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STRONG CONVERGENCE THEOREMS ON THE MODIFIED ITERATIVE ALGORITHM FOR A FAMILY OF FINITE NONEXPANSIVE NONSELF MAPPINGS

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Abstract. Let K be a nonempty closed convex subset of a real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \to E, i = 1, 2, \cdots, N$ be a family of nonexpansive mappings which are weakly inward with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $f : K \to K$ be a fixed contractive mapping. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q T_{n+1} x_n, \quad n \geq 0.$$

Some sufficient and necessary conditions are proved for a common fixed point of a family of nonexpansive mappings provided $T_i, i = 1, 2, \cdots, N$ satisfy some mild conditions.
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1. Introduction

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $f : K \to K$ is a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in K$. A mapping $T : K \to E$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. Let $T : K \to K$ be a nonexpansive self-mapping. For a sequence $\{\alpha_n\}$ of real numbers in $(0, 1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0. \tag{1.1}$$

Halpern [1] was the first to study the convergence of the algorithm (1.1) in the framework of Hilbert space. Lions [2] improved the result of Halpern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C3) $\lim_{n\to\infty}((\alpha_n - \alpha_{n-1})/\alpha_n^2) = 0$.

It was observed that both Halpern's and Lions's conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice $\alpha_n = \frac{1}{n}$. This was overcome by Wittmann [3] who proved the strong convergence of $\{x_n\}$ if $\{\alpha_n\}$ satisfies the control conditions (C1), (C2) and (C4)

(C4) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

Reich [4] extended this result of Wittmann to the class of Banach spaces which are uniformly smooth and have weakly sequentially continuous duality maps. Subsequently, Shioji and Takahashi [5] extended Wittmann's result to Banach spaces with uniformly Gâteaux differentiable norm and in which each nonempty closed convex subset of K has the fixed point property for nonexpansive mappings and $\{\alpha_n\}$ satisfies control conditions (C1), (C2) and (C4).

Xu [6] showed that the results of Halpern holds in uniformly smooth Banach space if $\{\alpha_n\}$ satisfies control conditions (C1), (C2) and (C5)

(C5) $\lim_{n\to\infty}((\alpha_n - \alpha_{n-1})/\alpha_n) = 0$.

As has been remarked in [6], control conditions (C3) and (C4) are not comparable. Also conditions (C4) and (C5) are not comparable. However, condition (C3) does not permit the natural choice $\alpha_n = 1/n$ for all integers $n \geq 0$. Hence, conditions (C4) and (C5) are preferred.

Next consider N nonexpansive mappings $T_1, T_2, \cdots, T_N$. For a sequence $\{\alpha_n\} \subset (0, 1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{n+1} x_n, \quad n \geq 0, \tag{1.2}$$

where $T_n = T_{\text{n mod N}}$.

In 1996, Bauschke [7] defined and studied the iterative process 1.2 in Hilbert spaces with control conditions (C1), (C2) and (C4) on the parameter $\{\alpha_n\}$.

Recently, Takahashi et al. [8] extended Bauschke's result to uniformly convex Banach spaces. More precisely, they proved the following result.

Theorem 1.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_i : K \to K, i = 1, 2, \cdots, N$, be a family of nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and*

$$\bigcap_{i=1}^{N} F(T_i) = F(T_N T_{N-1} \cdots T_1)$$

$$= F(T_1 T_N \cdots T_2) = \cdots = F(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{n+1} x_n, \quad n \geq 0,$$

where $T_n = T_{n \bmod N}$ and $\{\alpha_n\}$ is a real sequence which satisfies the control conditions (C1), (C2) and (C6)

(C6) $\sum_{n=0}^{\infty} |\alpha_{n+N} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \cdots, T_N$. Further, if $Px_0 = \lim_{n\to\infty} x_n$ for each $x_0 \in K$, then P is a sunny nonexpansive retraction of K onto F.

Remark 1.2. (1) Control conditions (C1) and (C2) are necessary for the strong convergence of algorithm (1.1) and (1.2) for nonexpansive mappings. It is unclear if they are sufficient.

