Fixed Point Theory, Volume 7, No. 1, 2006, 103-110
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

SELECTIONS AND COMMON FIXED POINTS FOR SOME MULTIVALUED MAPPINGS

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Abstract. We prove that a multivalued operator which satisfies a contraction type condition of Latif-Beg type has a selection which is a Caristi type operator. Another purpose of this paper is to give a common fixed point theorem for two multivalued mappings defined on a closed ball of a complete metric space with values in the set of all nonempty and closed subsets of this space, mappings which satisfy a contraction type condition of Latif-Beg type.
Key Words and Phrases: Multivalued mapping, selection, fixed point, common fixed point.
2000 Mathematics Subject Classification: 47H04, 47H10, 54H25.

1. Introduction

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of X, i. e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$. Let $T : X \to P(X)$ be a multivalued operator. We denote by F_T the fixed points set of T, i. e. $F_T := \{ x \in X \mid x \in T(x) \}$.
An operator $t : X \to X$, with the property that $t(x) \in T(x)$, for each $x \in X$, is called a *selection* of T.

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. Further on we shall use the notations $\overline{B}(x_0, r) := \{ x \in X \mid d(x_0, x) \leq r \}$ and $P_{cl}(X) := \{ Y \in P(X) \mid Y \text{ is a closed set} \}$. We also need the functional $D : P(X) \times P(X) \to \mathbb{R}_+$, defined by $D(A, B) = \inf \{ d(a, b) \mid a \in A,\ b \in B \}$, for each $A, B \in P(X)$.

J. R. Jachymski established in [6] that a multivalued contraction admits a selection, which is a Caristi type operator. A. Petruşel and A. Sîntămărian proved in [11] and [12] two selection theorems for multivalued operators which satisfy Reich type conditions. Two selection theorems for multivalued operators which satisfy more general conditions than those given in [6], [11] and [12] are proved in [16].
In Section 2 of this paper we give a selection theorem for a multivalued operator which satisfies a contraction type condition of Latif-Beg type.

Assuming that (X, d) is complete, M. Frigon and A. Granas proved in [5] a fixed point theorem for a multivalued contraction $T : \overline{B}(x_0, r) \to P_{cl}(X)$, which does not displace the center of the ball too far. A. Petruşel established in [9] fixed point theorems for multivalued non-self mappings which satisfy Reich type conditions. A fixed point theorem for a multivalued mapping $T : \overline{B}(x_0, r) \to P_{cl}(X)$, which satisfies a more general contraction type condition, was proved by R. P. Agarwal and D. O'Regan in [1]. A common fixed point theorem for two multivalued mappings $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$, which satisfy a contraction type condition and at least one of them does not displace the center of the ball too far, is proved in [16]. The corresponding fixed point theorem is also presented in [16]. We remark that fixed point and common fixed point theorems for singlevalued and multivalued non-self mappings on other spaces (Banach spaces or complete and convex metric spaces) are presented in Lj. B. Ćirić's monograph [3].
In Section 3 of the paper we give a common fixed point theorem for two multivalued mappings $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$, which satisfy a contraction type condition of Latif-Beg type and at least one of them does not displace the center of the ball too far. The corresponding fixed point theorem is presented in the same section.

2. A selection theorem

Theorem 2.1. *Let (X, d) be a metric space and $T : X \to P_{cl}(X)$ a multivalued operator with the property that there exist $a_1, \dots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that*

$$d(u_x, u_y) \leq a_1\, d(x, y) + a_2\, d(x, u_x) + a_3\, d(y, u_y) + a_4\, d(x, u_y) + a_5\, d(y, u_x).$$

Then there exist $t : X \to X$ a selection of T and a functional $\varphi : X \to \mathbb{R}_+$ so that

$$d(x, t(x)) \le \varphi(x) - \varphi(t(x)),$$

for each $x \in X$.

