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WEAKLY PICARD OPERATORS: EQUIVALENT DEFINITIONS, APPLICATIONS AND OPEN PROBLEMS

IOAN A. RUS, ADRIAN PETRUŞEL and MARCEL ADRIAN ŞERBAN

Department of Applied Mathematics
Babeş-Bolyai University
Cluj-Napoca 400084, Romania
E-mail: iarus@math.ubbcluj.ro
petrusel@math.ubbcluj.ro
mserban@math.ubbcluj.ro

Abstract. The purpose of this paper is to present several characterizations for the concept of weakly Picard operator in K-metric spaces. Some new characterizations and applications, as well as, several open questions are also discussed.
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1. Introduction

Let $A : X \to X$ be an operator. Then $A^0 := 1_X,\ A^1 := A, \dots, A^{n+1} = A \circ A^n,\ n \in \mathbb{N}$ denote the iterate operators of A. By $I(A)$ we will denote the set of all nonempty invariant subsets of A, i. e. $I(A) := \{Y \subset X | A(Y) \subseteq Y\}$, while $F_A := \{x \in X |\ x = A(x)\}$ will denote the fixed point set of the operator A. Also, by $Graf(A) := \{(x, y) \in X \times X | A(x) = y\}$ we will denote the graph of A and by $(AB)_A(x^*)$ the attraction basin of $x^* \in X$ with respect to A, i. e. $(AB)_A(x^*) := \{x \in X | A^n(x) \to x^*,\ \text{as } n \to +\infty\}$.

Let X be a nonempty set. Denote $s(X) := \{(x_n)_{n\in N} | x_n \in X,\ n \in N\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (Fréchet [16]) if the following conditions are satisfied:

(i) If $x_n = x,\ \forall\ n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $Lim(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $Lim(x_{n_i})_{i\in N} = x$.

By definition an element of $c(X)$ is a convergent sequence, $x := Lim(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$.

Recall now the following important concept.

Definition 1.1. (I.A. Rus [45]) Let $(X, \to)$ be an L-space. An operator $A : X \to X$ is, by definition, a Picard operator (briefly PO) if:

(i) $F_A = \{x^*\}$;

(ii) $(A^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

In terms of the Picard operators, some classical results in metrical fixed point theory have the following form ([32], [44]).

Example 1.2. (Contraction principle) *Let (X, d) be a complete metric space and $A : X \to X$ an a-contraction, i. e. $a \in]0, 1[$ and $d(A(x), A(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$. Then A is a PO.*

Example 1.3. (Nemytzki and Edelstein) *Let (X, d) be a compact metric space and $A : X \to X$ satisfying $d(A(x), A(y)) < d(x, y)$, for all $x, y \in X$ with $x \neq y$. Then A is a PO.*

Example 1.4. (Perov) *Let (X, d) be a complete generalized metric space ($d(x, y) \in \mathbb{R}^m_+$) and $S \in M_{mm}(\mathbb{R}_+)$, such that, $S^n \to 0$ as $n \to \infty$. If $A : X \to X$ is an A-**contraction**, i. e., $d(A(x), A(y)) \leq S \cdot d(x, y)$, for all $x, y \in X$, then A is a PO.*

Example 1.5. (Sehgal and Bharucha-Reid, [19]) *Let (X, F, min) be a complete probabilistic metric space. Let $A : X \to X$ be a continuous operator for which there exists $a \in [0, 1[$ such that $F_{A(x),A(y)}(a\lambda) \geq F_{x,y}(\lambda)$, for each $x, y \in X$ and each $\lambda > 0$. Then A is a PO.*

Another important concept is:

Definition 1.6. Let $(X, \to)$ be an L-space. By definition, $A : X \to X$ is called a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

Example 1.7. *Let (X, d) be a complete metric space and $A : X \to X$ be a closed operator such that there is $a \in]0, 1[$ with the property $d(A(x), A^2(x)) \leq a \cdot d(x, A(x))$, for each $x \in X$. Then A is a WPO.*

Example 1.8. *Let (X,d) be a complete metric space, $A : X \to X$ be a closed operator and $\varphi : X \to \mathbb{R}_+$. We suppose that A satisfies the Caristi condition with respect to φ, i. e, $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$, for each $x \in X$. Then A is a WPO.*

In I. A. Rus [45] the basic theory of Picard and weakly Picard operators is presented. For the multivalued case see A. Petruşel [39], as well as, A. Petruşel and I. A. Rus [40]. For both settings see also [48].

2. K-metrics generated by K-functionals

Let $\mathbb{B}$ be an ordered linear L-space with the cone K. Let X be a nonempty set. A function $d : X \times X \to K$ is called a K-metric on X if the following properties hold:

a) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$
b) $d(x_1, x_2) = d(x_2, x_1)$, for each $x_1, x_2 \in X$
c) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$, for each $x_1, x_2, x_3 \in X$.