(2) In the above work, the mappings $T_1, T_2, \cdots, T_N$ remain self-mappings of a nonempty closed convex subset K either of a Hilbert space or a uniformly convex space. If, however, the domain of $T_1, T_2, \cdots, T_N$, $D(T_i) = K, i = 1, 2, \cdots, N$, is a proper subset of E and T_i maps K into E, then the iteration process (1.2) may fail to be well defined.

The objective of this paper is to show another generalization of Mann and Halpern iterative algorithm to a setting of a finite family of nonexpansive nonself mappings. We deal with the iterative scheme: $x_0 \in K$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Q T_{n+1} x_n, \quad n \geq 0.$$

Using this iterative scheme, we can find a common fixed point of a finite family of nonexpansive nonself mappings under some type of control conditions.

2. Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued. In the sequel, we will denote the single valued normalized duality map by j.

The norm is said to be uniformly *Gâteaux differentiable* if for each $y \in S_1(0) = \{x \in E : \|x\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists uniformly for $x \in S_1(0)$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm. Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality map is norm to weak star uniformly continuous on bounded subsets of E.

A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach space E, we have that if $\|x\| = \|y\| = \|\lambda x + (1 - \lambda y\|$, for $x, y \in E$ and $\lambda \in (0, 1)$, then $x = y$.

Let K be a nonempty subset of a Banach space E. For $x \in K$, the *inward* set of x, $I_K(x)$, is defined by

$$I_K(x) = \{x + \lambda(u - x) : u \in K, \lambda \geq 1\}.$$

A mapping $T : K \to E$ is called *weakly inward* if $Tx \in cl[I_K(x)]$ for all $x \in K$, where $cl[I_K(x)]$ denotes the closure of the inward set. Every self-map is trivially weakly inward.

Let $K \subset E$ be closed convex and Q a mapping of E onto K. Then Q is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Qz = z$ for every $z \in R(Q)$, rang of Q. A subset K of E is said to be a *sunny nonexpansive retract* of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a *nonexpansive retract* of E if here exists a nonexpansive retraction of E onto K. If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed convex subset of H.

In the sequel, we will make use of the following lemmas.

Lemma 2.1. ([9]) *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. *Then,* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.2. ([10]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. ([11]) Let K be a nonempty closed convex subset of a reflexive Banach space E which has a uniformly Gâteaux differentiable norm. Let $T : K \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that every nonempty closed convex bounded subset of K has the fixed point property for nonexpansive mappings. Then there exists a continuous path $t \to z_t, 0 < t < 1$, satisfying $z_t = tu + (1-t)Tz_t$, for arbitrary but fixed $u \in K$, which converges strongly to a fixed point of T. Further, if $Qu = \lim_{t\to 0} z_t$ for each $u \in K$, then Q is a sunny nonexpansive retraction of K onto $F(T)$.

3. Main results

Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction and $f : K \to K$ be a fixed contractive mapping. For each $t \in (0,1)$ define a contraction mapping $T_t : K \to K$ by

$$T_t x = tf(x) + (1-t)QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x, \quad x \in K.$$

By Banach's contraction principle yields a unique fixed point $z_t \in K$ of T_t which is a unique solution of the equation:

$$z_t = tf(z_t) + (1-t)QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}z_t.$$

Theorem 3.1. *Let* K *be a nonempty closed convex subset of a real Banach space* E *which has a uniformly Gâteaux differentiable norm. Assume that* K *is a sunny nonexpansive retract of* E *with* Q *as the sunny nonexpansive* *retraction. Let* $T_i : K \to E, i = 1, 2, \cdots, N$ *be a family of nonexpansive mappings which are weakly inward with* $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and*

$$\bigcap_{i=1}^{N} F(QT_i) = F(QT_N QT_{N-1} \cdots QT_1)$$

$$= F(QT_1 QT_N \cdots QT_2) = \vdots = F(QT_{N-1} QT_{N-2} \cdots QT_1 QT_N).$$

Let $f : K \to K$ *be a fixed contractive mapping. Assume that* $\{z_t\}$ *converges strongly to a fixed point* z *of* $QT_{n+N} QT_{n+N-1} \cdots QT_{n+2} QT_{n+1}$ *as* $t \to 0$, *where* z_t *is the unique element of* K *which satisfies* $z_t = tf(z_t) + (1 - t)QT_{n+N} QT_{n+N-1} \cdots QT_{n+2} QT_{n+1} z_t$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be three real sequences in* $[0, 1]$ *which satisfies the following conditions:*

