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RELATIONAL BREZIS-BROWDER PRINCIPLES

MIHAI TURINICI

"A. Myller" Math. Seminar
"A. I. Cuza" University, 11, Copou Boulevard,
700506 Iaşi, Romania
E-mail: mturi@uaic.ro

Abstract. The relational type versions of the (quasi-order) Brezis-Browder principle are logical equivalent with it. Some applications of these facts to maximality principles are also discussed.

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1. Introduction

Let M be some nonempty set. By a *quasi-order* over M we shall understand any relation $(\leq)$ on this set which is *reflexive* $(x \leq x, \forall x \in M)$ and *transitive* $(x \leq y$ and $y \leq z$ imply $x \leq z)$. Assume that we fixed such an object; and let $x \vdash \varphi(x)$ be some function from M to $R_+ = [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$- *maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w). \tag{1.1}$$

A basic result involving such points is the 1976 Brezis-Browder principle [4]:

Theorem 1. *Suppose that*

$$\begin{array}{l} M \textit{ is sequentially inductive } (\textit{modulo } (\leq)) \\ (\textit{each ascending sequence in } M \textit{ has an upper bound}) \end{array} \tag{1.2}$$

$$\varphi \textit{ is } (\leq)\textit{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{1.3}$$

Then, for each $u \in M$ *there exists a* $(\leq, \varphi)$*-maximal* $v \in M$ *with* $u \leq v$.

In particular, when (1.3) is taken in the stronger sense

$$\varphi \text{ is strictly } (\leq)\text{-decreasing } (x \leq y, x \neq y \Rightarrow \varphi(x) > \varphi(y)) \tag{1.4}$$

the concept in (1.1) means

$$w \in M, z \leq w \Rightarrow z = w \text{ (referred to as: } z \text{ is } (\leq)\text{-maximal);} \tag{1.5}$$

and the Brezis-Browder principle includes directly the well known Ekeland's [7]. Note that the regularity condition $\mathrm{Codom}(\varphi) \subseteq R_+$ is not essential for the conclusions above. In fact, let $x \vdash \varphi(x)$ be some function from M to $\bar{R} = R \cup \{-\infty, \infty\}$. Take a certain order isomoprphism $t \vdash \chi(t)$ between $\bar{R}$ and a bounded subinterval of R_+; such as, e.g.,

$$\chi(t) = \pi/2 + \mathrm{arctg}(t), t \in R; \quad \chi(-\infty) = 0, \chi(\infty) = \pi.$$

The composed function $\varphi_1 = \chi \circ \varphi$ fulfills $\mathrm{Codom}(\varphi_1) \subseteq [0, \pi]$. Moreover, (1.3) (resp., (1.4)) holds for φ whenever it holds for φ_1 (and viceversa). Adding to this the generic property

$$(\leq, \varphi)\text{-maximal} \Longleftrightarrow (\leq, \varphi_1)\text{-maximal}$$

shows that if conclusions of Theorem 1 are true for φ_1 then these are also true for φ. This remark goes back to Carja and Ursescu [6].

Now, Theorem 1 found some useful applications to convex and nonconvex analysis; we refer to the quoted papers for details. So, it cannot be surprising that many extensions of Theorem 1 were proposed. Here, we shall concentrate on the relational way of enlargement. This may be described as a deduction of maximality results like Theorem 1 when the relation $(\leq)$ has not all the properties of a quasi-order. Two basic situations may occur:

(A). The considered relation is (only) transitive. Results of this type are implicitly deductible from the variational ones in Kada, Suzuki and Takahashi [10]. (These will be discussed in Section 2).

(B). The underlying relation is a general (=amorph) one. A specific result of this kind may be found in Gajek and Zagrodny [8]. (We refer to Section 3 for details). The particular situation of reflexivity being added enters in such a scheme. For a basic result of this type we refer to Bae, Cho and Yeom [2]. (This will be delineated in Section 4 below).

As we shall see, all such techniques are non-effective: i.e., they produce nothing but logical equivalents of Theorem 1. So, genuine extensions of this result must be based on a different approach; we shall discuss it elsewhere.