Proof. Let $\varepsilon > 0$ be such that $a_1 + a_2 + a_4 < \varepsilon < 1 - (a_3 + a_4)$. We denote $U_x = \{\ y \in T(x) \mid \varepsilon\ d(x, y) \le [1 - (a_3 + a_4)]\, D(x, T(x))\ \}$, for each $x \in X$. Obviously, for each $x \in X$, the set U_x is nonempty (otherwise, if $x \in X \backslash F_T$ and we suppose that for each $y \in T(x)$ we have $\varepsilon\, d(x, y) > [1-(a_3+a_4)]\, D(x, T(x))$, then we reach the contradiction $\varepsilon\ D(x, T(x)) \ge [1 - (a_3 + a_4)]\, D(x, T(x))$; if $x \in F_T$, then clearly $x \in U_x$).

So, we can define the singlevalued operator $t : X \to X$ such that $t(x) \in U_x$, for each $x \in X$, i. e. $t(x) \in T(x)$ and $\varepsilon\, d(x, t(x)) \le [1 - (a_3 + a_4)]\, D(x, T(x))$, for each $x \in X$.

For $x \in X$, taking into account that $t(x) \in T(x)$ and the metric condition from the hypothesis of the theorem, we have that there exists $u_{t(x)} \in T(t(x))$ such that

$$\begin{gathered} d(t(x), u_{t(x)}) \le a_1\, d(x, t(x)) + a_2\, d(x, t(x)) + a_3\, d(t(x), u_{t(x)}) + \\ + a_4\, d(x, u_{t(x)}) + a_5\, d(t(x), t(x)) = \\ = (a_1 + a_2)\, d(x, t(x)) + a_3\, d(t(x), u_{t(x)}) + a_4\, d(x, u_{t(x)}) \le \\ \le (a_1 + a_2 + a_4)\, d(x, t(x)) + (a_3 + a_4)\, d(t(x), u_{t(x)}) \end{gathered}$$

and hence

$$\begin{gathered} [1 - (a_3 + a_4)]\, D(t(x), T(t(x))) \le [1 - (a_3 + a_4)]\, d(t(x), u_{t(x)}) \\ \le (a_1 + a_2 + a_4)\, d(x, t(x)). \end{gathered}$$

Now we are able to write that

$$\begin{gathered} d(x, t(x)) = [\varepsilon - (a_1 + a_2 + a_4)]^{-1}\, [\varepsilon\, d(x, t(x)) - (a_1 + a_2 + a_4)\ d(x, t(x))] \\ \le [\varepsilon - (a_1 + a_2 + a_4)]^{-1}\, \{[1 - (a_3 + a_4)]\, D(x, T(x)) - [1 - (a_3 + a_4)]\, D(t(x), T(t(x)))\} \\ = [1 - (a_3 + a_4)]/[\varepsilon - (a_1 + a_2 + a_4)]\, [D(x, T(x)) - D(t(x), T(t(x)))], \end{gathered}$$

for each $x \in X$.

We define $\varphi : X \to \mathbb{R}_+$ by

$$\varphi(x) := [1 - (a_3 + a_4)]/[\varepsilon - (a_1 + a_2 + a_4)]\, D(x, T(x)),$$

for each $x \in X$, and we get

$$d(x, t(x)) \leq \varphi(x) - \varphi(t(x)),$$

for each $x \in X$. $\square$

Remark 2.1. *If the multivalued operator $T : X \to P_{cl}(X)$ from Theorem 2.1 is upper semicontinuous, then the functional $\varphi : X \to \mathbb{R}_+$ is lower semicontinuous.*

3. A common fixed point theorem for two multivalued mappings defined on closed balls

Theorem 3.1. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T_1, T_2 : \overline{B}(x_0, r) \to P_{cl}(X)$ two multivalued mappings. We suppose that:*