(C0) $\alpha_n + \beta_n + \gamma_n = 1$,

(D) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

For given $x_0 \in K$, *let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n QT_{n+1} x_n, \quad n \ge 0, \tag{3.1}$$

where $T_n = T_{n \bmod N}$, *then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \cdots, T_N$ *if and only if the real sequence* $\{\alpha_n\}$ *satisfies the control conditions (C1) and (C2) and* $\|QT_{n+2} x_n - QT_{n+1} x_n\| \to 0 \ (n \to \infty)$.

Proof. *"Sufficiency".* We show first that $\{x_n\}$ is bounded. For $p \in F$, we have

$$\|x_{n+1} - p\| \le \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|QT_{n+1} x_n - p\|)$$
$$\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\|$$
$$\le \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (\beta_n + \gamma_n) \|x_n - p\|$$
$$= (1 - \alpha_n + \alpha\alpha_n) \|x_n - p\| + \alpha_n \|f(p) - p\|$$
$$\le \max\{\|x_n - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\}.$$

By induction, we get

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{1}{1-\alpha} \|f(p) - p\|\},$$

for all $n \ge 0$. This shows that $\{x_n\}$ is bounded, so are $\{QT_{n+1} x_n\}$ and $\{f(x_n)\}$.

We show then that $\|x_{n+i} - x_n\| \to 0 \ (n \to \infty), i = 1, 2, \cdots, N$.

Define a sequence $\{y_n\}$ which satisfies

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n.$$

Observe that

$$\begin{aligned} y_{n+1} - y_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}QT_{n+2}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n QT_{n+1}x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n})f(x_n) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}(QT_{n+2}x_{n+1} - QT_{n+2}x_n) + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n})QT_{n+1}x_n \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}(QT_{n+2}x_n - QT_{n+1}x_n). \end{aligned}$$

It follows that

$$\begin{aligned} \|y_{n+1}-y_n\| - \|x_{n+1}-x_n\| &\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|f(x_n)\| \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1} - x_n\| + |\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|QT_{n+1}x_n\| \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|QT_{n+2}x_n - QT_{n+1}x_n\| - \|x_{n+1} - x_n\| \\ &\le |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|QT_{n+1}x_n\|) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}\|QT_{n+2}x_n - QT_{n+1}x_n\|. \end{aligned}$$

Since $\{QT_{n+1}x_n\}$ and $\{f(x_n)\}$ are bounded, from the conditions (C1) and $\lim_{n\to\infty}\|QT_{n+2}x_n - QT_{n+1}x_n\| = 0$, we obtain

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 2.1 we know that $\|y_n - x_n\| \to 0$ as $n \to \infty$ which implies $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Consequently, $\lim_{n\to\infty}\|x_{n+i} - x_n\| = 0, i = 1, 2, \cdots, N$.

Since

$$\begin{aligned} x_{n+1} - QT_{n+1}x_n &= \alpha_n(f(x_n) - QT_{n+1}x_n) + \beta_n(x_n - QT_{n+1}x_n) \\ &= \alpha_n(f(x_n) - QT_{n+1}x_n) + \beta_n(x_n - x_{n+1}) \\ &+ \beta_n(x_{n+1} - QT_{n+1}x_n), \end{aligned}$$

this together with conditions (C1) and (D) implies $\|x_{n+1} - QT_{n+1}x_n\| \to 0$ as $n \to \infty$, consequently, $\|x_n - QT_{n+1}x_n\| \to 0$ as $n \to \infty$.