The pair (X, d), where X is a nonempty set and d is a K-metric on X is said to be a K- metric space.

One of the most important class of K-metric spaces is the class of K-normed linear space. For more details on convergence structures and K-metrics see P. P. Zabreiko [54] and E. De Pascale, G. Marino and P. Pietromala [13].

Let X be a nonempty set and $(Y, +, \mathbb{R}, \|\cdot\|, \leq)$ be an ordered Banach space. Let $K := \{y \in Y | y \geq 0\}$ the cone of the positive elements of Y. Let $\psi : X \to K$ be a functional and $Z_\psi := \{x \in X | \psi(x) = 0\}$.

We have:

Lemma 2.1. *If card(Z_ψ) ≤ 1 then the functional $d_\psi : X \times X \to K$ defined by*

$$d_\psi(x,y) = \begin{cases} 0, & \text{if } x = y \\ \psi(x) + \psi(y), & \text{if } x \neq y. \end{cases}$$

is a K-metric on X.

If in addition, card(Z_ψ) $= 1$ then d_ψ is a complete K-metric on X.

Proof. It is easy to see that d_ψ satisfy all the properties of a K-metric. We will prove now that d_ψ is complete. For this purpose, let $(x_n)_{n\in\mathbb{N}}$ be such that $d_\psi(x_n, x_m) \to 0$, as $n, m \to +\infty$.

If $(x_n)_{n\in\mathbb{N}}$ contains a constant subsequence $x_{n_i} := \tilde{x}$, for $i \in \mathbb{N}$ then $d_\psi(x_i, x_{n_i}) = d_\psi(x_i, \tilde{x}) \to 0$, as $i \to +\infty$ and hence $(x_n)_{n\in\mathbb{N}}$ converges.

If there exists a subsequence $(x_{n_i})_{i\in\mathbb{N}}$ with distinct elements, then for $i \neq j$ we have: $d(x_{n_i}, x_{n_j}) = \psi(x_{n_i}) + \psi(x_{n_j}) \to 0$, as $i, j \to +\infty$. As consequence, $\psi(x_{n_i}) \to 0$, as $i \to +\infty$. Let $Z_\psi = \{y\}$. Then $d_\psi(x_i, y) \leq d_\psi(x_i, x_{n_i}) + d_\psi(x_{n_i}, y) = d_\psi(x_i, x_{n_i}) + \psi(x_{n_i}) \to 0$, as $i \to +\infty$. So $(x_n)_{n\in\mathbb{N}}$ converges again. □

Remark 2.1. For the particular case $K = \mathbb{R}_+$ see J. S. Wong [53], K. Deimling [14] and J. Jachymski [25].

Lemma 2.2. *Let* $X = \bigcup_{i\in I} X_i$ *be a partition of* X *and* $e \in K, e > 0$ *be arbitrary. Let* $\psi : X \to K$ *be such that card*$(Z_\psi \bigcap X_i) = 1$*, for each* $i \in I$*. Then the functional* $d_\psi : X \times X \to K$ *defined by*

$$d_\psi(x,y) = \begin{cases} 0, & \text{if } x = y \\ \psi(x) + \psi(y), & \text{if } x, y \in X_i, i \in I \\ \psi(x) + \psi(y) + e, & \text{if } x \in X_i,\ y \in X_j,\ i \neq j,\ i, j \in I. \end{cases}$$

is a complete K*-metric on* X*.*

Proof. We will prove again that d_ψ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be such that $d_\psi(x_n, x_m) \to 0$, as $n, m \to +\infty$. From the definition of d_ψ it follows that there exists $n_0 \in \mathbb{N}$ and $i \in I$ such that $x_n \in X_i$, for $n \geq n_0$. The proof follows now from Lemma 2.1. □

Remark 2.2. For the particular case $K = \mathbb{R}_+$, see I. A. Rus [43] and J. Jachymski [25].

3. Schröder's pair

Let X be a nonempty set, $(Y, +, \mathbb{R}, \|\cdot\|, \leq)$ be an ordered Banach space with the generating and regular cone K (see Krasnoselskii [33] and Eisenfeld-Lakchmikantham [15]). Let $A : X \to X$ and $\psi : X \to K$ be operators.

Definition 3.1. The pair (A, ψ) is called a Schröder's pair if there exists a linear increasing operator $Q : K \to K$ such that:

(a) the Neumann series $\sum_{n=0}^{+\infty} Q^n(y)$ converges for all $y \in Y$;

(b) the pair (A, ψ) is a solution of the Schröder's inequation:

$$\psi(A(x)) \leq Q(\psi(x)), \text{ for each } x \in X.$$

Remark 3.1. If (A, ψ) is a Schröder's pair then (A^n, ψ) is a Schröder's pair too, for all $n \in \mathbb{N}^*$.