(i$_1$) *there exist $a_{11}, \ldots, a_{15} \in \mathbb{R}_+$, with $a_{11} + a_{12} + a_{13} + 2a_{14} < 1$ such that for each $x \in \overline{B}(x_0, r)$, any $u_x \in T_1(x)$ and for all $y \in \overline{B}(x_0, r)$, there exists $u_y \in T_2(y)$ so that*

$$d(u_x, u_y) \leq a_{11}\ d(x, y) + a_{12}\ d(x, u_x) + a_{13}\ d(y, u_y) + a_{14}\ d(x, u_y) + a_{15}\ d(y, u_x);$$

(i$_2$) *there exist $a_{21}, \ldots, a_{25} \in \mathbb{R}_+$, with $a_{21} + a_{22} + a_{23} + 2a_{24} < 1$ such that for each $x \in \overline{B}(x_0, r)$, any $u_x \in T_2(x)$ and for all $y \in \overline{B}(x_0, r)$, there exists $u_y \in T_1(y)$ so that*

$$d(u_x, u_y) \leq a_{21}\ d(x, y) + a_{22}\ d(x, u_x) + a_{23}\ d(y, u_y) + a_{24}\ d(x, u_y) + a_{25}\ d(y, u_x);$$

(ii) *there exists $y_0 \in T_1(x_0) \cup T_2(x_0)$ such that*

$$d(x_0, y_0) \leq \left(1 - \max\left\{\frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})}, \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})}\right\}\right) r.$$

Then $F_{T_1} = F_{T_2} \in P_{cl}(X)$.

Proof. By an easy calculation we get that $F_{T_1} = F_{T_2}$.

We put $l := \max\left\{\frac{a_{11}+a_{12}+a_{14}}{1-(a_{13}+a_{14})}, \frac{a_{21}+a_{22}+a_{24}}{1-(a_{23}+a_{24})}\right\} < 1$ and we suppose, for example, that there exists $x_1 = y_0 \in T_1(x_0)$ such that $d(x_0, x_1) \leq (1 - l)r$.

It is clear that $x_1 \in \overline{B}(x_0, r)$.

Taking into account the condition (i_1) we have that there exists $x_2 \in T_2(x_1)$ such that

$$d(x_1, x_2) \leq a_{11}\ d(x_0, x_1) + a_{12}\ d(x_0, x_1) + a_{13}\ d(x_1, x_2) + a_{14}\ d(x_0, x_2) \leq$$

$$\leq (a_{11}+a_{12}+a_{14})\ d(x_0,x_1)+(a_{13}+a_{14})\ d(x_1,x_2).$$

From this we get that

$$d(x_1,x_2)\leq \frac{a_{11}+a_{12}+a_{14}}{1-(a_{13}+a_{14})}\ d(x_0,x_1)\leq l\ d(x_0,x_1)\leq l(1-l)r.$$

Using the triangle inequality we obtain

$$d(x_0,x_2)\leq d(x_0,x_1)+d(x_1,x_2)\leq (1-l)r+l(1-l)r=(1-l^2)r\leq r,$$

hence $x_2\in \overline{B}(x_0,r)$.

Now, taking into account the condition (i_2), we have that there exists $x_3\in T_1(x_2)$ such that

$$d(x_2,x_3)\leq a_{21}\ d(x_1,x_2)+a_{22}\ d(x_1,x_2)+a_{23}\ d(x_2,x_3)+a_{24}\ d(x_1,x_3)\leq$$

$$\leq (a_{21}+a_{22}+a_{24})\ d(x_1,x_2)+(a_{23}+a_{24})\ d(x_2,x_3).$$

From this we get that

$$d(x_2,x_3)\leq \frac{a_{21}+a_{22}+a_{24}}{1-(a_{23}+a_{24})}\ d(x_1,x_2)\leq l\ d(x_1,x_2)\leq l^2(1-l)r.$$

Because

$$d(x_0,x_3)\leq d(x_0,x_2)+d(x_2,x_3)\leq (1+l)(1-l)r+l^2(1-l)r=(1-l^3)r\leq r,$$

we have that $x_3\in \overline{B}(x_0,r)$.