We next show that

$$\lim_{n\to\infty} \|x_n - QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n\| = 0.$$

It suffices to show that

$$\lim_{n\to\infty} \|x_{n+1} - QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n\| = 0.$$

From

$$\begin{aligned}\|x_{n+1} - QT_{n+2}QT_{n+1}x_n\| &\le \|x_{n+1} - QT_{n+2}x_{n+1} \\ &\quad +\|QT_{n+2}x_{n+1} - QT_{n+2}QT_{n+1}x_n\| \\ &\le \|x_{n+1} - QT_{n+2}x_{n+1}\| + \|x_{n+1} - QT_{n+1}x_n\|,\end{aligned}$$

we also have

$$\lim_{n\to\infty} \|x_{n+1} - QT_{n+2}QT_{n+1}x_n\| = 0.$$

Similarly, we can obtain

$$\lim_{n\to\infty} \|x_{n+1} - QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n\| = 0,$$

which implies

$$\lim_{n\to\infty} \|x_n - QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n\| = 0.$$

Notice $\bigcap_{i=1}^N F(QT_i) = F(QT_{n+N}QT_{n+N-1}\cdots QT_{n+1})$ and hence as $T_i, i = 1, 2, \cdots, N$ is weakly inward, $z \in F$.

Now we show that

$$\limsup_{n\to\infty} \langle z - f(z), j(z - x_n)\rangle \le 0.$$

Let z_t be the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1-t)QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x.$$

Then

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-t)^2\|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}z_t - x_n\|^2 \\ &\quad +2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\ &\le (1-t)^2(\|QT_{n+N}\cdots QT_{n+1}z_t - QT_{n+N}\cdots QT_{n+1}x_n\| \\ &\quad +\|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - x_n\|)^2 \\ &\quad +2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2\end{aligned}$$

$$\leq (1-t)^2(\|z_t - x_n\| + \|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - x_n\|)^2$$
$$+2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2$$
$$\leq (1+t^2)\|z_t - x_n\|^2 + \|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - x_n\|$$
$$\times(2\|z_t - x_n\| + \|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - x_n\|)$$
$$+2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle.$$

The last inequality implies

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \leq \frac{t}{2}\|z_t - x_n\|^2 + \frac{\|QT_{n+N}\cdots QT_{n+1}x_n - x_n\|}{2t}$$
$$\times(2\|z_t - x_n\| + \|QT_{n+N}\cdots QT_{n+1}x_n - x_n\|).$$

It follows that

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle \leq 0.$$

Moreover, we have that

$$\langle z - f(z), j(z - x_n)\rangle = \langle z - f(z), j(z - x_n)\rangle - \langle z - f(z), j(z_t - x_n)\rangle$$
$$+\langle z - f(z), j(z_t - x_n)\rangle - \langle z_t - f(z), j(z_t - x_n)\rangle$$
$$+\langle z_t - f(z), j(z_t - x_n)\rangle - \langle z_t - f(z_t), j(z_t - x_n)\rangle$$
$$+\langle z_t - f(z_t), j(z_t - x_n)\rangle$$
$$= \langle z - f(z), j(z - x_n) - j(z_t - x_n)\rangle$$
$$+\langle z - z_t, j(z_t - x_n)\rangle + \langle f(z_t) - f(z), j(z_t - x_n)\rangle$$
$$+\langle z_t - f(z_t), j(z_t - x_n)\rangle.$$

Then, we obtain

$$\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle$$
$$\leq \sup_{n\in N}\langle z - f(z), j(z - x_n) - j(z_t - x_n)\rangle$$
$$+\|z - z_t\|\limsup_{n\to\infty}\|z_t - x_n\| + \|f(z_t) - f(z)\|\limsup_{n\to\infty}\|z_t - x_n\|$$
$$+\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle$$
$$\leq \sup_{n\in N}\langle z - f(z), j(z - x_n) - j(z_t - x_n)\rangle$$
$$+(1+\alpha)\|z - z_t\|\limsup_{n\to\infty}\|z_t - x_n\|$$
$$+\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle.$$