Lemma 3.1. *Let* $X = \bigcup_{i \in I} X_i$ *be a partition of* X *and* $\psi : X \to K$ *such that card*$(Z_\psi \bigcap X_i) = 1$*, for each* $i \in I$*. Let* $A : X \to X$ *be an operator such that* $A(X_i) \subset X_i$*, for all* $i \in I$ *and* (A, ψ) *is a Schröder's pair. Then:*

(1) $F_A = F_{A^n} = Z_\psi$*, for all* $n \in \mathbb{N}$

(2) $d_\psi(A^2(x), A(x)) \leq Q(d_\psi(A(x), x))$*, for each* $x \in X$

(3) A *is a WPO on* (X, d_ψ)*.*

Proof. (1) From Definition 3.1. (b) we have that:

a) If $x \in Z_\psi$ and $x \in X_i$ then $\psi(A(x)) = 0$ and $A(x) \in X_i$. Since card$(Z_\psi \bigcap X_i) = 1$, for each $i \in I$ we obtain $x \in F_A$.

b) If $x \in F_A$ and $x \in X_i$ then $\psi(x) \leq Q(\psi(x)) \leq \cdots \leq Q^n(\psi(x)) \to 0$, as $n \to +\infty$. Hence $\psi(x) = 0$ and so $x \in Z_\psi$. The first conclusion follows now from Remark 3.2.

(2) Let $x \in X$. Then there exists $i \in I$ such that $x \in X_i$. Hence $A(x), A^2(x) \in X_i$.

Suppose $A(x) \neq A^2(x)$. Then:

$$d_\psi(A^2(x), A(x)) = \psi(A^2(x)) + \psi(A(x)) \leq Q(\psi(A(x)) + \psi(x))$$

$$= Q(d_\psi(A(x), x)). \quad \square$$

(3) The conclusion follows from (2), the definition of the Schröder's pair and the same approach as in Example 1.7.

Remark 3.2. For B. Schröder's equations and inequalities see I. N. Baker [6], M. Kuczma [34], J. S. Wong [53], K. Deimling [14] and J. Jachymski [24], [25].

4. Caristi operators on K-metric spaces

Let $\mathbb{B}$ be an ordered Banach space with the generating and regular cone K.

We recall that a K-metric space X is said to be sequentially complete in Weierstrass sense (see [54]) if each sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) < +\infty$ is convergent in X. Let us remark that the above inequality means that the series is convergent in the space $\mathbb{B}$.

Theorem 4.1. *Let (X, d) be a sequentially complete (in Weierstrass' sense) K-metric space, $A : X \to X$ be a closed operator. Suppose that there exists a functional $\varphi : X \to K$ such that:*

$$d(x, A(x)) \leq \varphi(x) - \varphi(A(x)), \text{ for each } x \in X.$$

Then A is a WPO.

Proof. Denote by $x_n := A^n(x)$, for $n \in \mathbb{N}$ and $x \in X$. Then:

$$\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) = \sum_{n=0}^{+\infty} d(A^n(x), A^{n+1}(x)), \text{ for each } x \in X.$$

We will prove that the series $\sum_{n=0}^{+\infty} d(A^n(x), A^{n+1}(x))$ is convergent in the ordered Banach space $\mathbb{B}$. For this purpose we need to show that the sequence of its partial sums is convergent in $\mathbb{B}$. Denote by $s_n := \sum_{k=0}^{n} d(A^k(x), A^{k+1}(x))$. Then $s_{n+1} - s_n = d(A^{n+1}(x), A^{n+2}(x)) \geq 0$, for each $n \in \mathbb{N}$. Moreover $s_n = \sum_{k=0}^{n} d(A^k(x), A^{k+1}(x)) \leq \varphi(x)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{B}$. Then the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent in the ordered Banach space $\mathbb{B}$.

It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and, from the sequentially completeness of the space, convergent to a certain element $x^* \in X$. The conclusion follows from the fact that A is closed. □

Remark 4.1. For Caristi operators in the particular case $K = \mathbb{R}$ see A. Brøndsted [8], [9], F. E. Browder [10], J. Caristi [11], J. Jachymski [24]-[26], W. A. Kirk and L. M. Saliga [30], [31] and M. Turinici [50]-[52].

5. Invariant subsets

Let X be a nonempty set and $A : X \to X$ be an operator. Then, by definition, a subset Y of X is said to be:

(i) an invariant subset for A if $A(Y) \subset Y$
(ii) a fixed subset for A if $A(Y) = Y$
(iii) an invariant subset for A^{-1} if $A^{-1}(Y) \subset Y$
(iv) a fixed subset for A^{-1} if $A^{-1}(Y) = Y$

(v) a completely invariant subset for A if $A(Y) \subset Y = A^{-1}(Y)$, i. e. Y is invariant for A and fixed for A^{-1}.

