SELECTIONS AND COMMON FIXED POINTS
FOR SOME MULTIVALUED MAPPINGS

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Abstract. We prove that a multivalued operator which satisfies a contraction type condition of Latif-Beg type has a selection which is a Caristi type operator. Another purpose of this paper is to give a common fixed point theorem for two multivalued mappings defined on a closed ball of a complete metric space with values in the set of all nonempty and closed subsets of this space, mappings which satisfy a contraction type condition of Latif-Beg type.

Key Words and Phrases: Multivalued mapping, selection, fixed point, common fixed point.

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1. Introduction

Let $X$ be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of $X$, i.e. $P(X) := \{ Y \mid \emptyset \neq Y \subseteq X \}$. Let $T : X \to P(X)$ be a multivalued operator. We denote by $F_T$ the fixed points set of $T$, i.e. $F_T := \{ x \in X \mid x \in T(x) \}$. An operator $t : X \to X$, with the property that $t(x) \in T(x)$, for each $x \in X$, is called a selection of $T$.

Let $(X, d)$ be a metric space, $x_0 \in X$ and $r > 0$. Further on we shall use the notations $\overline{B}(x_0, r) := \{ x \in X \mid d(x_0, x) \leq r \}$ and $P_d(X) := \{ Y \in P(X) \mid Y \text{ is a closed set} \}$. We also need the functional $D : P(X) \times P(X) \to \mathbb{R}_+$, defined by $D(A, B) = \inf \{ d(a, b) \mid a \in A, b \in B \}$, for each $A, B \in P(X)$. 
J. R. Jachymski established in [6] that a multivalued contraction admits a selection, which is a Caristi type operator. A. Petruşel and A. Sîntămărian proved in [11] and [12] two selection theorems for multivalued operators which satisfy Reich type conditions. Two selection theorems for multivalued operators which satisfy more general conditions than those given in [6], [11] and [12] are proved in [16].

In Section 2 of this paper we give a selection theorem for a multivalued operator which satisfies a contraction type condition of Latif-Beg type.

Assuming that \((X, d)\) is complete, M. Frigon and A. Granas proved in [5] a fixed point theorem for a multivalued contraction \(T : \overline{B}(x_0, r) \to P_d(X)\), which does not displace the center of the ball too far. A. Petruşel established in [9] fixed point theorems for multivalued non-self mappings which satisfy Reich type conditions. A fixed point theorem for a multivalued mapping \(T : \overline{B}(x_0, r) \to P_d(X)\), which satisfies a more general contraction type condition, was proved by R. P. Agarwal and D. O’Regan in [1]. A common fixed point theorem for two multivalued mappings \(T_1, T_2 : \overline{B}(x_0, r) \to P_d(X)\), which satisfy a contraction type condition and at least one of them does not displace the center of the ball too far, is proved in [16]. The corresponding fixed point theorem is also presented in [16]. We remark that fixed point and common fixed point theorems for singlevalued and multivalued non-self mappings on other spaces (Banach spaces or complete and convex metric spaces) are presented in Lj. B. Ćirić’s monograph [3].

In Section 3 of the paper we give a common fixed point theorem for two multivalued mappings \(T_1, T_2 : \overline{B}(x_0, r) \to P_d(X)\), which satisfy a contraction type condition of Latif-Beg type and at least one of them does not displace the center of the ball too far. The corresponding fixed point theorem is presented in the same section.

2. A selection theorem

**Theorem 2.1.** Let \((X, d)\) be a metric space and \(T : X \to P_d(X)\) a multivalued operator with the property that there exist \(a_1, \ldots, a_5 \in \mathbb{R}_+\), with \(a_1 + a_2 + a_3 + 2a_4 < 1\) such that for each \(x \in X\), any \(u_x \in T(x)\) and for all \(y \in X\), there exists \(u_y \in T(y)\) so that

\[
d(u_x, u_y) \leq a_1 d(x, y) + a_2 d(x, u_x) + a_3 d(y, u_y) + a_4 d(x, u_y) + a_5 d(y, u_x).
\]
Then there exist \( t : X \to X \) a selection of \( T \) and a functional \( \varphi : X \to \mathbb{R}_+ \) so that
\[
d(x, t(x)) \leq \varphi(x) - \varphi(t(x)),
\]
for each \( x \in X \).

**Proof.** Let \( \varepsilon > 0 \) be such that \( a_1 + a_2 + a_4 < \varepsilon < 1 - (a_3 + a_4) \). We denote \( U_x = \{ y \in T(x) \mid \varepsilon \cdot d(x, y) \leq [1 - (a_3 + a_4)] D(x, T(x)) \} \), for each \( x \in X \).