2. Transitive relations

Let M be a nonempty set; and $\triangleleft$, some *transitive* relation ($x \triangleleft y, y \triangleleft z \Rightarrow x \triangleleft z$) over it. Denote by $(\leq)$ the associated quasi-order

$$x \leq y \text{ iff either } x = y \text{ or } x \triangleleft y. \tag{2.1}$$

Further, take a function $\varphi : M \to R_+$. The $(\triangleleft)$-decreasing property for it is that of (1.3) (with $(\triangleleft)$ in place of $(\leq)$). Note that, by (2.1),

$$\varphi \text{ is } (\triangleleft)\text{-decreasing} \Longleftrightarrow \varphi \text{ is } (\leq)\text{-decreasing}. \tag{2.2}$$

Call the point $z \in M$, $(\triangleleft, \varphi)$-*maximal*, provided

$$w \in M \text{ and } z \triangleleft w \text{ imply } \varphi(z) = \varphi(w). \tag{2.3}$$

Again by (2.1), the generic relation holds

$$(\triangleleft, \varphi)\text{-maximal} \Longleftrightarrow (\leq, \varphi)\text{-maximal}. \tag{2.4}$$

This, along with (2.2), shows that maximality results involving the transitive relation $(\triangleleft)$ are deductible from the Brezis-Browder principle involving its associated quasi-order $(\leq)$. The key moment of this approach is that of (1.2) being assured. It would be useful to have expressed this condition in terms of the initial transitive relation. This necessitates a few conventions and auxiliary facts. Call the sequence (x_n), *ascending* (modulo $(\triangleleft)$) when

$$x_n \triangleleft x_{n+1}, \forall n \text{ (or, equivalently: } x_n \triangleleft x_m \text{ if } n < m). \tag{2.5}$$

Note the generic (sequential) relation

$$\text{ascending (modulo } (\triangleleft)) \Longrightarrow \text{ascending (modulo } (\leq)).$$

The reciprocal is not in general true. For example, the constant sequence $(x_n = a; n \in N)$ is ascending (modulo $(\leq)$); but not ascending (modulo $(\triangleleft)$), whenever $a \triangleleft a$ is false. Further, given the sequence (x_n) in M, let us say that $u \in M$ is an *upper bound* (modulo $(\triangleleft)$) of it provided

$$x_n \triangleleft u, \forall n \text{ (written as: } (x_n) \triangleleft u). \tag{2.6}$$

If u is generic in this convention, we say that (x_n) is *bounded above* (modulo $(\lhd)$). As before, the relation below is clear

$$(\forall u)[(x_n) \lhd u] \Longrightarrow [(x_n) \le u]; \quad \text{wherefrom}$$
$$\text{bounded above (modulo } (\lhd)) \Longrightarrow \text{bounded above (modulo } (\le)).$$

(The converse is not in general valid). Finally, let the concept of sequential inductivity modulo $(\lhd)$ be that of (1.2), with $(\lhd)$ in place of $(\le)$.

We may now give an appropriate answer to the posed question.

Theorem 2. *Let the transitive relation* $(\lhd)$ *and the function* $\varphi : M \to R_+$ *be such that* M *is sequentially inductive modulo* $(\lhd)$ *and* φ *is* $(\lhd)$*-decreasing. Then, for each* $u \in M$ *there exists a* $(\lhd, \varphi)$*-maximal* $v \in M$ *with*

$$\textit{either } u = v \ (\textit{hence } u \textit{ is } (\lhd, \varphi)\textit{-maximal}) \textit{ or } u \lhd v \tag{2.7}$$

$$u \lhd v \quad \textit{when, in addition, } u \lhd u. \tag{2.8}$$

Proof. Let $(\le)$ stand for the quasi-order (2.1). We claim that M is sequentially inductive modulo $(\le)$. In fact, let (x_n) be an ascending (modulo $(\le)$) sequence in M

$$x_n \le x_{n+1}, \forall n \ \ (\text{hence } x_n \le x_m \text{ whenever } n \le m).$$

If this sequence is stationary beyond a certain rank

$$\exists k \text{ such that: } \forall n > k \text{ one has } x_n = x_k$$

we are done; because $(x_n) \le u (= x_k)$. Otherwise,

$$\forall p, \exists q > p \text{ such that } x_p \ne x_q \ \ (\text{hence } x_p \lhd x_q).$$

Consequently, a subsequence $(y_n = x_{r(n)})$ of (x_n) may be constructed with

$$(y_n) \text{ is ascending (modulo } (\lhd)); \quad \text{wherefrom}$$
$$(y_n) \lhd t, \text{ for some } t \in M \quad \text{(cf. the hypothesis).}$$