We have:

Lemma 5.1. *If $X = \bigcup_{i\in I} X_i$ is a partition of X and $X_i, i \in I$ are invariant subsets for the operator $A : X \to X$, then X_i $(i \in I)$ are completely invariant subsets for A.*

Lemma 5.2. *Let $(X, \to)$ be an L-space and $A : X \to X$ be a WPO. Then for $x \in F_A$, the attraction basin, $(AB)_A(x)$, of x with respect to A is a completely invariant subset of A.*

Proof. It is obvious that $(AB)_A(x)$ is an invariant subset of A. Let $y \in A^{-1}((AB)_A(x))$ be arbitrary. This means that there exists $u \in (AB)_A(x)$ such that $y \in A^{-1}(u)$. Hence $A(y) = u$ and $A^n(u) \to x$, as $n \to +\infty$. As consequence, $A^{n+1}(y) = A^n(y) \to x$, as $n \to +\infty$. Hence $y \in (AB)_A(x)$. □

Remark 5.1. Obviously $X = \bigcup_{x\in F_A} (AB)_A(x)$.

Lemma 5.3. (S. Leader [35]) *Let X be a compact metric space and $A : X \to X$ be a continuous operator. Then $\bigcap_{n\in\mathbb{N}} A^n(X)$ is a fixed set for A.*

6. Some suggestive results

Let (X, d) be a complete K-metric space, where K is a generating and normal cone of an ordered Banach space Y. Let $Q : Y \to Y$ be a linear positive operator. Then $A : X \to X$ is said to be a Q-contraction if $\|Q\| < 1$ and $d(A(x_1), A(x_2)) \le Q(d(x_1, x_2))$, for each $x_1, x_2 \in X$.

The first result of this section is:

Theorem 6.1. *Let (X, d) be a complete K-metric space, where K is a generating and regular cone. Let $A : X \to X$ be a Q-contraction. Then we have:*

(i) $F_A = F_{A^n} = \{x^*\}$, for $n \in \mathbb{N}^*$

(ii) If (X, d) is bounded then $\bigcap_{n\in\mathbb{N}} A^n(X) = \{x^*\}$

(iii) If we consider $\psi : X \to K$, $\psi(x) := d(x, x^*)$ then the pair (A, ψ) is a Schröder pair

(iv) $A^n(x) \to x^*$, as $n \to +\infty$, for each $x \in X$

(v) $d(A^n(x), x^*) \leq (I-Q)^{-1} \cdot Q^n \cdot d(x, A(x))$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$.

(vi) $d(A^n(x), x^*) \leq Q^n \cdot d(x, x^*)$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$.

(vii) $d(A^n(x), x^*) \leq (I-Q)^{-1} \cdot d(A^n(x), A^{n+1}(x))$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$.

(viii) $d(x, x^*) \leq (I-Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$

(ix) $\sum_{n=0}^{+\infty} d(A^n(x), A^{n+1}(x)) \leq (I-Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$

(x) there exists a neighborhood U *of* x^* *such that* $A^n(U) \to \{x^*\}$, *as* $n \to +\infty$.

Proof. The proof is essentially the same as that given in [12], [32] and [44], in the case $K = \mathbb{R}_+$. $\square$

Remark 6.1. By definition, an operator which satisfies (i) in the above theorem is said to be a Bessaga operator ([44]).

An operator $A : X \to X$ is said to be a Q-graph contraction if the Q-contraction condition holds for each $(x_1, x_2) \in Graf A$.

In a similar way to the case $K = \mathbb{R}^m_+$ and $K = \mathbb{R}_+$ (see [44], [45], [5], [13], [54]) one can prove the following:

Theorem 6.2. *Let* (X, d) *be a complete K-metric space and* $A : X \to X$ *be a closed Q-graph contraction. Then we have:*

(i) $F_A = F_{A^n} \neq \emptyset$, *for* $n \in \mathbb{N}^*$

(ii) $A^n(x) \to A^\infty(x)$, *as* $n \to +\infty$, *for each* $x \in X$

(iii) $d(A^n(x), A^\infty(x)) \leq (I-Q)^{-1} \cdot Q^n \cdot d(x, A(x))$, *for each* $n \in \mathbb{N}^*$ *and each* $x \in X$.

(iv) $d(x, A^\infty(x)) \leq (I-Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$.

(v) $\sum_{n=0}^{+\infty} d(A^n(x), A^{n+1}(x)) \leq (I-Q)^{-1} \cdot d(x, A(x))$, *for each* $x \in X$

(vi) there exists a partition $X = \bigcup_{i \in I} X_i$ *of* X *such that:*

(a) $A(X_i) \subset X_i$, *for* $i \in I$

(b) $A|_{X_i} : X_i \to X_i$ *is a PO, for each* $i \in I$.