Obviously, for each \( x \in X \), the set \( U_x \) is nonempty (otherwise, if \( x \in X \setminus F_T \) and we suppose that for each \( y \in T(x) \) we have \( \varepsilon \cdot d(x, y) > [1 - (a_3 + a_4)] D(x, T(x)) \), then we reach the contradiction \( \varepsilon \cdot D(x, T(x)) \geq [1 - (a_3 + a_4)] D(x, T(x)) \); if \( x \in F_T \), then clearly \( x \in U_x \).

So, we can define the single-valued operator \( t : X \to X \) such that \( t(x) \in U_x \), for each \( x \in X \), i.e. \( t(x) \in T(x) \) and \( \varepsilon \cdot d(x, t(x)) \leq [1 - (a_3 + a_4)] D(x, T(x)) \), for each \( x \in X \).

For \( x \in X \), taking into account that \( t(x) \in T(x) \) and the metric condition from the hypothesis of the theorem, we have that there exists \( u_{t(x)} \in T(t(x)) \) such that
\[
d(t(x), u_{t(x)}) \leq a_1 d(x, t(x)) + a_2 d(x, t(x)) + a_3 d(t(x), u_{t(x)}) + a_4 d(x, u_{t(x)}) + a_5 d(t(x), t(x)) = (a_1 + a_2) d(x, t(x)) + a_3 d(t(x), u_{t(x)}) + a_4 d(x, u_{t(x)}) \leq (a_1 + a_2 + a_4) d(x, t(x)) + (a_3 + a_4) d(t(x), u_{t(x)})
\]
and hence
\[
[1 - (a_3 + a_4)] D(t(x), T(t(x))) \leq [1 - (a_3 + a_4)] d(t(x), u_{t(x)}) \leq (a_1 + a_2 + a_4) d(x, t(x)).
\]

Now we are able to write that
\[
d(x, t(x)) = [\varepsilon - (a_1 + a_2 + a_4)]^{-1} [\varepsilon d(x, t(x)) - (a_1 + a_2 + a_4) d(x, t(x))] \leq [\varepsilon - (a_1 + a_2 + a_4)]^{-1} \{ [1 - (a_3 + a_4)] D(x, T(x)) - [1 - (a_3 + a_4)] D(t(x), T(t(x))) \} = [1 - (a_3 + a_4)]/[\varepsilon - (a_1 + a_2 + a_4)] \cdot [D(x, T(x)) - D(t(x), T(t(x)))],
\]
for each \( x \in X \).

We define \( \varphi : X \to \mathbb{R}_+ \) by
\[
\varphi(x) := [1 - (a_3 + a_4)]/[\varepsilon - (a_1 + a_2 + a_4)] D(x, T(x)),
\]
for each \( x \in X \), and we get
\[
d(x, t(x)) \leq \varphi(x) - \varphi(t(x)),
\]
for each \( x \in X \). \( \square \)

**Remark 2.1.** If the multivalued operator \( T : X \to P_d(X) \) from Theorem 2.1 is upper semicontinuous, then the functional \( \varphi : X \to \mathbb{R}_+ \) is lower semicontinuous.

3. A COMMON FIXED POINT THEOREM FOR TWO MULTIVALUED MAPPINGS DEFINED ON CLOSED BALLS

**Theorem 3.1.** Let \((X, d)\) be a complete metric space, \( x_0 \in X \), \( r > 0 \) and \( T_1, T_2 : \overline{B}(x_0, r) \to P_d(X) \) two multivalued mappings. We suppose that:

(i) there exist \( a_{11}, \ldots, a_{15} \in \mathbb{R}_+ \), with \( a_{11} + a_{12} + a_{13} + 2a_{14} < 1 \) such that for each \( x \in \overline{B}(x_0, r) \), any \( u_x \in T_1(x) \) and for all \( y \in \overline{B}(x_0, r) \), there exists \( u_y \in T_2(y) \) so that
\[
d(u_x, u_y) \leq a_{11} d(x, y) + a_{12} d(x, u_x) + a_{13} d(y, u_y) + a_{14} d(x, u_y) + a_{15} d(y, u_x);
\]

(ii) there exist \( a_{21}, \ldots, a_{25} \in \mathbb{R}_+ \), with \( a_{21} + a_{22} + a_{23} + 2a_{24} < 1 \) such that for each \( x \in \overline{B}(x_0, r) \), any \( u_x \in T_2(x) \) and for all \( y \in \overline{B}(x_0, r) \), there exists \( u_y \in T_1(y) \) so that
\[
d(u_x, u_y) \leq a_{21} d(x, y) + a_{22} d(x, u_x) + a_{23} d(y, u_y) + a_{24} d(x, u_y) + a_{25} d(y, u_x);
\]

(iii) there exists \( y_0 \in T_1(x_0) \cup T_2(x_0) \) such that
\[
d(x_0, y_0) \leq \left( 1 - \max \left\{ \frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})}, \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})} \right\} \right) r.
\]

Then \( F_{T_1} = F_{T_2} \in P_d(X) \).