But then, t acts as an upper bound (modulo $(\le)$) of (x_n); hence the claim. In addition (cf. (2.2)), φ is $(\le)$-decreasing. By Theorem 1 it follows that, for the starting point $u \in M$ there exists $v \in M$ such that

$$u \le v \text{ and } v \text{ is } (\le, \varphi)\text{-maximal.}$$

The latter of these yields v is $(\triangleleft, \varphi)$-maximal, if we take (2.4) into account. And the former one gives (2.7)+(2.8), by the very definition of $(\leq)$. The proof is complete. ■

Clearly, the Brezis-Browder principle (i.e., Theorem 1) follows from Theorem 2. The reciprocal inclusion also holds, by the argument above. Hence

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 2 (from a logical viewpoint).} \tag{2.9}$$

Nevertheless, a direct use of Theorem 2 is more profitable; because the transitive relation $(\triangleleft)$ is "very similar" to its induced quasi-order $(\leq)$.

An interesting completion of Theorem 2 is to be given under the lines of Section 1. Precisely, after the model of (1.5), we may introduce the concept

$$w \in M, z \triangleleft w \Rightarrow z = w \ \ \text{(referred to as: } z \text{ is } (\triangleleft)\text{-maximal).} \tag{2.10}$$

This is a stronger version of the concept (2.3). To get a corresponding form of Theorem 2 with (2.10) in place of (2.3), we need that $(\triangleleft)$ be *φ-sufficient*:

$$x \triangleleft y, \ y \triangleleft z \text{ and } \varphi(x) = \varphi(y) = \varphi(z) \text{ imply } y = z \tag{2.11}$$

Precisely, we have

Theorem 3. *Let the conditions of Theorem 2 be in use and $(\triangleleft)$ be φ-sufficient. Then, for each $u \in M$ there exists a $(\triangleleft)$-maximal $w \in M$ with*

$$\textit{either } u = w \ (\textit{hence } u \textit{ is } (\triangleleft)\textit{-maximal}) \textit{ or } u \triangleleft w \tag{2.12}$$

$$u \triangleleft w \quad \textit{if, in addition, } u \triangleleft u. \tag{2.13}$$

Proof. By Theorem 2, we have promised some $(\triangleleft, \varphi)$-maximal $v \in M$ with the properties (2.8)+(2.9). We claim that there exists a $(\triangleleft)$-maximal $w \in M$ so that

$$\text{either } v = w \text{ (hence } v \text{ is } (\triangleleft)\text{-maximal) or } v \triangleleft w; \tag{2.14}$$

and this will complete the argument. The first alternative is clear; so, it remains to discuss the second one:

$$v \triangleleft w \text{ (hence } \varphi(v) = \varphi(w)\text{), for some } w \in M \setminus \{v\}. \tag{2.15}$$

We claim that, necessarily, w is $(\triangleleft)$-maximal. Assume not:

$$w \triangleleft y, \quad \text{for some } y \in M \setminus \{v\}. \tag{2.16}$$

Combining with (2.15) yields

$$v \lhd y; \quad \text{hence } \varphi(v) = \varphi(y) \text{ (by the choice of } v).$$

Summing up, $v \lhd w$, $w \lhd y$ and $\varphi(v) = \varphi(w) = \varphi(y)$; wherefrom (cf. (2.11))

$$w = y; \quad \text{in contradiction with the choice of } y.$$

Hence, (2.16) cannot hold; and our claim follows. ■

The obtained statement is nothing but a "transitive" form of the Zorn maximality principle (cf. Bourbaki [3]) for such structures. A basic particular case of it may be described under the lines below. Let (M, d) be a complete metric space; and $F : M \to R \cup \{\infty\}$, some function with

$$F \text{ is proper } (\mathrm{Dom}(F) \neq \emptyset), \text{ bounded below } (\inf[F(M)] > -\infty) \tag{2.17}$$

$$F \text{ is lsc on } M \ (F(x) \leq \liminf_n F(x_n), \text{ if } x_n \to x). \tag{2.18}$$

Denote for simplicity

$$M_u = \{x \in M; F(x) \leq F(u)\} \ \text{ where } u \in \mathrm{Dom}(F) \text{ is arbitrary fixed;}$$

and let the function $\varphi = \varphi(u, F)$ from M_u to R_+ be given as

$$\varphi(x) = F(x) - F_*, \ x \in M_u, \quad \text{where } F_* = \inf[F(M)]. \tag{2.19}$$

