SEHGAL CONTRACTIONS ON MENDER SPACES

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Dedicated to Professor Ioan A. Rus on the occasion of his 70th birthday

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Abstract. We present some remarks and comments on four proofs of Sehgal & Bharucha-Reid fixed point principle for probabilistic B-contractions.

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1. Introduction

A Sehgal-contraction, or a $B$-contraction, on a probabilistic metric space $(S, \mathcal{F})$ is a self-mapping $A$ of $S$ such that

$F_{A p A q}(L x) \geq F_{p q}(x), \forall p, q \in S$

for some $L \in (0, 1)$ and every $x$. As it is well known [1, 2, 4, 10, 11, 12, 19, 20], every $B$-contraction on a complete Menger space $(S, \mathcal{F}, Min)$ has a unique fixed point, which is globally attractive. Therefore $B$-contractions on complete Menger spaces $(S, \mathcal{F}, Min)$ belong to the class of Picard operators, extensively studied by I. A. Rus (see [15], [16] and [9]). In fact, the following more general result holds:

Theorem 1.1. Every t-norm of Hadžić-type has the fixed point property for $B$-contractions.

Indeed, let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T$ is of Hadžić-type and consider a $B$-contraction $A : S \rightarrow S$. Without loss of generality, we can

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suppose $L \in (0, \frac{1}{2}]$. Let $p_0 \in S$ and $x \in (0, \infty)$ be fixed. If $m$ is a positive integer, then
\[
F_{p_0,A^mp_0}(2x) \geq T(F_{p_0,Ap_0}(x), F_{Ap_0,A^{m-1}p_0}(2x))
\]
and we can see by induction that
\[
F_{p_0,A^mp_0}(2x) \geq T_m(F_{p_0,Ap_0}(x)), \forall m \geq 1. \tag{1.1}
\]
Thus we have, for all integers $n, m$:
\[
F_{A^n p_0 A^{n+m} p_0}(2x) \geq F_{p_0,A^mp_0}(2x/L^n) \geq T_m(F_{p_0,Ap_0}(x/L^n)). \tag{1.2}
\]
Since $T$ is of Hadžić-type (that is the iterates $\{T_m\}$ are equicontinuous at $a = 1$) and $F_{p_0,Ap_0} \in D^+$ (so that $\lim_{t \to \infty} F_{p_0,Ap_0}(t) = 1$), then
\[
\lim_{n \to \infty} F_{A^n p_0 A^{n+m} p_0}(2x) = 1, \tag{1.3}
\]
uniformly in $m$, for each $x \in (0, \infty)$. By definition, this means that $\{A^n p_0\}$ is $\mathcal{F}$-Cauchy and the conclusion follows.

The proof of the following result of type Sherwood (compare with [20]) is easy to reproduce:

**Lemma 1.2.** Let $T$ be an lc-t-norm and fix an $F$ in $D^+$. Let $S = \{1, 2, \ldots\}$ and define a probabilistic metric by
\[
\begin{align*}
F_{mn+m}(x) &= T^m[F(2^n x), F(2^{n+1} x), \ldots, F(2^{n+m} x)], n, m \in S \\
F_{nn} &= \varepsilon_0
\end{align*}
\]
Then $(S, \mathcal{F}, T)$ is a Menger space and the mapping $n \xrightarrow{A} n + 1$ is a $B-$contraction with $L = \frac{1}{2}$.

The following theorem is a partial converse to Theorem 1.1:

**Theorem 1.3.** If $T$ is a continuous t-norm with the fixed point property for $B-$contractions, then $T$ is of Hadžić-type.

**Proof.** Suppose that $T$ is not of Hadžić-type. Then there exists $a \in (0, 1)$ such that for each $1 > b > a$ there is $m_b \geq 1$ for which $T_{m_b}(b) < a$. Now let $b_n \in (a, 1)$ be increasing to 1. Then there exists $m_n \geq 1$ such that
\[
T_{m_n}(b_n) < a, n = 1, 2, \ldots \tag{1.5}
\]
Obviously we can suppose that $m_n$ is increasing on $n$. Let $F \in D^+$ be defined by

$$F(x) = \begin{cases} 
0 & \text{if } x \leq 1 \\
b_1 & \text{if } x \in (1, 2^{2+m_1}] \\
b_{n+1} & \text{if } x \in (2^{2n+m_n}, 2^{2n+m_n+1}], \ n \geq 1
\end{cases} \tag{1.6}$$

If we consider the Menger space as in Lemma 1.2 with this $F$, then we have successively:

$$F_{n+m_n}(1) \leq F_{n+m_n}(2^n) = T^{m_n}[F(2^n), F(2^{2n+1}), \ldots, F(2^{2n+m_n})] \leq$$

$$\leq T^{m_n}[F(2^{2n+m_n}), \ldots, F(2^{2n+m_n})] \leq T_{m_n}(b_n) < a$$

Therefore the sequence $\{A^n\}$ is not Cauchy, so that $T$ does not have the fixed point property for $B-$contractions.

