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A NOTE ON DIRECTIONALLY NON-EXPANSIVE MAPPINGS

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Abstract. In this paper, we establish a fixed point theorem for a strongly directionally non-expansive mapping on a non-empty closed convex subset of a Banach space.
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1. Introduction

The notion of directionally non-expansive mapping on a non-empty subset of a Banach space was introduced by W.A.Kirk in [1]. In this paper,we introduce the concept of strongly directionally non-expansive map and use it to prove a fixed point theorem for the map which takes R_x into itself where $R_x := \{(1-\lambda)x + \lambda Tx : \lambda \in R\}$.

Notations

Let X be a Banach space and K be a non-empty subset of X. Let $T : K \to K$ be a mapping. Let $[x, y] := \{(1-\lambda)x + \lambda y : \lambda \in [0, 1]\}$ and $(x, y) := \{(1-\lambda)x + \lambda y : \lambda \in (0, 1)\}$. For any $x_1, x_2 \in R_x$, we say that $x_1 \leq x_2$ when $\lambda_1 \leq \lambda_2$ where $x_1 = (1-\lambda_1)x + \lambda_1 Tx$ and $x_2 = (1-\lambda_2)x + \lambda_2 Tx$ for some λ_1 , $\lambda_2 \in R$.

Basic definitions

Definition 1.1. *Let X be a Banach space and K be a non-empty closed bounded convex subset of X. A mapping $T : K \to K$ is said to be non-expansive if for each pair of elements x and y of K, we have*
$\|Tx - Ty\| \leq \|x - y\|$.

Definition 1.2. *(W.A. Kirk [1])[Directionally non-expansive mapping] Let X be a Banach space and K be a non-empty subset of X. A mapping $T : K \to K$ is said to be directionally non-expansive if*
$\|Tx - Ty\| \leq \|x - y\|$ *for all $x \in K$ and $y \in [x, Tx]$.*

We now give the following definition:

Definition 1.3. *Let X be a Banach space and K be a non-empty closed convex subset of X. A mapping $T : K \to K$ is said to be strongly directionally non-expansive if for each $x \in K$, there exist $y_x, z_x \in X$ such that (i). $(y_x, z_x) \bigcap K \supseteq [x, Tx]$ and (ii). $\|Tx - Tm\| \leq \|x - m\|$ for all $m \in (y_x, z_x) \bigcap K$.*

Remark: Every strongly directionally non-expansive mapping is a directionally non-expansive mapping. But the following simple example tells us that the converse is not true.
Example: Define a map $T : [0, 1] \to [0, 1]$ by $T(0) = 0$ and $T(x) = 1$ for $0 < x \leq 1$. We observe that T is directionally non-expansive but not strongly directionally non-expansive.

2. Main Results

Theorem 2.1. *Let X be a Banach space and K be a non-empty closed bounded convex subset of X. Suppose $T : K \to K$ is a strongly directionally non-expansive mapping such that for every $x \in K$, $R_x \bigcap K$ is invariant under T. Then T has a fixed point.*

Proof. Assume $Tx \neq x$ for every $x \in K$. Now, it is sufficient to prove that for some $x \in K$, $[x, Tx]$ is invariant under T.
Suppose not, then for every $x_0 \in [x, Tx]$, there exists $x_1 \in [x_0, Tx_0]$ such that $Tx_1 \notin [x_0, Tx_0]$. Let $G_{x_0} = R_{x_0} \cap K$. Now we can easily prove that $A = \{\lambda \in R : (1-\lambda)x_0 + \lambda Tx_0 \in K\}$ is bounded. Let $\alpha = \inf A$ and $\beta = \sup A$.

Let $a = (1-\alpha)x_o + \alpha Tx_o$ and $b = (1-\beta)x_o + \beta Tx_o$. Therefore, there exists a sequence $\{\alpha_n\} \in A$ such that $\{\alpha_n\}$ converges to α. Hence $(1-\alpha_n)x_o + \alpha_n Tx_o$ converges to $(1-\alpha)x_o + \alpha Tx_o$. Therefore it is easy to see that $a \in G_{x_o}$ and $b \in G_{x_o}$. Hence
$G_{x_o} = \{(1-\lambda)a + \lambda b : 0 \leq \lambda \leq 1\}$. Let $x_o = (1-\lambda_o)a + \lambda_o b$ and $Tx_o = (1-\alpha_o)a + \alpha_o b$, where $\alpha \leq \lambda_o \leq \beta$, $\alpha \leq \alpha_o \leq \beta$. We first prove that $\lambda_o < \alpha_o$.

$$\begin{aligned}
\|b - Tx_o\| &= \|[(1-\beta)x_o + \beta Tx_o] - Tx_o\| \\
&= \|[(1-\beta)x_o + \beta Tx_o] - Tx_o\| \\
&= |1-\beta|\|x_o - Tx_o\| \\
\|b - x_o\| &= \|[(1-\beta)x_o + \beta Tx_o] - x_o\| \\
&= |\beta|\|x_o - Tx_o\|
\end{aligned}$$