The quasi-order $(\leq)$ over M_u introduced under

$$(x, y \in M_u) \colon \ x \leq y \text{ iff } d(x, y) \leq \varphi(x) - \varphi(y)$$

fulfills (1.2)+(1.4); and then, an application of Theorem 3 to these data yields the well known Ekeland's variational principle [7]. For a non-metrical version of it, one may proceed as follows. By an *almost pseudometric* over M we shall mean any map $e : M \times M \to R_+$. We shall say that this object is a KST-*distance* (modulo d) over M provided

$$e \text{ is triangular } (e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in M) \tag{2.20}$$

$$y \vdash e(x, y) \text{ is lsc over } M \text{ (see above), for each } x \in M \tag{2.21}$$

$$\text{each } e\text{-Cauchy sequence is a } d\text{-Cauchy sequence too} \tag{2.22}$$

$$e \text{ is transitively sufficient } (e(x, y) = e(x, z) = 0 \text{ imply } y = z). \tag{2.23}$$

The following variational result is then available.

Theorem 4. *Let the function $F : M \to R \cup \{\infty\}$ be taken as in (2.17)+(2.18); and $e : M \times M \to R_+$ be some KST-distance (modulo d). Then, for each $u \in \mathrm{Dom}(F)$ there exists $w \in \mathrm{Dom}(F)$ with*

$$\begin{array}{l} e(w,x) > F(w) - F(x), \text{ for each } x \in M \setminus \{w\} \\ (\text{referred to as: } w \text{ is } (e,F)\text{-variational}) \end{array} \tag{2.24}$$

in such a way that

$$\text{either } u = w \ (u \text{ is } (e,F)\text{-variational}) \text{ or } e(u,w) \le F(u) - F(w) \tag{2.25}$$

$$e(u,w) \le F(u) - F(w) \quad \text{if, in addition, } e(u,u) = 0. \tag{2.26}$$

Proof. Let (M_u, φ) be introduced as before. Clearly, $M_u \neq \emptyset$ (since it contains u); moreover, by (2.18), M_u is closed (hence complete) in M. Let $(\triangleleft)$ stand for the transitive relation (over M_u)

$$(x, y \in M_u)\colon\ x \triangleleft y \text{ iff } e(x,y) \le \varphi(x) - \varphi(y). \tag{2.27}$$

We claim that conditions of Theorem 3 hold for the pair $(\triangleleft, \varphi)$ over M_u. Clearly, φ is $(\triangleleft)$-decreasing on M_u; so, it remains to establish that M_u is sequentially inductive modulo $(\triangleleft)$. Let (x_n) be an ascending (modulo $(\triangleleft)$) sequence in M_u:

$$e(x_n, x_m) \le \varphi(x_n) - \varphi(x_m) (= F(x_n) - F(x_m)), \text{ if } n < m. \tag{2.28}$$

The sequence $(\varphi(x_n))$ is descending in R_+; hence a Cauchy sequence; and $\lim_n \varphi(x_n) = \inf_n \varphi(x_n)$ exists. This, along with (2.28), tells us that (x_n) is an e-Cauchy sequence; wherefrom (by (2.22)), a d-Cauchy one. By completeness and (2.18), there must be some $y \in M$ with

$$x_n \to y \ \text{ and } \lim_n F(x_n) \ge F(y) \text{ (hence } \lim_n \varphi(x_n) \ge \varphi(y)). \tag{2.29}$$

This firstly gives $F(y) \le F(u)$ (hence $y \in M_u$); because $(x_n) \subseteq M_u$. Secondly, fix a certain rank n. By (2.28)+(2.29)

$$e(x_n, x_m) \le \varphi(x_n) - \varphi(y), \quad \text{for all } \ m > n.$$

Passing to limit upon m yields (via (2.21))

$$e(x_n, y) \le \varphi(x_n) - \varphi(y) \quad \text{(i.e., } x_n \triangleleft y).$$

As n was arbitrarily, it results that y is an upper bound (modulo $(\triangleleft)$) of (x_n); hence the claim. By Theorem 3, we get a $(\triangleleft)$-maximal $w \in M_u$ so that

(2.12)+(2.13) be retainable. This is our desired element for the conclusions in the statement. The proof is thereby complete. ■

Let again $e : M \times M \to R_+$ stand for an almost pseudometric over M; we term it a *w-distance* provided (2.20)+(2.21) hold, as well as

$$\begin{aligned}&e \text{ is strongly transitively } d\text{-sufficient}\\&(\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } e(x,y), e(x,z) \leq \delta \Rightarrow d(y,z) \leq \varepsilon).\end{aligned} \tag{2.30}$$

Clearly, this last condition gives (2.22)+(2.23); hence this object is a KST-distance. As a consequence, Theorem 4 is applicable to w-distances. This tells us that Theorem 4 includes the variational principle in Kada, Suzuki and Takahashi [10]. In addition, the proposed argument shows that the recursion to the nonconvex minimization principle in Takahashi [14] is not necessary. On the other hand, our statement includes the variational principle (involving *τ-distances*) due to Suzuki [12]; we do not give details. Finally, when $e : M \times M \to R_+$ is a *pseudometric* ($e(x,x) = 0, \forall x \in M$) Theorem 4 is comparable with the variational statement in Tataru [15]; see also Turinici [16]. In fact, an almost pseudometric version of the quoted statement is available so as to cover all these; we shall develop such facts elsewhere.