7. Picard operators

The result of this section is the following:

Theorem 7.1. *Let X be a nonempty set and $A : X \to X$ be an operator. Then the following statements are equivalent:*

(P_1) *there exists an L-space structure on the set X, denoted by $\to$, such that $A : (X, \to) \to (X, \to)$ is PO;*

(P_2) *the operator A is Bessaga;*

(P_3) *there exist $\alpha \in]0, 1[$ and $\chi : X \to \mathbb{R}_+$ such that:*

(i) $card(Z_\chi) = 1$

(ii) $\chi(A(x)) \leq \alpha \cdot \chi(x)$, for each $x \in X$;

(P_4) *there exist $\alpha \in]0, 1[$ and a complete metric d on X such that $A : (X, d) \to (X, d)$ is an α-contraction;*

(P_5) *there exist $x^* \in F_A$, $\alpha \in]0, 1[$ and a metric d on X such that $d(A(x), x^*) \leq \alpha \cdot d(x, x^*)$, for each $x \in X$;*

(P_6) *there exist $x^* \in F_A$ and a Hausdorff topology on X such that if $Y \in I_{cl}(A)$ then $x^* \in Y$;*

(P_7) *there exists $n_0 \in \mathbb{N}^*$ such that $A^{n_0} : (X, d) \to (X, d)$ is a Bessaga operator;*

(P_8) *there exist $n_0 \in \mathbb{N}^*$, $\alpha \in]0, 1[$ and a complete metric d on X such that $A^{n_0} : (X, d) \to (X, d)$ is an α-contraction;*

(P_9) *there exist $n_0 \in \mathbb{N}^*$ and an L-space structure on X, denoted by $\to$, such that $A^{n_0} : (X, \to) \to (X, \to)$ is PO.*

Proof.

$(P_1) \Rightarrow (P_2)$ Let $F_A = \{x^*\}$ and $y^* \in F_{A^m}$. Then $A^n(y^*) \to x^*$, as $n \to +\infty$. Since $A^{k_m}(y^*) = y^*$, for $k \in \mathbb{N}$ we have $x^* = y^*$.

$(P_2) \Rightarrow (P_3)$ This implication is a theorem by J. Jachymski [25].

$(P_3) \Rightarrow (P_4)$ Let $Z_\chi = \{x^*\}$. We define $d(x, y) := \chi(x) + \chi(y)$.

$(P_4) \Rightarrow (P_5)$ Let (X, d) be a complete metric space and $A : X \to X$ be an α-contraction. Then $F_A = \{x^*\}$ and $d(A(x), x^*) \leq \alpha \cdot d(x, x*)$, for each $x \in X$.

$(P_5) \Rightarrow (P_6)$ We consider on X the topology defined by the metric d. Let $Y \in I_{cl}(A)$ $and x \in Y$. We have $A^n(x) \in Y$ and $d(A^n(x), x^*) \leq \alpha^n \cdot d(x, x*)$, for each $\mathbb{N}$. Hence $A^n(x) \to x^*$, as $n \to +\infty$ and $x^* \in Y$.

$(P_6) \Rightarrow (P_7)$ Let us remark first that $F_A = \{x^*\}$. Indeed, if there exists $y^* \in F_A$ with $x^* \neq y^*$ then taking $Y := \{y^*\}$ and using (P_6) we get $x^* = y^*$. Further let $y^* \in F_{A^n}$ with $n > 1$ and $x^* \neq y^*$. Then if we choose $Y :=$

$\{y^*, A(y^*), A^2(y^*), \cdots, A^{n-1}(y^*)\}$ we obtain again $x^* = y^*$. Hence A^n is a Bessaga operator.

$(P_7) \Rightarrow (P_8)$ This implication follows from Bessaga's theorem.

$(P_8) \Rightarrow (P_9)$ Define $\to := \xrightarrow{d}$. From the contraction principle the operator $A^{n_0} : (X, d) \to (X, d)$ is Picard.

$(P_9) \Rightarrow (P_2)$ $F_A^{n_0} = \{x^*\}$. We have $A^{n_0}(x) \to x^*$, as $n \to +\infty$, for each $x \in X$. Obviously $x^* \in F_A$. Since $F_A \subset F_{A^{n_0}}$ and $F_{A^n} \subset F_{A^{nn_0}}$ we get that A is Bessaga.

$(P_4) \Rightarrow (P_1)$ Let us define $\to := \xrightarrow{d}$. Then the proof follows from the contraction principle. $\square$

Remark 7.1. For other equivalent statements of PO definition see V. G. Angelov [3], [4], B. Fuchssteiner [17], O. Hadžić, E. Pap and V. Radu [18], O. Hadžić and E. Pap [19], J. Jachymski [24], L. Janos [28], P. R. Meyers [37], V. I. Opoizev [38], V. Radu [41], I. A. Rus [47], [44], B. Schweizer, H. Sherwood and R. M. Tardiff [49].