By hypothesis $z_t \to z \in F$ as $t \to 0$ and j is norm-to-weak* uniformly continuous on bounded subset of E, we obtain

$$\lim_{t\to 0}\sup_{n\in N}\langle z - f(z), j(z - x_n) - j(z_t - x_n)\rangle = 0.$$

Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle \\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - f(z_t), j(z_t - x_n)\rangle \\ &\le 0.\end{aligned}$$

Finally, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(f(x_n) - z) + \beta_n(x_n - z) \\ &\quad + \gamma_n(QT_{n+N}\cdots QT_{n+1}x_n - z)\|^2 \\ &\le \|\beta_n(x_n - z) + \gamma_n(QT_{n+N}\cdots QT_{n+1}x_n - z)\|^2 \\ &\quad + 2\alpha_n\langle f(x_n) - z, j((x_{n+1} - z)\rangle \\ &\le \beta_n^2\|x_n - z\|^2 + \gamma_n^2\|QT_{n+N}\cdots QT_{n+1}x_n - z\|^2 \\ &\quad + 2\beta_n\gamma_n\|x_n - z\|\|QT_{n+N}\cdots QT_{n+1}x_n - z\| \\ &\quad + 2\alpha_n\langle f(x_n) - f(z), j(x_{n+1} - x)\rangle \\ &\quad + 2\alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\ &\le (\beta_n + \gamma_n)^2\|x_n - z\|^2 + 2\alpha\alpha_n\|x_n - z\|\|x_{n+1} - z\| \\ &\quad + 2\alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle \\ &\le (1 - \alpha_n)^2\|x_n - z\|^2 + \alpha\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - x\|^2) \\ &\quad + 2\alpha_n\langle f(z) - z, j(x_{n+1} - z)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \frac{1 - (2-\alpha)\alpha_n + \alpha_n^2}{1 - \alpha\alpha_n}\|x_n - z\|^2 \\ &\quad + \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(z) - z, j(x_{n+1} - z)\rangle \\ &\le \frac{1 - (2-\alpha)\alpha_n}{1 - \alpha\alpha_n}\|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(z) - z, j(x_{n+1} - z)\rangle \\ &\quad + \frac{\|x_n - z\|^2}{1 - \alpha\alpha_n}\alpha_n^2.\end{aligned}$$

Apply Lemma 2.2 to conclude that $x_n \to z$ as $n \to \infty$. The sufficiency of Theorem 3.1 is proved.

"Necessity". Suppose that sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point $p \in F$. Hence we have

$$\begin{aligned}&\|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - x_n\|\\ &\le \|QT_{n+N}QT_{n+N-1}\cdots QT_{n+1}x_n - p\| + \|x_n - p\|\\ &\le 2\|x_n - p\| \to 0 \ \ (n \to \infty).\end{aligned}$$

Since each $QT_i : K \to K, i = 1, 2, \cdots, N$ is nonexpansive, it is continuous, and so

$$\|QT_{n+1}x_n - p\| \le \|x_n - p\|,$$

i.e.,

$$QT_{n+1}x_n \to p \ \ (n \to \infty).$$

Again it follows that

$$\begin{aligned}\alpha_n\|f(x_n) - QT_{n+1}x_n\| \le \|x_{n+1} - QT_{n+1}x_n\|&\\ + \beta_n\|x_n - QT_{n+1}x_n\| \to 0 \ \ (n \to \infty).&\end{aligned}$$

Therefore we have

$$\limsup_{n\to\infty} \alpha_n\|f(x_n) - QT_{n+1}x_n\| = \limsup_{n\to\infty} \alpha_n\|f(p) - p\| = 0.$$

This implies

$$\limsup_{n\to\infty} \alpha_n = 0.$$

Hence we have

$$\lim_{n\to\infty} \alpha_n = 0.$$

The necessity of condition (C1) is proved.