3. General case

The next objective of the program sketched in Section 1 is to give appropriate versions of Theorem 1 when the ambient relation has no regularity properties at all. As we shall see, the natural approach consists in reducing this case to the transitive one.

Let M be a nonempty set; and $(\perp)$, some general(=amorph) relation over it. Denote by $(\lhd)$ the transitive relation on M attached to $(\perp)$

$$\begin{aligned}&x \lhd y \text{ iff } x = u_1 \perp ... \perp u_k = y \text{ (i.e.: } u_i \perp u_{i+1}, \forall i \in \{1, ..., k-1\})\\&\text{for some } k \geq 2 \text{ and certain points } u_1, ..., u_k \in M.\end{aligned} \tag{3.1}$$

Further, take a function $\varphi : M \to R_+$. The $(\perp)$-decreasing property for it is that of (1.3) (with $(\perp)$ in place of $(\leq)$). Note that, by (3.1) above

$$\varphi \text{ is } (\perp)\text{-decreasing} \Longrightarrow \varphi \text{ is } (\lhd)\text{-decreasing}. \tag{3.2}$$

Call the point $z \in M$, $(\perp, \varphi)$-maximal, in case

$$w \in M \text{ and } z \perp w \text{ imply } \varphi(z) = \varphi(w). \tag{3.3}$$

The following simple fact is evident

$$z \text{ is } (\triangleleft, \varphi)\text{-maximal} \Longrightarrow z \text{ is } (\perp, \varphi)\text{-maximal}; \tag{3.4}$$

the converse relation is not in general valid. This, along with (3.2), shows that maximality results involving the general relation $(\perp)$ are deductible from the ones in Section 2 concerning its associated transitive relation $(\triangleleft)$. As before, it would be desirable to have expressed these "transitive" conditions in terms of our initial "amorph" relation. Call the sequence (x_n), *ascending* (modulo $(\perp)$) when

$$x_n \perp x_{n+1}, \quad \text{for all ranks } n. \tag{3.5}$$

Note the generic (sequential) relation

$$\text{ascending (modulo } (\perp)) \Longrightarrow \text{ascending (modulo } (\triangleleft));$$

the reciprocal is not in general true. Further, given the sequence (x_n) in M, let us say that $u \in M$ is an *asymptotic upper bound* of it (written as: $(x_n) \perp\perp u$) provided

$$\begin{aligned} &\forall n, \exists m \geq n \text{ such that } x_m \perp u; \text{ or, equivalently:} \\ &\text{there exists a subsequence } (y_n = x_{p(n)}) \text{ of } (x_n) \text{ with} \\ &y_n \perp u, \forall n \text{ (written as: } (y_n) \perp u). \end{aligned} \tag{3.6}$$

When u is generic, we say that (x_n) is *asymptotic bounded above* (modulo $(\perp)$). The relation below is clear, for ascending (modulo $(\perp)$) sequences

$$\begin{aligned} &(\forall u)[(x_n) \perp\perp u \text{ implies } (x_n) \triangleleft u]; \quad \text{wherefrom} \\ &\text{asymptotic bd. above (modulo } (\perp)) \Rightarrow \text{bd. above (modulo } (\triangleleft)). \end{aligned} \tag{3.7}$$

(The converse implication is false, in general). Finally, call the ambient set M, *sequentially inductive* (modulo $(\perp)$) when

$$\begin{aligned} &\text{each ascending (modulo } (\perp)) \text{ sequence in } M \text{ is} \\ &\text{asymptotic bounded above (modulo } (\perp)). \end{aligned} \tag{3.8}$$

We are now in position to give an appropriate answer to the posed question.