It is worth noting that in [1], G. L. Cain & R. Kasriel proved the Sehgal’s result (for $T = \text{Min}$) by using the Nishiura pseudo-metrics $d_\lambda$, defined by

$$d_\lambda(p, q) = \sup\{x \mid F_{pq}(x) \leq 1 - \lambda\}, \ \lambda \in (0, 1), \ p, q \in S.$$ 

As a matter of fact, the family $\{d_\lambda\}_{\lambda \in (0, 1)}$ generates the $(\varepsilon, \lambda)-$topology on $S$. Moreover, for every $\lambda \in (0, 1),

$$d_\lambda(Ap, Aq) \leq Ld_\lambda(p, q), \ \forall p, q \in S.$$ 

For if $d_\lambda(p, q) < r$ then $F_{pq}(r) > 1 - \lambda$ and the contraction condition $(BC_L)$ implies $F_{Ap,Aq}(Lr) > 1 - \lambda$, which shows that $d_\lambda(Ap, Aq) < Lr$. Hence one can apply the Banach contraction principle in the uniform space $(S, \{d_\lambda\}_{\lambda \in (0,1)})$.

As we will see, the result in (Sehgal & Bharucha-Reid, [19]) is also a consequence of the "fixed point alternative" in generalized (Luxemburg) complete metric spaces (see [6] or [13]). Incidentally, this method offers a sort of converse of the fixed point principle in Menger spaces under the t-norm $T_M = \text{Min},$ by giving a suitable family of (generalized) metric topologies on such a space. The method also suggests order theoretic proofs of the contraction principle for probabilistic $B-$contractions.

For terminology and notations, we refer to [2], [4] and [18].
2. SOME GENERALIZED METRICS ON MENERG SPACES

Recall that $D^+$ denotes the set of the distribution functions of all nonnegative (real) random variables, which then are nondecreasing and left-continuous on $(0, \infty)$ and have limit 1 at $\infty$. Let we are given a probabilistic metric $\mathcal{F} : S \to D^+$, such that $(S, \mathcal{F}, Min)$ is a Menger space, and consider an element $G$ of $D^+$ which will be fixed. Recall that $\mathcal{F}(p, q)$ is usually denoted by $F_{pq}$.

**Theorem 2.1.** If we define the two-place function $d_G$ by

$$d_G(p, q) = \inf \{a \mid a > 0 \text{ and } F_{pq}(ax) \geq G(x), \forall x \in \mathbb{R}\},$$

then

(i) $d_G$ is a Luxemburg metric on $S$;

(ii) The $d_G$-topology is stronger than the $(\varepsilon, \lambda)$-topology;

(iii) If $(S, \mathcal{F})$ is complete, then $(S, d_G)$ is complete.

**Proof.** (i) Clearly $d_G$ is symmetric and $d_G(p, p) = 0$. If $d_G(p, q) = 0$ then, for each $a > 0$, $F_{pq}(ax) \geq G(x)$ for all $x$. Now, if $y = ax$ is fixed and $x \to +\infty$, then $F_{pq}(y) \geq \lim_{x \to +\infty} G(x) = 1$, that is $p = q$.

Suppose that $d_G(p, r) < \infty$ and $d_G(r, q) < \infty$. If $d_G(p, r) < a' < a$ and $d_G(r, q) < b' < b$, then

$$F_{pq}[(a' + b')x] \geq \text{Min}\{F_{pr}(a'x), F_{rq}(b'x)\} \geq G(x)$$

which shows that $d_G(p, q) \leq a' + b' < a + b$. Therefore $d_G(p, q) \leq d_G(p, r) + d_G(r, q)$, and it follows that $d_G$ is a Luxemburg metric.

(ii) Now, suppose that $\{p_n\}$ is $d_G$-convergent to $p$. Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Since $G \in D^+$, then there exists $x_0$ such that $G(x_0) > 1 - \lambda$. For $a < \frac{\varepsilon}{x_0}$, we choose $n_0$ such that $d_G(p_n, p) < a$ for all $n \geq n_0$. Therefore $F_{p_n, p}(\varepsilon) \geq F_{p_n, p}(ax_0) \geq G(x_0) > 1 - \lambda$, and we see that $\{p_n\}$ is $\mathcal{F}$-convergent to $p$.