Take $f = 0, K = \{x \in E : \|x\| \le 1\}$ and $QT_1 = QT_2 = \cdots = QT_N = I : K \to K$, where I is the identity mapping. Since each $QT_i, i = 1, 2, \cdots, N$ is nonexpansive and 0 is the unique common fixed point of $QT_1, QT_2, \cdots, QT_N$ in K, hence we have

$$x_{n+1} = (1 - \alpha_n)x_n = \prod_{i=0}^{n+1}(1 - \alpha_i)x_0.$$

If $x_n \to 0 \in \bigcap_{i=1}^N F(QT_i)$, we have

$$\lim_{n\to\infty} \prod_{i=0}^{n+1}(1-\alpha_i)\|x_0 - 0\| = 0.$$

This implies that

$$\prod_{i=0}^{\infty}(1-\alpha_i) = 0,$$

i.e.,

$$\sum_{i=0}^{\infty} \alpha_n = \infty.$$

The necessity of condition (C2) is proved. This completes the proof. □

If in Theorem 3.1, $f = u \in K$ is a fixed constant. By Lemma 2.3 and Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let K be a nonempty closed convex subset of a reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Assume that every nonempty closed bounded convex subset of K has the fixed point property for nonexpansive mappings. Let $T_i : K \to E, i = 1, 2, \cdots, N$ be a family of nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and*

$$\begin{aligned}\bigcap_{i=1}^N F(QT_i) &= F(QT_N QT_{N-1} \cdots QT_1)\\ &= F(QT_1 QT_N \cdots QT_2)\\ &\vdots\\ &= F(QT_{N-1} QT_{N-2} \cdots QT_1 QT_N).\end{aligned}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ which satisfies the following conditions:

(C0) $\alpha_n + \beta_n + \gamma_n = 1$,

(D) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

For given $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n QT_{n+1} x_n, \quad n \ge 0, \tag{3.2}$$

where $T_n = T_{n \bmod N}$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \cdots, T_N$ if and only if the real sequence $\{\alpha_n\}$ satisfies the *control conditions (C1) and (C2) and* $\|QT_{n+2}x_n - QT_{n+1}x_n\| \to 0 \ (n \to \infty)$*. Further, if* $\{x_n\}$ *converges strongly to some common fixed point and if* $Qu = \lim_{n\to\infty} x_n$ *for each* $u \in K$*, then* Q *is a sunny nonexpansive retraction of* K *onto* $\bigcap_{i=1}^N F(T_i)$*.*

If in Theorem 3.1, $T_i, i = 1, 2, \cdots, N$ are self-mappings then the projection operator Q is replaced with I, the identity map on E. Thus, we have the following corollary.

Corollary 3.3. *Let* K *be a nonempty closed convex subset of a real Banach space* E *which has a uniformly Gâteaux differentiable norm. Let* $T_i : K \to K, i = 1, 2, \cdots, N$ *be a family of nonexpansive mappings with* $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and*

$$\begin{aligned}\bigcap_{i=1}^N F(T_i) &= F(T_N T_{N-1} \cdots T_1) \\ &= F(T_1 T_N \cdots T_2) \\ &\ \vdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Let $f : K \to K$ *be a fixed contractive mapping. Assume that* $\{z_t\}$ *converges strongly to a fixed point* z *of* $T_{n+N}T_{n+N-1} \cdots T_{n+2}T_{n+1}$ *as* $t \to 0$*, where* z_t *is the unique element of* K *which satisfies* $z_t = tf(z_t) + (1-t)T_{n+N}T_{n+N-1} \cdots T_{n+2}T_{n+1}z_t$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be three real sequences in* $[0, 1]$ *which satisfies the following conditions:*

(C0) $\alpha_n + \beta_n + \gamma_n = 1$*,*

(D) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*

For given $x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{n+1}x_n, \quad n \geq 0, \tag{3.3}$$

where $T_n = T_{n \bmod N}$*, then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \cdots, T_N$ *if and only if the real sequence* $\{\alpha_n\}$ *satisfies the control conditions (C1) and (C2) and* $\|T_{n+2}x_n - T_{n+1}x_n\| \to 0 \ (n \to \infty)$*.*

Remark 3.4. We note that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm. By Xu [12,Theorem 4.1], we know that $\{z_t\}$ converges strongly to a fixed point of $T_{n+N}T_{n+N-1} \cdots T_{n+1}$ as $t \to$

0, where z_t is the unique element of C which satisfies $z_t = tf(z_t) + (1 - t)T_{n+N}T_{n+N-1} \cdots T_{n+1}z_t$.