8. Weakly Picard operators

For weakly Picard operator we have:

Theorem 8.1 *Let X be a nonempty set and $A : X \to X$ an operator. Then the following statements are equivalent:*

(WP_1) *there exists an L-space structure on the set X, denoted by $\to$, such that $A : (X, \to) \to (X, \to)$ is WPO*

(WP_2) $F_A = F_{A^n} \neq \emptyset$, *for each $n \in \mathbb{N}^*$*

(WP_3) *there exists a partial ordering, let say $\leq$, such that the set of all maximal elements of X, denoted by $Max(X)$, is nonempty and $A : (X, \leq) \to (X, \leq)$ is progressive*

(WP_4) *there exists a complete metric d on X and a number $\alpha \in]0, 1[$ such that:*

(i) $A : (X, d) \to (X, d)$ *is closed*

(ii) $d(A^2(x), A(x)) \leq \alpha \cdot d(A(x), x)$, *for each $x \in X$*

(WP_5) *there exist a complete metric d on X and a lower semicontinuous functional $\varphi : X \to \mathbb{R}_+$ such that $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$, for each $x \in X$*

(WP_6) *there exist a complete metric d on X and a functional $\varphi : X \to \mathbb{R}_+$ such that* *(i) A is closed*

(ii) $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$, for each $x \in X$

(WP_7) *there exists a partition $X = \bigcup_{i \in I} X_i$ of X such that $A(X_i) \subset X_i$ and $A|_{X_i} : X_i \to X_i$ is a Bessaga operator for all $i \in I$*

(WP_8) *there exists a partition $X = \bigcup_{i \in I} X_i$ of X such that $A(X_i) \subset X_i$ and $A|_{X_i} : X_i \to X_i$ satisfies (P_3) in Theorem 7.1.*

$(WP9)$ *there exists a complete metric d on X and a number $\alpha \in]0, 1[$ such that:*

(i) $A : (X, d) \to (X, d)$ is continuous

(ii) $d(A^2(x), A(x)) \leq \alpha \cdot d(A(x), x)$, for each $x \in X$

(WP_{10}) *there exists a complete metric d on X such that:*

(i) $A : (X, d) \to (X, d)$ is continuous

(ii) $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) < +\infty$, for each $x \in X$

(WP_{11}) *there exists a complete metric d on X such that:*

(i) $A : (X, d) \to (X, d)$ is closed

(ii) $\sum_{n \in \mathbb{N}} d(A^n(x), A^{n+1}(x)) < +\infty$, for each $x \in X$

(WP_{12}) *there exist a complete metric d on X and a functional $\varphi : X \to \mathbb{R}_+$ such that*

(i) A is continuous

(ii) $d(x, A(x)) \leq \varphi(x) - \varphi(A(x))$, for each $x \in X$

Proof. $(WP1) \Rightarrow (WP2)$. The definition of weakly Picard operator implies that $F_A \neq \emptyset$. The convergence of all sequences of successive approximation with the limits in F_A, implies that $F_A = F_{A^n}$, for all $n \in \mathbb{N}$.

$(WP2) \Rightarrow (WP4)$ Since $F_A = F_{A^n}$, for all $n \in \mathbb{N}$, then there exist a partition of X, $X = \bigcup_{i \in I} X_i$ such that $X_i \in I\,(A)$, $card\,(F_A \cap X_i) = 1$ and $A\,|_{X_i}$ is a Bessaga mapping (see Rus [49]). From theorem of Bessaga there exists a complete metric d_i on X_i such that $A\,|_{X_i}$ is an $\alpha-$contraction for all $i \in I$. We define a complete metric on X. Let $x_i^* \in X_i \cap F_A$, $i \in I$, we take

$$d : X \times X \to X$$

$$d\,(x, y) = \begin{cases} d_i\,(x, y)\,, \text{ if } x, y \in X_i \\ d_i\,(x, x_i^*) + d_j\left(y, x_j^*\right) + 1, \text{ if } x \in X_i,\, y \in X_j,\, i \neq j \end{cases}$$

The completeness of (X, d) follows from the following remark:

$$d(x, y) < 1 \Rightarrow \exists i \in I,\ x, y \in X_i.$$

If $x \in X_i$ then $A(x), A^2(x), ..., A^n(x) \in X_i$ since $X_i \in I(A)$ and

$$d\left(A^2(x), A(x)\right) = d_i\left(A^2(x), A(x)\right) \leq \alpha \cdot d_i(A(x), x) = \alpha \cdot d(A(x), x).$$

The conclusion (i) follows from remark that $A|_{X_i}$ is continuous.

$(WP4) \Rightarrow (WP6)$ We define $\varphi : X \to \mathbb{R}_+, \varphi(x) = \frac{1}{1-\alpha} \cdot d(x, A(x))$.

$(WP6) \Rightarrow (WP3)$ See J. Jachymski [26].

$(WP3) \Rightarrow (WP2)$ See J. Jachymski [26].

$(WP4) \Rightarrow (WP1)$ We take on X, $\to := \xrightarrow{d}$.The proof follows from condition (ii) and (i).

$(WP4) \Rightarrow (WP5)$ We take $\varphi : X \to \mathbb{R}_+, \varphi(x) = \frac{1}{1-\alpha} \cdot d(x, A(x))$.

