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UNIFORM CONVERGENCE OF ITERATES FOR A CLASS OF ASYMPTOTIC CONTRACTIONS

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Abstract. We provide sufficient conditions for the iterates of certain asymptotic contractions on a complete metric space X to converge to their unique fixed points, uniformly on each bounded subset of X.

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1. Introduction

Let (X, d) be a complete metric space. According to Banach's fixed point theorem, the iterates of any strict contraction on X converge to its unique fixed point. Since this classical theorem has found numerous important applications, it has also been extended in several directions. Perhaps the first such extension is due to Rakotch [10]. Several years later Boyd and Wong [3] obtained even a more general result. See [8] for a comprehensive survey of the results available regarding various types of contractive mappings up to 2001.

More recently, Kirk [9] has introduced the notion of an asymptotic contraction and proved the following fixed point theorem for such mappings by using ultrapower techniques. His theorem can be considered an asymptotic version of the Boyd-Wong fixed point theorem [3] mentioned above.

Theorem 1.1. *Let $T : X \to X$ be a continuous mapping such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x, y))$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on the range of d. Suppose that ϕ and all ϕ_n are continuous and that $\phi(t) < t$ for all $t > 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$ for all $x \in X$.

Arandelović [1] then provided a short and simple proof of Kirk's Theorem 1.1. Jachymski and Jóźwik [7] extended this result with a constructive proof and obtained a complete characterization of asymptotic contractions on a compact metric space.

The following theorem is the main result of Chen [6]. It improves upon Kirk's original theorem [9].

Theorem 1.2. *Let $T : X \to X$ be such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x, y))$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and that $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$ for all $x \in X$.

Note that Theorem 1.2 does not provide us with uniform convergence of the iterates of T on bounded subsets of X, although this does hold for many classes of mappings of contractive type (e.g., [4, 10]). This property is important because it yields stability of the convergence of iterates even in the presence of computational errors [5]. In [2] we show that this conclusion can be derived in the setting of Theorem 1.2. To this end, we first prove in [2] a somewhat more general result (Theorem 1.3 below) which, when combined with Theorem 1.2, yields our strengthening of Chen's result (Theorem 1.4).

Theorem 1.3. *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that*

$$d(T^n x, x_*) \leq \phi_n(d(x, x_*)) \text{ for all } x \in X \text{ and all natural numbers } n,$$

where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and $\phi(t) < t$ for all $t > 0$. Then $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X.

Theorem 1.4. *Let $T : X \to X$ be such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x, y))$$

for all $x, y \in X$ and all natural numbers n, where $\phi_n : [0, \infty) \to [0, \infty)$ and $\lim_{n\to\infty} \phi_n = \phi$, uniformly on any bounded interval $[0, b]$. Suppose that ϕ is upper semicontinuous and $\phi(t) < t$ for all $t > 0$. Furthermore, suppose that there exists a positive integer n_* such that ϕ_{n_*} is upper semicontinuous and $\phi_{n_*}(0) = 0$. If there exists $x_0 \in X$ which has a bounded orbit $O(x_0) = \{x_0, Tx_0, T^2x_0, \dots\}$, then T has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} T^n x = x_*$, uniformly on each bounded subset of X.

We now state the main result of the present paper. In contrast with Theorem 1.3, here we only assume that a subsequence of $\{\phi_n\}_{n=1}^{\infty}$ converges to ϕ.

Theorem 1.5. *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that*

$$d(T^n x, x_*) \leq \phi_n(d(x, x_*)) \tag{1.1}$$

for all $x \in X$ and all natural numbers n, where the functions $\phi_n : [0, \infty) \to [0, \infty)$, $n = 1, 2, \dots$, satisfy the following conditions:

(i) For each $b > 0$, there is a natural number n_b such that

$$\sup\{\phi_n(t) : \ t \in [0, b] \text{ and all } n \geq n_b\} < \infty;$$

(ii) there exist an upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ and a strictly increasing sequence of natural numbers $\{m_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \phi_{m_k} = \phi$, uniformly on any bounded interval $[0, b]$.

Then $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X.

We prove this theorem in Section 2. In the last section of our paper we present and prove a consequence of Theorem 1.5 (Theorem 3.1). We also show there that if the iterates of a self-mapping T of X converge uniformly on any bounded subset of X, then T necessarily satisfies all the assumptions of Theorem 1.5 with an appropriate sequence $\{\phi_n\}_{n=1}^{\infty}$.

2. Proof of Theorem 1.5

Set $T^0 x = x$ for all $x \in X$. For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{z \in X : \ d(x, z) \leq r\}. \tag{2.1}$$

Let $M > 0$ and $\epsilon \in (0, 1)$. By (i), there are $M_1 > M$ and an integer $n_1 \geq 1$ such that

$$\phi_i(t) \leq M_1 \text{ for all } t \in [0, M+1] \text{ and all integers } i \geq n_1. \tag{2.2}$$

In view of (1.1) and (2.2), for each $x \in B(x_*, M)$ and each integer $n \geq n_1$,

$$d(T_n x, x_*) \leq \phi_n(d(x, x_*)) \leq M_1. \tag{2.3}$$

Since the function $t - \phi(t)$ is lower semicontinuous, there is $\delta > 0$ such that

$$\delta < \epsilon/8 \tag{2.4}$$

and

$$t - \phi(t) \geq 2\delta, \ t \in [\epsilon/8, 4M_1 + 4]. \tag{2.5}$$

By (ii), there is an integer $n_2 \geq 2n_1 + 2$ such that

$$|\phi_{n_2}(t) - \phi(t)| \leq \delta, \ t \in [0, 4M_1 + 4]. \tag{2.6}$$

Assume that

$$x \in B(x_*, M_1 + 4). \tag{2.7}$$

If $d(x, x_*) \leq \epsilon/8$, then it follows from (1.1), (2.4), (2.6) and (2.7) that

$$d(T^{n_2} x, x_*) \leq \phi_{n_2}(d(x, x_*)) \leq \phi(d(x, x_*)) + \delta \leq d(x, x_*) + \delta < \epsilon/4.$$