Theorem 5. *Let the amorph structure $(M, \perp)$ and the function $\varphi : M \to R_+$ be such that M is sequentially inductive (modulo $(\perp)$) and φ is $(\perp)$-decreasing. Then, for each $u \in M$ there exists a $(\perp, \varphi)$-maximal $v \in M$ in such a way that*

$$\text{either } u = v \text{ (hence } u \text{ is } (\perp, \varphi)\text{-maximal) or } u \triangleleft v. \tag{3.9}$$

$$u \lhd v \quad \textit{when, in addition, } u \perp u. \tag{3.10}$$

Proof. By (3.2)+(3.7), it is clear that Theorem 2 applies to our data, where ($\lhd$) is that of (3.1). This, along with (3.4), ends the reasoning. ■

Clearly, Theorem 5 includes Theorem 2, to which it reduces when ($\perp$) is transitive. The reciprocal inclusion is also true, by the argument above; hence

$$\text{Theorem 5} \Longleftrightarrow \text{Theorem 2 (from a logical viewpoint).} \tag{3.11}$$

This, along with (2.9), shows that Theorem 5 is also logical equivalent with the Brezis-Browder principle (subsumed to Theorem 1).

The following variant of this result is to be noted. Let again M be a momempty set; and ($\top$), some general (=amorph) relation over it. Denote by (∇) the associated transitive relation

$$x \nabla y \text{ iff } x = v_1 \top ... \top v_k = y, \text{ for some } k \geq 2 \text{ and } v_1, ..., v_k \in M. \tag{3.12}$$

Further, let $\varphi : M \to R_+$ be a function. Call the point $z \in M$, $(\top, \varphi)$-*Maximal* if

$$w \in M,\ z \top w \quad \text{imply} \quad \varphi(z) \leq \varphi(w). \tag{3.13}$$

An interesting statement involving such points is the one due to Gajek and Zagrodny [8]:

Theorem 6. *Suppose that*

$$\begin{array}{l} \textit{for each sequence } (x_n) \subseteq M \textit{ with } [x_n \top x_{n+1}, \varphi(x_n) \geq \varphi(x_{n+1}), \forall n] \\ \textit{there exist a subsequence } (y_n = x_{p(n)}) \textit{ of it and a point } z \in M \\ \textit{in such a way that } [y_n \top z, \varphi(y_n) \geq \varphi(z), \forall n]. \end{array} \tag{3.14}$$

Then, for each $u \in M$ there exists a $(\top, \varphi)$-Maximal $v \in M$ with

$$\textit{either } u = v \ (\textit{hence } u \textit{ is } (\top, \varphi)\textit{-Maximal}) \textit{ or } \ u \nabla v, \varphi(u) \geq \varphi(v) \tag{3.15}$$

$$u \nabla v, \varphi(u) \geq \varphi(v) \quad \textit{when, in addition, } u \top u. \tag{3.16}$$

[As a matter of fact, the original result is with Codom(φ) $\subseteq R$. This, however, is not a restriction if we remember the arguments in Section 1].

For the moment, it is clear that Theorem 5 is reductible to this principle. In fact, let the amorph structure $(M, \top)$ and the function $\varphi : M \to R_+$ be such that M is sequentially inductive modulo ($\top$) and φ is ($\top$)-decreasing. Then, evidently, (3.14) holds; i.e., Theorem 6 is applicable to our data. This, added to the generic relation

$$(\top, \varphi)\text{-Maximal} \Longleftrightarrow (\top, \varphi)\text{-maximal when } \varphi \text{ is } (\top)\text{-decreasing} \tag{3.17}$$

gives, via (3.15)+(3.16), the conclusion we need. The reverse is also true:

$$\text{Theorem 5} \Longrightarrow \text{Theorem 6 (from a logical viewpoint).} \tag{3.18}$$

This will follow from the

Proof of Theorem 6. Let $(\perp)$ stand for the amorph relation over M

$$x \perp y \text{ iff } x \top y \text{ and } \varphi(x) \geq \varphi(y). \tag{3.19}$$

Denote also by $(\triangleleft)$ the transitive relation over M attached to $(\perp)$, under the model of (3.1). By (3.14), M is sequentially inductive modulo $(\perp)$; and, by (3.19), φ is $(\perp)$-decreasing. Summing up, Theorem 5 applies to the precised data. And this, in conjunction with the generic relation

$$(\perp, \varphi)\text{-maximal} \Longleftrightarrow (\top, \varphi)\text{-Maximal} \tag{3.20}$$

gives the desired conclusion. ■

As a consequence, such statements are logical equivalents of the Brezis-Browder principle. Some "abstract" counterparts of these may be found in Sonntag and Zalinescu [11]; see also Hazen and Morrin [9].