By induction, we obtain that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

$x_{2n-1}\in T_1(x_{2n-2})$, $x_{2n}\in T_2(x_{2n-1})$,
$d(x_{n-1},x_n)\leq l^{n-1}(1-l)r$,
$d(x_0,x_n)\leq (1-l^n)r$, which means that $x_n\in \overline{B}(x_0,r)$,
for each $n\in\mathbb{N}^*$.

The inequality $d(x_{n-1},x_n)\leq l^{n-1}(1-l)r$, which holds for each $n\in\mathbb{N}^*$, implies that $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence, because $l<1$ and (X,d) is a complete metric space. Let $x^*=\lim\limits_{n\to\infty} x_n$. Obviously $x^*\in \overline{B}(x_0,r)$.

We shall prove that x^* is a fixed point of T_1, for example. From $x_{2n}\in T_2(x_{2n-1})$ we have that there exists $u_n\in T_1(x^*)$ such that

$$d(x_{2n},u_n)\leq a_{21}\ d(x_{2n-1},x^*)+a_{22}\ d(x_{2n-1},x_{2n})+a_{23}\ d(x^*,u_n)+$$

$$+a_{24}\ d(x_{2n-1},u_n)+a_{25}\ d(x^*,x_{2n}),$$

for each $n\in\mathbb{N}^*$.

Using the triangle inequality we get

$$d(x^*, u_n) \le [1 - (a_{23} + a_{24})]^{-1}[(1 + a_{25})\ d(x^*, x_{2n}) +$$

$$+ (a_{21} + a_{24})\ d(x^*, x_{2n-1}) + a_{22}\ d(x_{2n-1}, x_{2n})],$$

for each $n \in \mathbb{N}^*$.

This implies that $d(x^*, u_n) \to 0$, as $n \to \infty$. Since $u_n \in T_1(x^*)$, for all $n \in \mathbb{N}^*$ and $T_1(x^*)$ is a closed set, it follows that $x^* \in T_1(x^*)$. So $x^* \in F_{T_1} = F_{T_2}$.

Let us prove now that $F_{T_1} = F_{T_2}$ is a closed set. For this purpose let $y_n \in F_{T_1} = F_{T_2}$, for each $n \in \mathbb{N}^*$, such that $y_n \to y^*$, as $n \to \infty$. Clearly $y^* \in \overline{B}(x_0, r)$. For example, from $y_n \in T_1(y_n)$ we have that there exists $v_n \in T_2(y^*)$ so that

$$d(y_n, v_n) \le a_{11}\ d(y_n, y^*) + a_{13}\ d(y^*, v_n) + a_{14}\ d(y_n, v_n) + a_{15}\ d(y^*, y_n),$$

for each $n \in \mathbb{N}^*$.

Using the triangle inequality we obtain

$$d(y^*, v_n) \le (1 + a_{11} + a_{14} + a_{15})/[1 - (a_{13} + a_{14})]\ d(y^*, y_n),$$

for all $n \in \mathbb{N}^*$.

This implies that $d(y^*, v_n) \to 0$, as $n \to \infty$. Since $v_n \in T_2(y^*)$, for each $n \in \mathbb{N}^*$ and $T_2(y^*)$ is a closed set, it follows that $y^* \in T_2(y^*)$. Therefore $F_{T_1} = F_{T_2}$ is a closed set. □

The following fixed point theorem for a multivalued mapping defined on a closed ball can be proved.

Theorem 3.2. *Let (X, d) be a complete metric space, $x_0 \in X$, $r > 0$ and $T : \overline{B}(x_0, r) \to P_{cl}(X)$ a multivalued mapping for which there exist $a_1, \ldots, a_5 \in \mathbb{R}_+$, with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that:*

(i) *for each $x \in \overline{B}(x_0, r)$, any $u_x \in T(x)$ and for all $y \in \overline{B}(x_0, r)$, there exists $u_y \in T(y)$ so that*

$$d(u_x, u_y) \le a_1\ d(x, y) + a_2\ d(x, u_x) + a_3\ d(y, u_y) + a_4\ d(x, u_y) + a_5\ d(y, u_x);$$

(ii) *there exists $y_0 \in T(x_0)$ such that $d(x_0, y_0) \le \left[1 - \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)}\right] r$.*

Then $F_T \in P_{cl}(X)$.