If $d(x, x_*) \geq \epsilon/8$, then relations (1.1), (2.5), (2.6) and (2.7) imply that

$$d(T^{n_2} x, x_*) \leq \phi_{n_2}(d(x, x_*)) \leq \phi(d(x, x_*)) + \delta \leq d(x, x_*) - 2\delta + \delta = d(x, x_*) - \delta.$$

Thus in both cases we have

$$d(T^{n_2} x, x_*) \leq \max\{d(x, x_*) - \delta, \epsilon/4\}. \tag{2.8}$$

Now choose a natural number $q > 2$ such that

$$q > (8 + 2M_1)\delta^{-1}. \tag{2.9}$$

Assume that

$$x \in B(x_*, M_1 + 4) \text{ and } T^{in_2}x \in B(x_*, M_1 + 4),\ i = 1, \dots, q-1. \tag{2.10}$$

We claim that

$$\min\{d(T^{jn_2}x, x_*) :\ j = 1, \dots, q\} \le \epsilon/4. \tag{2.11}$$

Assume the contrary. Then by (2.8) and (2.10), for each $j = 1, \dots, q$, we have

$$d(T^{jn_2}x, x_*) \le d(T^{(j-1)n_2}x, x_*) - \delta$$

and

$$d(T^{qn_2}x, x_*) \le d(T^{(q-1)n_2}x, x_*) - \delta \le \dots \le d(x, x_*) - q\delta \le M_1 + 4 - q\delta.$$

This contradicts (2.9). The contradiction we have reached proves (2.11).

Assume that an integer j satisfies $1 \le j \le q-1$ and

$$d(T^{jn_2}x, x_*) \le \epsilon/4.$$

When combined with (2.8) and (2.10), this implies that

$$d(T^{(j+1)n_2}x, x_*) \le \max\{d(T^{jn_2}x, x_*) - \delta, \epsilon/4\} \le \epsilon/4.$$

It follows from this inequality and (2.11) that

$$d(T^{qn_2}x, x_*) \le \epsilon/4 \tag{2.12}$$

for all x satisfying (2.7).

Assume now that $x \in B(x_*, M)$ and let an integer s be such that $s \ge n_1 + qn_2$. By (2.3),

$$T^i x \in B(x_*, M_1) \text{ for all integers } i \ge n_1$$

and

$$T^{s-qn_2}x \in B(x_*, M_1). \tag{2.13}$$

Since $T^s x = T^{qn_2}(T^{s-qn_2}x)$, it follows from (2.12) and (2.13) that

$$d(T^s x, x_*) = d(T^{qn_2}(T^{s-qn_2}x), x_*) < \epsilon/4.$$

This completes the proof of Theorem 1.5. □

3. Concluding remarks

In this section we present an extension of Theorem 1.5 (Theorem 3.1) and point out that Theorem 1.5 also has a converse.

Theorem 3.1. *Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that $\{m_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers such that*

$$d(T^{m_k}x, x_*) \leq \phi_{m_k}(d(x, x_*))$$

for all $x \in X$ and all natural numbers k, where T and the functions ϕ_{m_k} : $[0, \infty) \to [0, \infty)$, $k = 1, 2, \dots$, satisfy the following conditions:

(i) For each $M > 0$, there is $M_1 > 0$ such that

$$T^i(B(x_*, M)) \subset B(x_*, M_1) \text{ for each integer } i \geq 0;$$

(ii) there exists an upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ such that $\lim_{k\to\infty} \phi_{m_k} = \phi$, uniformly on any bounded interval $[0, b]$.

Then $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X.

Proof. Let i be a natural number such that $i \neq m_k$ for all natural numbers k. For each $t \geq 0$, set

$$\phi_i(t) = \sup\{d(T^i x, x_*) : \; x \in B(x_*, t)\}.$$

Clearly, $\phi_i(t)$ is finite for all $t \geq 0$. It is easy to see that all the assumptions of Theorem 1.5 hold. Therefore Theorem 1.5 implies that $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X. Theorem 3.1 is proved. □

Assume now that $T : X \to X$, $x_* \in X$, $T^n x \to x_*$ as $n \to \infty$, uniformly on each bounded subset of X, and that $T(C)$ is bounded for any bounded $C \subset X$. We claim that T necessarily satisfies all the hypotheses of Theorem 1.5 with an appropriate sequence $\{\phi_n\}_{n=1}^{\infty}$.

Indeed, fix a natural number n and for all $t \geq 0$, set

$$\phi_n(t) = \sup\{d(T^n x, x_*) : \; x \in B(x_*, t)\}.$$

Clearly, $\phi_n(t)$ is finite for all $t \geq 0$ and all natural numbers n, and

$$d(T^n x, x_*) \leq \phi_n(d(x, x_*))$$

for all $x \in X$ and all natural numbers n. It is also obvious that $\phi_n \to 0$ as $n \to \infty$, uniformly on each bounded subinterval of $[0, \infty)$, and that for any

$b > 0$,

$$\sup\{\phi_n(t) : t \in [0, b], n \geq 1\} < \infty.$$

Thus all the assumptions of Theorem 1.5 hold with $\phi(t) = 0$ identically, as claimed. □

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