4. Reflexive relations

A basic particular case of these developments corresponds to the underlying relation being in addition *reflexive*. As we shall see, the motivation of treating it separately is practical in nature.

Let M be a nonempty set; and $(\perp)$, some *reflexive* relation $(x \perp x, \forall x \in M)$ over it. Let $(\leq)$ stand for the transitive relation associated to $(\perp)$ under (3.1); note that, by the admitted hypothesis, $(\leq)$ is reflexive too; hence a quasi-order. Furher, let $\varphi : M \to R_+$ be a function. The remaining concepts and auxiliary facts are the ones in Section 3. As a direct consequence, the following version of Theorem 5 is available.

Theorem 7. *Let the reflexive relation* $(\perp)$ *and the function* φ *be such that* M *is sequentially inductive* (*modulo* $(\perp)$) *and* φ *is* $(\perp)$*-decreasing. Then, for* *each* $u \in M$ *there exists a* $(\perp, \varphi)$*-maximal* $v \in M$ *with*

$$u \leq v \ [u = x_1 \perp ... \perp x_k = v, \text{ for some } k \geq 2 \text{ and } x_1, ..., x_k \in M]. \tag{4.1}$$

For the moment, the Brezis-Browder principle follows from Theorem 7; because the sequential inductivity (modulo $(\perp)$) becomes the one of (1.2) when $(\perp)$ is, in addition, transitive (hence a quasi-order). On the other hand, the reciprocal inclusion is also true, by the developments in Section 3. Hence

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 7 (from a logical viewpoint).} \tag{4.2}$$

In particular, suppose that the $(\perp)$-decreasing property of φ is to be substituted by its stronger counterpart

$$\varphi \text{ is strongly } (\perp)\text{-decreasing } (x \perp y,\ x \neq y \text{ imply } \varphi(x) > \varphi(y)). \tag{4.3}$$

Then, the point $v \in M$ assured by Theorem 7 fulfils the (stronger than $(\perp, \varphi)$-maximal) property

$$z \in M, v \perp z \Rightarrow v = z \text{ (referred to as: } v \text{ is } (\perp)\text{-maximal).} \tag{4.4}$$

Hence, this variant of Theorem 7 incorporates the basic ordering principle in Bae, Cho and Yeom [2] obtained via similar methods.

By the above developments it results that, in all maximality principles based on Theorem 7, an alternate use of Theorem 1 is always possible. In fact, this is the most profitable approach; because the ambient (reflexive) relation $(\perp)$ is "very distinct" from its induced quasi-order. The following example will illustrate our claim. But, prior to this, we need some preliminaries.

Let $c : R_+ \to R_+$ be some function; we call it *right locally bounded above* at $r \in R_+$ if

$$\text{there exists } \delta > 0 \text{ such that } \sup c([r, r + \delta]) < \infty. \tag{4.5}$$

If r is generic in this convention, then $t \vdash c(t)$ will be referred to as *right locally bounded above* on R_+. A basic situation when this property holds may be described as below. Call the function $c : R_+ \to R_+$, *right usc* at $r \in R_+$ provided

$$\limsup_n c(t_n) \leq c(r), \text{ whenever } t_n \to r+. \tag{4.6}$$

(Here, $t_n \to r+$ means: $t_n \to r$ and $t_n > r, \forall n$). If r is generic in this convention, then $t \vdash c(t)$ will be termed *right usc* on R_+. We now have

$$(\forall r)\text{: } c \text{ is right usc at } r \Longrightarrow c \text{ is right locally bounded above at } r;$$
$$c \text{ is right usc on } R_+ \Longrightarrow c \text{ is right locally bounded above on } R_+.$$

(The reciprocals are not in general true).

Let in the following (M, d) stand for a complete metric space. Take some lsc function $\varphi : M \to R_+$ (in the sense of (2.18)); and let the function $c : R_+ \to R_+$ be right locally bounded above on R_+. Finally, take a function $H : R_+^2 \to R_+$ with the property of being *locally bounded*; i.e.:

$$\text{the image of each bounded part in } R_+^2 \text{ is bounded (in } R). \tag{4.7}$$

The following variational principle involving our data may be formulated:

Theorem 8. *Let $u \in M$ be arbitrary fixed. There exists then $v = v(u) \in M$ such that $\varphi(u) \geq \varphi(v)$ and*

$$d(v, x) > H(c(\varphi(v)), c(\varphi(x)))[\varphi(v) - \varphi(x)], \forall x \in M \setminus \{v\}. \tag{4.8}$$

As already precised, a "direct" approach for getting this result is possible (via Theorem 7) by starting from the reflexive (over M) relation

$$(x, y \in M)\text{: } x \perp y \text{ iff } d(x, y) \leq H(c(\varphi(x)), c(\varphi(y)))[\varphi(x) - \varphi(y)]. \tag{4.9}$$

However, such developments are technically complicated. So, we shall use an "indirect" approach (by a reduction to Theorem 1).

