x_n, f(x_n)) \leq (1 - \eta^b(c)/2) \cdot d(x_0, f(x_0)) \leq (1 - \eta^b(c)/2) \cdot b$$

for $n \geq K_c$. Let now

$$N := \beta^b_{b/2}(\eta^b(b/2)/2),$$
$$\alpha := \eta^b(b/2) \cdot b/8.$$

Notice that $\alpha/N < b/(2N)$, since $0 < \eta^b(b/2) < 1$. Assume that for some integer $M \geq 0$ we have $d(x_n, f(x_n)) \leq \alpha/N$ for all $K_c \leq n \leq K_c + M$. Let $M = k \cdot N + m$, with $k \geq 0$ and with $m \geq 0$ an integer strictly smaller than N. If $k < 2$ then it follows by the triangle inequality that $d(x_n, x_{n'}) < b$ for $K_c \leq n, n' \leq K_c + M + 1$, since $\alpha/N < b/(2N)$. If $k = 2$ then likewise

$$d(x_n, x_{n'}) < \frac{3b}{2}$$

for $K_c \leq n, n' \leq K_c + M + 1$. Assume $k > 2$, and assume that for some integer $k' > 0$ such that $k \geq k' + 2$ we have

$$d(x_{K_c}, x_{K_c + k'N}) \leq \frac{b}{2}.$$

(Notice that this holds for $k' = 1$.) Then

$$d(x_{K_c + k'N}, x_{K_c + (k'+1)N}) \leq \frac{b}{2},$$

and so

$$d(x_{K_c}, x_{K_c + (k'+1)N}) \leq b.$$

We also have

$$d(x_{K_c}, x_{K_c + N}) \leq \alpha$$

and

$$d(x_{K_c+(k'+1)N}, x_{K_c+(k'+2)N}) \leq \alpha,$$

and thus Lemma 1.5 gives that

$$d(x_{K_c}, x_{K_c+(k'+1)N}) \leq \frac{b}{2}.$$

Thus we have $d(x_n, x_{n'}) \leq 3b/2$ for $K_c \leq n, n' \leq K_c + (k-1)N$, and hence $d(x_n, x_{n'}) < 5b/2$ for $K_c \leq n, n' \leq K_c + M + 1$. If $d(x_n, f(x_n)) \leq \alpha/N$ for all $n \geq K_c$, then we get that $d(x_n, x_{n'}) \leq b$ for all $n, n' \geq K_c$. Namely, if we assume

$$d(x_{K_c}, x_{K_c+kN}) \leq \frac{b}{2}$$

for some $k \in \mathbb{N}$ (notice that this holds for $k = 1$), then

$$d(x_{K_c}, x_{K_c+(k+1)N}) \leq b,$$

and also

$$d(x_{K_c}, x_{K_c+N}) \leq \alpha$$

and

$$d(x_{K_c+(k+1)N}, x_{K_c+(k+2)N}) \leq \alpha.$$

So by Lemma 1.5 we have

$$d(x_{K_c}, x_{K_c+(k+1)N}) \leq \frac{b}{2},$$

and hence $d(x_n, x_{n'}) \leq b$ for all $n, n' \geq K_c$. (Then $d(x_n, x_{n+N}) \leq \alpha$ and $d(x_{n'}, x_{n'+N}) \leq \alpha$ in fact implies $d(x_n, x_{n'}) \leq b/2$ for all $n, n' \geq K_c$.)

Thus by letting $x'_0 := x_{K_c}$ we can conclude that either (x'_n) is bounded by b or else we have

$$\alpha/N \leq d(x'_m, f(x'_m)) \leq (1 - \eta^b(c)/2) \cdot b$$

for an m such that $(x'_n)_{n \leq m}$ is bounded by $5b/2$. So by Proposition 1.4 we get an $N_1 \in \mathbb{N}$ such that

$$d(x'_n, f(x'_n)) \leq (1 - \eta^b(\alpha/N)/2) \cdot (1 - \eta^b(c)/2) \cdot b < (1 - \eta^b(\alpha/N)/2) \cdot b$$

for $n \geq N_1$ and such that $(x'_n)_{n \leq N_1}$ is bounded by $5b/2 + b \cdot K_{\alpha/N}$. (Where $K_{\alpha/N} = \beta^b_{\alpha/N}(\eta^b(\alpha/N)/2)$.) By considering x'_{N_1} as the starting point of a Picard iteration sequence (x''_n) with the property that $d(x''_m, f(x''_m)) < (1 -$

$\eta^b(\alpha/N)/2) \cdot b$ for all $m \geq 0$, we get by the above argument that either (x''_n) is bounded by b or else we get an $N_2 \in \mathbb{N}$ such that

$$d(x''_m, f(x''_m)) < (1 - \eta^b(\alpha/N)/2)^2 \cdot b$$

for $m \geq N_2$ and such that $(x''_n)_{n \leq N_2}$ is bounded by $5b/2 + b \cdot K_{\alpha/N}$. Thus either (x'_n) is bounded by $5b/2 + b \cdot K_{\alpha/N} + b$ or else we have that

$$d(x'_m, f(x'_m)) < (1 - \eta^b(\alpha/N)/2)^2 \cdot b$$

for $m \geq N_1 + N_2$, and furthermore that $(x'_n)_{n \leq N_1 + N_2}$ is bounded by $2 \cdot (5b/2 + b \cdot K_{\alpha/N})$. Solving the inequality

$$(1 - \eta^b(\alpha/N)/2)^k \cdot b \leq \alpha/N$$

with respect to k leads us to consider

$$K' := \left\lceil \frac{\lg(\alpha/N) - \lg(b)}{\lg(1 - \eta^b(\alpha/N)/2)} \right\rceil.$$

We get that either (x'_n) is bounded by $(K' - 1) \cdot (5b/2 + b \cdot K_{\alpha/N}) + b$, or else we get an $N' \in \mathbb{N}$ such that

$$d(x'_m, f(x'_m)) < (1 - \eta^b(\alpha/N)/2)^{K'} \cdot b \leq \alpha/N$$

for $m \geq N'$, and furthermore such that $(x'_n)_{n \leq N'}$ is bounded by $K' \cdot (5b/2 + b \cdot K_{\alpha/N})$. Hence (x'_n) is bounded by $K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b$. Let $b' := \max\{\varepsilon, K' \cdot (5b/2 + b \cdot K_{\alpha/N}) + b\}$. Then Corollary 2.4 gives that $d(x'_n, z) \leq \varepsilon$ for $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon)$, so $d(x_n, z) \leq \varepsilon$ for $n \geq \Phi(b', \eta^{b'}, \beta^{b'}, \varepsilon) + K_c$. (Here Φ is as in the remarks preceding Proposition 2.6.) $\square$

We remark that in the above proposition there is room for some numerical improvement. We can also easily adapt the proposition to cover the case where $\lim_{n\to\infty} x_n$ does not exist, as explained in the remarks following Theorem 3.2 in [2].

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