Proof. We put $l := \frac{a_1+a_2+a_4}{1-(a_3+a_4)} < 1$. Using a similar argument as in the proof of Theorem 3.1, we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ with the following properties:

$x_n \in T(x_{n-1})$,

$d(x_{n-1}, x_n) \leq l^{n-1}(1-l)r$,

$d(x_0, x_n) \leq (1-l^n)r$, which means that $x_n \in \overline{B}(x_0, r)$,

for each $n \in \mathbb{N}^*$.

The sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T. Also, it can be shown that F_T is a closed set. □

Remark 3.1. *If in Theorem* 3.2 *we take* $a_4 = a_5 = 0$*, then the fact that* $F_T \neq \emptyset$ *is a result mentioned in* [9]*, but there the condition* (ii) *is*

$$D(x_0, T(x_0)) < \left(1 - \frac{a_1+a_2}{1-a_3}\right) r.$$

References

[1] R. P. Agarwal, D. O'Regan, *Fixed point theory for acyclic maps between topological vector spaces having sufficiently many linear functionals, and generalized contractive maps with closed values between complete metric spaces*, R. P. Agarwal (ed.) et al., Set valued mappings with applications in nonlinear analysis, Taylor & Francis, London, Ser. Math. Anal. Appl., **4**(2002), 17-26.

[2] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Am. Math. Soc., **215**(1976), 241-251.

[3] Lj. B. Ćirić, *Some Recent Results in Metrical Fixed Point Theory*, C-Print, Belgrade, 2003.

[4] J. Dugundji, A. Granas, *Fixed Point Theory*, Polish Scientific Publishers, Warszawa, 1982.

[5] M. Frigon, A. Granas, *Résultats du type de Leray-Schauder pour des contractions multivoques*, Topol. Methods Nonlinear Anal., **4**(1994), 197-208.

[6] J. R. Jachymski, *Caristi's fixed point theorem and selections of set-valued contractions*, J. Math. Anal. Appl., **227**(1998), 55-67.

[7] A. Latif, I. Beg, *Geometric fixed points for single and multivalued mappings*, Demonstratio Math., **30**(1997), 791-800.

[8] A. Petruşel, *Operatorial Inclusions*, House of the Book of Science, Cluj-Napoca, 2002.

[9] A. Petruşel, *On Frigon-Granas type multifunctions*, Nonlinear Anal. Forum, **7**(2002), 113-121.

[10] A. Petruşel, *Multivalued weakly Picard operators and applications*, Sci. Math. Jpn., **59**(2004), 169-202.

[11] A. Petruşel, A. Sîntămărian, *On Caristi-type operators*, Proceedings of the "Tiberiu Popoviciu" Itinerant Seminar of Functional Equations, Approximation and Convexity, Cluj-Napoca, Romania, 22-26 May 2001, Editura Srima, Cluj-Napoca, 2001, 181-190.

[12] A. Petruşel, A. Sîntămărian, *Single-valued and multi-valued Caristi type operators*, Publ. Math. Debrecen, **60**(2002), 167-177.

[13] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[14] I. A. Rus, A. Petruşel, A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Anal., Theory Meth. Appl., **52**(2003), 1947-1959.

[15] A. Sîntămărian, *Common fixed point theorems for multivalued mappings*, Semin. Fixed Point Theory Cluj-Napoca, **1**(2000), 93-102.

[16] A. Sîntămărian, *Selections and common fixed points for some generalized multivalued contractions*, Demonstratio Math., **30**(2006), 609-617.

Received 14.02.2006; Revised 23.03.2006.