Proof of Theorem 8. Denote for simplicity

$$M[u] = \{x \in M; \varphi(x) \leq \varphi(u)\}; \quad r = \inf[\varphi(M)](= \inf[\varphi(M[u])]).$$

Clearly, $M[u]$ is nonempty (since it contains u); moreover, as φ is lsc on M, it results that $M[u]$ is closed (hence complete) in M. If $r = \varphi(u)$, we are done (with $v = u$); so, without loss, one may assume $r < \varphi(u)$. Since $c : R_+ \to R_+$ is right locally bounded above at r, there must be some δ in $]0, \varphi(u) - r[$ such that $\mu := \sup\{c(t); r \leq t \leq r + \delta\} < \infty$. Given this μ there exists, by the local boundedness of $(t, s) \vdash H(t, s)$, some $\nu > 0$ such that

$$H(\tau, \sigma) \leq \nu, \quad \text{whenever } 0 \leq \tau, \sigma \leq \mu.$$

Finally, take some $u^* \in M$ with the property

$$r \leq \varphi(u^*) < r + \delta < \varphi(u) \quad (\text{hence } u^* \in M[u]);$$

this is evidently possible, by the definition of r. Put $N = M[u^*]$; and let $(\leq)$ stand for the quasi-order (on M)

$$(x, y \in M)\colon\ x \leq y \text{ iff } d(x, y) \leq \nu[\varphi(x) - \varphi(y)].$$

It is not hard to see that (1.2)+(1.3) holds for the pair $(\leq, \varphi)$ over N. So, for the starting point $u^* \in N$ there exists $v \in N$ such that

$$u^* \leq v \text{ and } v \text{ is } (\leq, \varphi)\text{-maximal (relative to } N).$$

The former of these gives $\varphi(u^*) \geq \varphi(v)$ (by the very definition of $(\leq)$); hence $\varphi(u) \geq \varphi(v)$. And the latter one yields

$$x \in N, d(v, x) \leq \nu[\varphi(v) - \varphi(x)] \Rightarrow \varphi(v) = \varphi(x) \text{ (hence } v = x).$$

But, from this, (4.8) follows at once (by contradiction) if we note that

$$x, y \in M \text{ and } x \perp y \text{ imply } \varphi(x) \geq \varphi(y).$$

The proof is thereby complete. ∎

In particular, when $c = 1$, Theorem 8 becomes the variational Ekeland's principle [7] (in short: EVP). Moreover, the argument above shows that the converse inclusion holds too; hence Theorem 8 is logically equivalent with EVP. This may have a theoretical impact upon it; but, from a practical perspective, the situation may be reversed.

Now, by the well known Bourbaki methodological scheme [3], Theorem 8 may be written as a fixed point statement. In fact, let the general conditions above be in use.

Theorem 9. *Let the selfmap T of X be such that*

$$d(x, Tx) \leq H(c(\varphi(x)), c(\varphi(Tx)))[\varphi(x) - \varphi(Tx)], \quad \forall x \in M; \tag{4.10}$$

and let $u \in M$ be arbitrary fixed. There exists then a fixed point (relative to T) $v = v(u) \in M$ such that conclusions of Theorem 8 be retainable.

In particular, a sufficient condition for the right locally bounded above property of $c : R_+ \to R_+$ is (by a preceding remark) the right usc property of the same. This shows that Theorem 9 incorporates the statement in Suzuki [13]; as well as (under $H(t, s) = \max\{t, s\}$) the one due to Bae, Cho and Yeom [op. cit.]. But, these results extend in a direct way the Caristi-Kirk fixed point theorem [5] (in short: CK-FPT); hence, so does Theorem 9. On the other hand, the involved argument shows that the converse inclusion also holds; wherefrom Theorem 9 is logically equivalent with CK-FPT. Although non-effective (from a theoretical viewpoint) these extensions have useful applications in practice. For example, Theorems 8 and 9 are handy tools in deriving fixed point theorems for weakly contractive multivalued maps; and, moreover, the obtained statements extend a related one in Bae [1]. We shall discuss these in a separate paper.

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