Corollary 3.5. *Let K be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T_i : K \to K, i = 1, 2, \cdots, N$ be a family of nonexpansive mappings with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and*

$$\begin{aligned}\bigcap_{i=1}^N F(T_i) &= F(T_N T_{N-1} \cdots T_1) \\ &= F(T_1 T_N \cdots T_2) \\ &\vdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Let $f : K \to K$ be a fixed contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ which satisfies the following conditions:

(C0) $\alpha_n + \beta_n + \gamma_n = 1$,

(D) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{n+1} x_n, \quad n \geq 0,$$

where $T_n = T_{n \bmod N}$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \cdots, T_N$ if and only if the real sequence $\{\alpha_n\}$ satisfies the control conditions (C1) and (C2) and $\|T_{n+2}x_n - T_{n+1}x_n\| \to 0 \ (n \to \infty)$.

4. Applications to the feasibility problem

As an application, we shall utilize Theorem 3.1 to study the strong convergence theorem connected with the feasibility problem.

Let H be a Hilbert space, $K_1, K_2, \cdots, K_N$ be N closed convex subset of H with $\bigcap_{i=1}^N K_i \neq \emptyset$. Then the feasibility problem in H can be stated as follows.

The original image z is known a priori to belong to the intersection $K_0 = \bigcap_{i=1}^N K_i$, given only the metric projections P_{K_i} of H onto $K_i (i = 1, 2, \cdots, N)$, recover z by an iterative scheme.

In [13], Crombez proved the following result: Let $T = \alpha_0 I + \sum_{i=1}^N \alpha_i T_i$ with $T_i = (1 - \lambda_i)I + \lambda_i P_{K_i}$ for all $0 < \lambda_i < 1, \alpha_i \geq 0$ for $i = 1, 2, \cdots, N$ and $\sum_{i=0}^N \alpha_i = 1$, where $K_0 = \bigcap_{i=1}^N K_i$ is nonempty. Then for any given $x_0 \in H$, the sequence $\{T^n x_0\}$ converges weakly to some point in K_0. Later Kitahara and Takahashi [14], Takahashi et al. [8,15] dealt with the feasibility problem by convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces.

In this section, we shall utilize Theorem 3.1 to study the strong convergence theorem connected with the feasibility problem. For the purpose we first give the following Lemma.

Lemma 4.1. ([16, Lemma 3.3]) *Let K be a nonempty closed convex subset of a strictly convex real Banach space E. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \to E, i = 1, 2, \cdots, N$ be a family of nonexpansive mappings which are weakly inward with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $S_i : K \to E, i = 1, 2, \cdots, N$ be a family of mappings defined by $S_i = (1 - \lambda_i)I + \lambda_i T_i, 0 < \lambda_i < 1$ for each $i = 1, 2, \cdots, N$. Then $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N F(S_i) = \bigcap_{i=1}^N F(QS_i)$ and*

$$\begin{aligned}\bigcap_{i=1}^N F(S_i) &= F(QS_N QS_{N-1} \cdots QS_1)\\ &= F(QS_1 QS_N \cdots QS_2)\\ &\vdots\\ &= F(QS_{N-1} QS_{N-2} \cdots QS_1 QS_N).\end{aligned}$$