$(WP5) \Rightarrow (WP1)$ Follows from Caristi's theorem [11] and a remark of A. Brøndsted [8].

$(WP1) \Rightarrow (WP7)$ see Rus [49].

$(WP7) \Rightarrow (WP8)$. The condition $(WP7)$ implies $(WP2)$ and thus we obtain $(WP4)$. Now we define

$$\begin{gathered}\chi : X \to \mathbb{R}_+ \\ \chi(x) = d(x, A(x))\end{gathered}$$

It is obvious to see that $A|_{X_i}$ satisfies the condition $(P3)$ from Theorem 7.1.

$(WP8) \Rightarrow (WP1)$. It is obvious.

$(WP7) \Rightarrow (WP9)$. We know that $(WP7)$ implies $(WP2)$. The proof is similar to $(WP2) \Rightarrow (WP4)$, but we will additionally prove that the operator A is nonexpansive with respect to d. Since $A|_{X_i}$ is an $\alpha-$contraction for all $i \in I$, hence nonexpansive, it suffices to consider the case for $x \in X_i$ and $y \in X_j$, $i \neq j$. Since $x \in X_i$ and $y \in X_j$ then $A(x) \in X_i$ and $A(y) \in X_j$, hence

$$\begin{aligned} d(A(x), A(y)) &= d_i(A(x), x_i^*) + d_j\left(A(y), x_j^*\right) + 1 \leq \\ &\leq \alpha \cdot d_i(x, x_i^*) + \alpha \cdot d_j\left(y, x_j^*\right) + 1 \leq \\ &\leq d_i(x, x_i^*) + d_j\left(y, x_j^*\right) + 1 = \\ &= d(x, y) \end{aligned}$$

thus the operator A is continuous.

$(WP9) \Rightarrow (WP8)$ It is obvious, since A is continuous then A is closed.

$(WP9) \Rightarrow (WP10)$ Condition (ii) from $(WP9)$ implies

$$\sum_{n\in\mathbb{N}} d\left(A^n(x), A^{n+1}(x)\right) \leq \frac{1}{1-\alpha} \cdot d(x, A(x)), \ \forall x \in X$$

which proves $(WP10)$.

$(WP10) \Rightarrow (WP11)$ It is obvious.

$(WP11) \Rightarrow (WP1)$ Condition (ii) from $(WP11)$ implies that every successive approximation sequence is Cauchy, therefore convergent to $x^* \in X$. From the condition that operator A is closed we deduce that $x^* \in F_A$ which implies that operator A is WPO. The $L-$space structure is generated by the metric d.

$(WP9) \Rightarrow (WP12)$ The proof is the same as in $(WP4) \Rightarrow (WP6)$.

$(WP12) \Rightarrow (WP11)$ From condition (ii) of $(WP12)$ we obtain

$$\sum_{n\in\mathbb{N}} d\left(A^n(x), A^{n+1}(x)\right) \leq \varphi(x) < \infty, \ \forall x \in X$$

and thus the proof is complete. □

9. Weakly Picard operators on compact metric spaces

For Picard operators on compact metric spaces we have the following:

Theorem 9.1. (S. Leader [35], I. A. Rus [44] pp. 49) *Let (X, d) be a compact metric space and $A : X \to X$ be a continuous operator. Then the following statements are equivalent:*

(i) A is a Janos operator, i. e. $\bigcap_{n\in\mathbb{N}} A^n(X) = \{x^*\}$

(ii) there exists $n_0 \in \mathbb{N}$ such that A^{n_0} is an Janos operator

(iii) A is a Picard operator and $A^n \xrightarrow{u} A^\infty$, as $n \to +\infty$, where $\xrightarrow{u}$ stands for uniform convergence

(iv) A is a contraction with respect to some metric equivalent to d

(v) A is contractive with respect to some metric equivalent to d

(vi) the sequence $(d(A^n(x), A^n(y)))_{n\in\mathbb{N}}$ converges to 0, uniformly on $X \times X$.

(vii) there exist $\alpha \in]0, 1[$ and ρ an equivalent metric with d such that $\rho(A(x), x^*) \leq \alpha\rho(x, x^*)$.

Proof. The proof is organized as follows:

$(ii) \Rightarrow (i)$ Let $\bigcap_{n\in\mathbb{N}} A^n(X) = \{x^*\}$ and $n_0 \in \mathbb{N}$. We have $X \supset A(X) \supset \cdots \supset A^n(X) \supset \cdots \supset \{x^*\}$. Hence $\bigcap_{k\in\mathbb{N}} A^{n_0 k}(X) = \{x^*\}$.

$(ii) \Rightarrow (i)$ This implication is Janos theorem, [27].

$(iv) \Rightarrow (ii)$ Let ρ be an equivalent metric with d such that A is an α-contraction with respect to ρ. Then $F_A = \{x^*\}$ and $\delta(A^{n_0}(X)) \leq \alpha^{n_0 n}\cdot\delta(X) \to 0$, as $n \to +\infty$, for $n_0 \in \mathbb{N}$.