**Proof.** By an easy calculation we get that \( F_{T_1} = F_{T_2} \).

We put \( l := \max \left\{ \frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})}, \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})} \right\} < 1 \) and we suppose, for example, that there exists \( x_1 = y_0 \in T_1(x_0) \) such that \( d(x_0, x_1) \leq (1 - l)r \). It is clear that \( x_1 \in \overline{B}(x_0, r) \).

Taking into account the condition (i) we have that there exists \( x_2 \in T_2(x_1) \) such that
\[
d(x_1, x_2) \leq a_{11} d(x_0, x_1) + a_{12} d(x_0, x_1) + a_{13} d(x_1, x_2) + a_{14} d(x_0, x_2) \leq \frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})} d(x_0, x_1) + d(x_1, x_2).
\]

Hence, we get
\[
d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1 - l)r + d(x_1, x_2).
\]

Therefore, \( F_{T_1} = F_{T_2} \in P_d(X) \).
For each \( x \), from this we get that 
\[
d(x_1, x_2) \leq \frac{a_{11} + a_{12} + a_{14}}{1 - (a_{13} + a_{14})} d(x_0, x_1) \leq l \ d(x_0, x_1) \leq (1 - l)r.
\]
Using the triangle inequality we obtain 
\[
d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) \leq (1 - l)r + l(1 - l)r = (1 - l^2)r \leq r,
\]
hence \( x_2 \in \overline{B}(x_0, r) \).

Now, taking into account the condition \((i_2)\), we have that there exists \( x_3 \in T_1(x_2) \) such that 
\[
d(x_2, x_3) \leq a_{21} \ d(x_1, x_2) + a_{22} \ d(x_1, x_2) + a_{23} \ d(x_2, x_3) + a_{24} \ d(x_1, x_3) \leq
\leq (a_{21} + a_{22} + a_{24}) \ d(x_1, x_2) + (a_{23} + a_{24}) \ d(x_2, x_3).
\]
From this we get that 
\[
d(x_2, x_3) \leq \frac{a_{21} + a_{22} + a_{24}}{1 - (a_{23} + a_{24})} d(x_1, x_2) \leq l \ d(x_1, x_2) \leq l^2(1 - l)r.
\]
Because 
\[
d(x_0, x_3) \leq d(x_0, x_2) + d(x_2, x_3) \leq (1 + l)(1 - l)r + l^2(1 - l)r = (1 - l^3)r \leq r,
\]
we have that \( x_3 \in \overline{B}(x_0, r) \).

By induction, we obtain that there exists a sequence \((x_n)_{n \in \mathbb{N}}\) with the following properties:
\[
x_{2n-1} \in T_1(x_{2n-2}), \ x_{2n} \in T_2(x_{2n-1}),
\]
\[
d(x_{n-1}, x_n) \leq l^{n-1}(1 - l)r,
\]
\[
d(x_0, x_n) \leq (1 - l^n)r, \text{ which means that } x_n \in \overline{B}(x_0, r),
\]
for each \( n \in \mathbb{N}^* \).

The inequality \( d(x_{n-1}, x_n) \leq l^{n-1}(1 - l)r \), which holds for each \( n \in \mathbb{N}^* \), implies that \((x_n)_{n \in \mathbb{N}}\) is a convergent sequence, because \( l < 1 \) and \((X, d)\) is a complete metric space. Let \( x^* = \lim_{n \to \infty} x_n \). Obviously \( x^* \in \overline{B}(x_0, r) \).