(iii) Suppose that $\{p_n\}$ is $d_G$-Cauchy and $(S, \mathcal{F})$ is complete. Then, as above, we obtain that $\{p_n\}$ is $\mathcal{F}$-Cauchy and thus there exists $p \in S$ such that $\{p_n\}$ is $\mathcal{F}$-convergent to $p$. Let $a, \delta > 0$ be given. Then there exists $n_0$ such that

$$F_{p_n, p}(a + \delta)x \geq G(x) \text{ for all } n > n_0, \text{ all } m \geq 1 \text{ and each } x.$$  

Let $n > n_0$ and $x > 0$ be fixed. Since

$$F_{p_n, p}(a + \delta)x \geq \text{Min}\{F_{p_n, p}(ax), F_{p_n, p}(\delta x)\} \geq$$
The proof of the following result is easy to reproduce:

**Lemma 3.1.** Every Sehgal-contraction on \((S, F)\) is a Banach-contraction on \((S, d_G)\). Namely, if \(A : S \to S\) verifies the condition \((BC_L)\), then
\[
d_G(Ap, Aq) \leq Ld_G(p, q), \forall p, q \in S.
\]
By using this lemma and the alternative of fixed point, we can prove the Sehgal & Bharucha-Reid theorem ([19]).

**Theorem 3.2.** Let \( A \) be a Sehgal-contraction on the complete Menger space \( (S, \mathcal{F}, \text{Min}) \). Then \( A \) has a unique fixed point \( p^* \) and, for each \( p \in S \), 
\[
p^* = \lim_{n \to \infty} A^n p \quad \text{in the} \quad (\varepsilon, \lambda)-\text{topology}.
\]

**Proof.** Let \( p \in S \) and \( G := F_{pAp} \). By Lemma 3.1, \( A \) is \( d_G \)-contractive. Moreover, \( d_G(p, Ap) = 1 < \infty \). By the fixed point alternative (see [6] or [7]), the sequence \( A^n(p) \) converges to a fixed point \( p^* \) of \( A \), in the metric \( d_G \), and so in the \( (\varepsilon, \lambda) \)-topology. Clearly \( p^* \) is unique.

**Remark 3.3.** For \( G \) as in the above proof, \( S_p := \{ q, d_G(p, q) < \infty \} \) is a complete metric space. And \( A^n p \in S_p \) for all \( n \geq 1 \). Therefore \( A \) has a unique fixed point in \( S \), the theorem follows also in this way.

**Remark 3.4.** Let \( (S, \mathcal{F}, \text{Min}) \) be a complete Menger space and suppose that, for some \( G \in D^+ \), the \( d_G \)-topology and the \( (\varepsilon, \lambda) \)-topology are identical. Then, for every Sehgal-contraction \( A \) on \( S \), we have:

(i) For every \( p \in S \), \( A^n p \) is convergent to the unique fixed point of \( A \);
(ii) For each \( p \in S \) there exists \( n \geq 0 \) such that 
\[
F_{A^{n+1}pA^n p}(x) \geq G(x), \quad \forall x.
\]

Indeed, the first assertion follows from Theorem 3.2. The second assertion follows from the fixed point alternative, since (i) is always true. In fact, this assertions indicate, to a certain extent, the behavior of the values of \( \mathcal{F} \):

**Example 3.5.** For \( \beta > 0 \), let \( G(x) = \begin{cases} 0, & x \leq 1 \\ 1 - \frac{1}{x^\beta}, & x > 1 \end{cases} \). It is easy to see that \( d_G(p, q) = \sup \alpha^\beta d_\alpha(p, q) \). If \( d_G \) induces the \( (\varepsilon, \lambda) \)-topology on \( S \), then for each \( p \in S \) there exists \( m \geq 0 \) such that 
\[
F_{A^{n+1}pA^n p}(x) \geq 1 - \frac{1}{x^\beta}, \quad \forall x \geq 1, \quad \forall n \geq m.
\]

Generally, from the fixed point alternative we obtain the following.

**Theorem 3.6.** If \( A \) is a \( B \)-contraction on a complete Menger space \( (S, \mathcal{F}, \text{Min}) \) then, for each \( G \in D^+ \) and each \( p \in S \), either

\( (A_1) \ A^n p \) is \( d_G \) convergent to the unique fixed point of \( A \), or
(A2) for each \( n \geq 0 \) and each \( a > 0 \) there exists \( x_{n,a} > 0 \) such that
\[
F_{A^p, A^{n+1}p}(ax_{n,a}) < G(x_{n,a}).
\]

**Remark 3.7.** Having in mind the above results as well as the methods in [3], [5], [9] and [21], one can introduce the following relation on \( S \times \mathbb{R} \):
\[
(p, \lambda) \leq_G (q, \mu) \iff \lambda \leq \mu \text{ and } F_{pq} \geq (\mu - \lambda) \circ G.
\]
Recall that \( \nu \circ G(x) = G(\xi) \) for \( \nu \neq 0 \) and \( \nu \circ G = \varepsilon_0 \iff \nu = 0. \) Since \( (a + b) \circ G = \tau_{M}(a \circ G, b \circ G), \forall a, b \geq 0, \) then \( \leq_G \) is a partial order for every Menger space \((S, \mathcal{F}, M)\) and any \( G \in D^+. \) Now, the method of DeMarr can be applied to the monotone mapping \( B(p, \lambda) := (Ap, L\lambda) \) and we have an alternative proof of Theorem 3.2.

**References**


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