Therefore, $\|b - Tx_o\| \leq \|b - x_o\|$
That is, $\|b - [(1-\alpha_o)a + \alpha_o b]\| \leq \|b - [(1-\lambda_o)a + \lambda_o b]\|$
which implies $\|a - b\|(1-\alpha_o) \leq \|a - b\|(1-\lambda_o)$ and therefore,
$\lambda_o \leq \alpha_o$. Since $\lambda_o \neq \alpha_o$, we get $\lambda_o < \alpha_o$. Let $x_1 = (1-\lambda_1)a + \lambda_1 b$ where $\lambda_0 < \lambda_1 \leq \alpha_0$, $x_1 \in [x_0, Tx_0]$. Let $Tx_1 = (1-\alpha_1)a + \alpha_1 b$. Now $Tx_1 \notin [x_0, Tx_0]$ implies either

$$\alpha_1 < \lambda_0 \text{ or } \alpha_0 < \alpha_1 \tag{2.1}$$

$\|Tx_0 - Tx_1\| \leq \|x_0 - x_1\|$ since T is directionally non-expansive in $(y, z) \supset [x_0, Tx_0] \ni x_1$. This gives,

$$|\alpha_1 - \alpha_0| \leq (\lambda_1 - \lambda_0) \tag{2.2}$$

From (2.1) and (2.2) we get $\alpha_0 < \alpha_1$. Thus $\lambda_0 < \lambda_1 < \alpha_0 < \alpha_1$. Now choose $x_1 \in [x_0, Tx_0]$ such that

$$d(Tx_1, [x_0, Tx_0]) \geq \sup_{x \in [x_0, Tx_0],\ Tx \notin [x_0, Tx_0]} d(Tx, [x_0, Tx_0]) - 1$$

Proceeding this way, choose $x_2 \in [x_1, Tx_1]$ such that $Tx_2 \notin [x_1, Tx_1]$ and

$$d(Tx_2, [x_1, Tx_1]) \geq \sup_{x \in [x_1, Tx_1], Tx \notin [x_1, Tx_1]} d(Tx, [x_1, Tx_1]) - \frac{1}{2}$$

Having chosen x_n in a similar way, choose

$$x_{n+1} \in [x_n, Tx_n]$$

such that $Tx_{n+1} \notin [x_n, Tx_n]$ and

$$d(Tx_{n+1}, [x_n, Tx_n]) \geq \sup_{x\in[x_n,Tx_n], Tx\notin[x_n,Tx_n]} d(Tx, [x_n, Tx_n]) - \frac{1}{n+1}.$$

Let $x_n = (1-\lambda_n)a + \lambda_n b$ and $Tx_n = (1-\alpha_n)a + \alpha_n b$
where $\lambda_n < \lambda_{n+1} < \alpha_n < \alpha_{n+1}$ and λ_n, α_n being bounded increasing sequences converge to some μ and ν respectively.
Let

$$u = (1-\mu)a + \mu b$$

and

$$v = (1-\nu)a + \nu b$$

Since $\lambda_n < \alpha_n$ for every n, we have $\mu \leq \nu$.
Case 1: $\mu < \nu$. There exists a positive integer N such that
$\lambda_n \leq \mu < \nu_n$, for all $n \geq N$, hence $u \in [x_n, Tx_n]$, for all $n \geq N$. Therefore $\|Tu - Tx_n\| \leq \|u - x_n\|$ for all $n \geq N$.
This implies $Tx_n \to Tu$. Let $w \in [u, Tu]$. Then there exists a positive integer N_0 such that $w \in [x_n, Tx_n]$, for all $n \geq N_0$.
Now $d(Tw, [x_n, Tx_n]) \leq sup_{x\in[x_n,Tx_n]} d(Tx, [x_n, Tx_n]) \leq d(Tx_{n+1}, [x_n, Tx_n]) + \frac{1}{n+1}$. Taking limit as $n \to \infty$, we get $d(Tw, [u, Tu]) = 0$. Therefore, $Tw \in [u, Tu]$. This implies $[u, Tu]$ is invariant under T, contradicting our assumption.
Case 2: $\mu = \nu$. That is, $u = v$.
There exists a positive integer N_1 such that $x_n \in (y_u, z_u)$, for all $n \geq N_1$ and $\|Tx_n - Tu\| \leq \|x_n - u\|$, for all $n \geq N_1$.
Therefore, $Tx_n \to Tu$.
Hence $v = Tu = u$, contradicting our assumption. This proves the theorem. □

Example 2.2. Let K be a closed bounded convex subset of a strictly convex space X. Let A and B be two disjoint closed convex subsets of K. By Hahn Banach separation theorem, there exists a hyper plane
$\{x \in X : Ref(x) = \alpha\}$ such that $A \subseteq \{x : Ref(x) \leq \alpha\}$ and
$B \subseteq \{x : Ref(x) > \alpha\}$.
Let $A_1 = \{x : Ref(x) < \alpha\} \bigcap K$, $B_1 = \{x : Ref(x) > \alpha\} \bigcap K$ and $C_1 = \{x : Ref(x) = \alpha\} \bigcap K$.
Define $T : K \longrightarrow K$ by $T(x) = P_A(x)$, for all $x \in A_1$, $T(x) = P_B(x)$, for all

$x \in B_1$, $T(x) = I(x)$, for all $x \in C_1$, where $P_A(x)$ is the unique nearest point in A to x. Then T is a strongly directionally non-expansive mapping which is also discontinuous.

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