We shall prove that \( x^* \) is a fixed point of \( T_1 \), for example. From \( x_{2n} \in T_2(x_{2n-1}) \) we have that there exists \( u_n \in T_1(x^*) \) such that 
\[
d(x_{2n}, u_n) \leq a_{21} \ d(x_{2n-1}, x^*) + a_{22} \ d(x_{2n-1}, x_{2n}) + a_{23} \ d(x^*, u_n) +
+a_{24} \ d(x_{2n-1}, u_n) + a_{25} \ d(x^*, x_{2n}),
\]
for each \( n \in \mathbb{N}^* \).
Using the triangle inequality we get
\[ d(x^*, u_n) \leq [1 - (a_{23} + a_{24})]^{-1}[(1 + a_{25}) d(x^*, x_{2n}) +
+ (a_{21} + a_{24}) d(x^*, x_{2n-1}) + a_{22} d(x_{2n-1}, x_{2n})], \]
for each \( n \in \mathbb{N}^* \).

This implies that \( d(x^*, u_n) \to 0 \), as \( n \to \infty \). Since \( u_n \in T_1(x^*) \), for all \( n \in \mathbb{N}^* \) and \( T_1(x^*) \) is a closed set, it follows that \( x^* \in T_1(x^*) \). So \( x^* \in F_{T_1} = F_{T_2} \).

Let us prove now that \( F_{T_1} = F_{T_2} \) is a closed set. For this purpose let \( y_n \in F_{T_1} = F_{T_2} \), for each \( n \in \mathbb{N}^* \), such that \( y_n \to y^* \), as \( n \to \infty \). Clearly \( y^* \in \overline{B}(x_0, r) \). For example, from \( y_n \in T_1(y_n) \) we have that there exists \( v_n \in T_2(y^*) \) so that
\[ d(y_n, v_n) \leq a_{11} d(y_n, y^*) + a_{13} d(y^*, v_n) + a_{14} d(y_n, v_n) + a_{15} d(y^*, y_n), \]
for each \( n \in \mathbb{N}^* \).

Using the triangle inequality we obtain
\[ d(y^*, v_n) \leq (1 + a_{11} + a_{14} + a_{15})/[1 - (a_{13} + a_{14})] d(y^*, y_n), \]
for all \( n \in \mathbb{N}^* \).

This implies that \( d(y^*, v_n) \to 0 \), as \( n \to \infty \). Since \( v_n \in T_2(y^*) \), for each \( n \in \mathbb{N}^* \) and \( T_2(y^*) \) is a closed set, it follows that \( y^* \in T_2(y^*) \). Therefore \( F_{T_1} = F_{T_2} \) is a closed set. \( \square \)

The following fixed point theorem for a multivalued mapping defined on a closed ball can be proved.

**Theorem 3.2.** Let \( (X, d) \) be a complete metric space, \( x_0 \in X \), \( r > 0 \) and \( T : \overline{B}(x_0, r) \to P_{cl}(X) \) a multivalued mapping for which there exist \( a_1, \ldots, a_5 \in \mathbb{R}^+ \), with \( a_1 + a_2 + a_3 + 2a_4 < 1 \) such that:

(i) for each \( x \in \overline{B}(x_0, r) \), any \( u_x \in T(x) \) and for all \( y \in \overline{B}(x_0, r) \), there exists \( u_y \in T(y) \) so that
\[ d(u_x, u_y) \leq a_1 d(x, y) + a_2 d(x, u_x) + a_3 d(y, u_y) + a_4 d(x, u_y) + a_5 d(y, u_x); \]

(ii) there exists \( y_0 \in T(x_0) \) such that
\[ d(x_0, y_0) \leq \left[ 1 - \frac{a_1 + a_3 + a_4}{a_2} \right] r. \]

Then \( F_T \in P_{cl}(X) \).
Proof. We put 
\[ l := \frac{a_1 + a_2 + a_4}{1 - (a_3 + a_4)} < 1. \]
Using a similar argument as in the proof of Theorem 3.1, we obtain a sequence \((x_n)_{n \in \mathbb{N}}\) with the following properties:
- \(x_n \in T(x_{n-1})\),
- \(d(x_{n-1}, x_n) \leq l^{n-1}(1 - l)r\),
- \(d(x_0, x_n) \leq (1 - l^n)r\), which means that \(x_n \in \overline{B}(x_0, r)\),
for each \(n \in \mathbb{N}^*\).

The sequence \((x_n)_{n \in \mathbb{N}}\) is convergent and its limit is a fixed point of \(T\). Also, it can be shown that \(F_T\) is a closed set. □

Remark 3.1. If in Theorem 3.2 we take \(a_4 = a_5 = 0\), then the fact that \(F_T \neq \emptyset\) is a result mentioned in [9], but there the condition (ii) is

\[ D(x_0, T(x_0)) < \left(1 - \frac{a_1 + a_2}{1 - a_3}\right)r. \]

References


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