Theorem 4.2. *Let K be a nonempty closed convex subset of a strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Assume that K is a sunny nonexpansive retract of E with Q as the sunny nonexpansive retraction. Let $T_i : K \to E, i = 1, 2, \cdots, N$ be a family of nonexpansive mappings which are weakly inward with $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $S_i : K \to E, i = 1, 2, \cdots, N$ be a family of mappings defined by $S_i = (1 - \lambda_i)I + \lambda_i T_i$ for all $0 < \lambda_i < 1 (i = 1, 2, \cdots, N)$. Let $f : K \to K$ be a fixed contractive mapping. Assume that $\{z_t\}$ converges strongly to a fixed point z of $QS_{n+N} QS_{n+N-1} \cdots QS_{n+2} QS_{n+1}$ as $t \to 0$, where z_t is the unique element of K which satisfies $z_t = tf(z_t) + (1 - t)QS_{n+N} QS_{n+N-1} \cdots QS_{n+2} QS_{n+1} z_t$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $[0, 1]$ which satisfies the following conditions:*

(C0) $\alpha_n + \beta_n + \gamma_n = 1$,

(D) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

For given $x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n QS_{n+1}x_n, \quad n \geq 0,$$

where $S_n = S_{n \bmod N}$*, then the sequence* $\{x_n\}$ *converges strongly to a common fixed point of* $T_1, T_2, \cdots, T_N$ *if and only if the real sequence* $\{\alpha_n\}$ *satisfies the control conditions (C1) and (C2) and* $\|QS_{n+2}x_n - QS_{n+1}x_n\| \to 0 \ (n \to \infty)$*.*

Proof. By Lemma 4.1, $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N F(S_i) = \bigcap_{i=1}^N F(QS_i)$ and

$$\begin{aligned}\bigcap_{i=1}^N F(S_i) &= F(QS_N QS_{N-1} \cdots QS_1)\\ &= F(QS_1 QS_N \cdots QS_2)\\ &\vdots\\ &= F(QS_{N-1}QS_{N-2} \cdots QS_1 QS_N).\end{aligned}$$

Thus, the conclusion of Theorem 4.2 can be obtained from Theorem 3.1 immediately. This completes the proof. □

Theorem 4.3. *Let* K *be a nonempty closed convex subset of a strictly convex real Banach space* E *which has a uniformly Gâteaux differentiable norm. Let* $K_1, K_2, \cdots, K_N$ *be nonexpansive retracts of* K *onto itself such that the set* $\bigcap_{i=1}^N K_i \neq \emptyset$*. Define a family of mappings* $T_i, i = 1, 2, \cdots, N$ *by* $T_i = (1 - \lambda_i)I + \lambda_i P_{K_i}$ *for all* $0 < \lambda_i < 1 (i = 1, 2, \cdots, N)$*. Let* $f : K \to K$ *be a fixed contractive mapping. Assume that* $\{z_t\}$ *converges strongly to a fixed point* z *of* $T_{n+N}T_{n+N-1} \cdots T_{n+2}T_{n+1}$ *as* $t \to 0$*, where* z_t *is the unique element of* K *which satisfies* $z_t = tf(z_t) + (1 - t)T_{n+N}T_{n+N-1} \cdots T_{n+2}T_{n+1}z_t$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be three real sequences in* $[0, 1]$ *which satisfies the following conditions:*

(C0) $\alpha_n + \beta_n + \gamma_n = 1$*,*

(D) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$*.*

For given $x_0 \in K$*, let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_{n+1}x_n, \quad n \geq 0,$$

where $T_n = T_{n \bmod N}$*, then the sequence* $\{x_n\}$ *converges strongly to a point* $z \in \bigcap_{i=1}^N K_i$ *if and only if the real sequence* $\{\alpha_n\}$ *satisfies the control conditions (C1) and (C2) and* $\|T_{n+2}x_n - T_{n+1}x_n\| \to 0 \ (n \to \infty)$*.*

Proof. By Lemma 4.1, $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N F(P_{K_i}) = \bigcap_{i=1}^N K_i$ and

$$\begin{aligned}\bigcap_{i=1}^N F(T_i) &= F(T_N T_{N-1} \cdots T_1) \\ &= F(T_1 T_N \cdots T_2) \\ &\vdots \\ &= F(T_{N-1} T_{N-2} \cdots T_1 T_N).\end{aligned}$$

Thus, the conclusion of Theorem 4.3 can be obtained from Theorem 3.1 immediately. This completes the proof. □

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