$(iv) \to (v)$ This is obvious.

$(v) \to (iii)$ The implication is Nemytzki-Edelstein theorem, see [32], [35].

$(iii) \to (vi)$ Let $F_A = \{x^*\}$. Then $A^n(x) \overset{d}{\to} x^*$, as $n \to +\infty$, uniformly on X and $d : X \times X \to \mathbb{R}_+$ is uniformly continuous. Hence $(d(A^n(x), A^n(y)))_{n\in\mathbb{N}} \to d(x^*, x^*) = 0$, uniformly on $X \times X$.

$(vi) \to (iv)$ There exists a sequence $(\alpha_n)_{n\in\mathbb{N}} \to 0$, as $n \to +\infty$, such that $d(A^n(x), A^n(y)) \leq \alpha_n$, for each $x, y \in X$ and $n \in \mathbb{N}$. It follows that $\delta(A^n(X)) \to 0$, as $n \to +\infty$. So $(vi) \Rightarrow (i)$ and $(i) \Rightarrow (iv)$.

$(iv) \to (vi)$ is obviously.

$(vi) \to (iv)$ From $\rho(A^n(x), \{x^*\}) \to 0$, as $n \to +\infty$ we have, taking the supremum and making $n \to +\infty$, that $H(A^n(X), \{x^*\}) \to 0$. $\square$

Remark 9.1. For other considerations on Janos operators see L. Janos [27]- [29], I. A. Rus [44] and A. Iwanik, L. Janos, F. A. Smith [23].

10. Sequences of iterates and fixed points

In this section we will consider the following open question:

Open question 1. *Use the above equivalent statements for the study of the sequence of iterates of an operator.*

As an example, we will consider the iterates of some linear operators on the space of continuous functions.

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded domain, D be a nonempty closed subset of $\overline{\Omega}$ and $(X, +, \mathbb{R}, \|\cdot\|)$ be a Banach space. Let $(C(\overline{\Omega}, X), \|\cdot\|_C)$ be the Banach space of continuous functions $f : \overline{\Omega} \to X$ endowed with the Chebysev supremum norm, i. e. $\|x\|_C := \max_{t\in\overline{\Omega}} |x(t)|$.

We have:

Theorem 10.1. *Let $A : C(\overline{\Omega}, X) \to C(\overline{\Omega}, X)$ be an operator such that:*

(i) A is linear

(ii) $A(x)|_D = x|_D$, for each $x \in C(\overline{\Omega}, X)$

(iii) there exists $\alpha \in]0,1[$ such that $\|A(x)\|_C \leq \alpha \cdot \|x\|_C$, for all $x \in C(\overline{\Omega}, X)$, with $x|_D = 0$.

Then A is a WPO.

Proof. Consider on $C(\overline{\Omega}, X)$ the following equivalence relation:

$$x \sim y \Leftrightarrow x|_D = y|_D.$$

As usually, this equivalence relation induces on $C(\overline{\Omega}, X)$ a partition, let say $C(\overline{\Omega}, X) = \bigcup_{\lambda \in C(\overline{\Omega}, X)} X_\lambda$. From (ii) we have that $A(X_\lambda) \subset X_\lambda$, for all $\lambda \in C(D, X)$. From (iii) and (i) it follows that the operator A is an α-contraction on X_λ, for all $\lambda \in C(D, X)$. On the other hand, X_λ is a closed subset of $C(\overline{\Omega}, X)$. From the Banach contraction principle we obtain that $F_A \cap X_\lambda = \{x^*_\lambda\}$ and $A^n(x) \to x^*_\lambda$ as $n \to +\infty$, for all $x \in X_\lambda$ and all $\lambda \in C(D, X)$. Hence A is a WPO. □

Remark 10.1. It is obviously that the hypothesis of the above result imply

$$\|A^2(x) - A(x)\|_C \leq \alpha \cdot \|x - A(x)\|_C, \text{ for all } x \in C(\overline{\Omega}, X).$$

Remark 10.2. If we consider $X = \mathbb{R}, \overline{\Omega} = [0,1], D = \{0,1\}$ and $A = B_n, n \in \mathbb{N}^*$ (where B_n is the Bernstein operator), then we obtain the result in [46].

If in the above result $X = \mathbb{R}^p$, then we obtain the following improvement of Theorem 10.1.

Theorem 10.2. *Let $A = (A_1, A_2, \cdots, A_n) : C(\overline{\Omega}, \mathbb{R}^p) \to C(\overline{\Omega}, \mathbb{R}^p)$ be an operator such that:*

(i) A is linear

(ii) $A(x)|_D = x|_D$, for each $X \in C(\overline{\Omega}, \mathbb{R}^p)$

(iii) there exists a matrix $S \in \mathcal{M}_p(\mathbb{R}_+)$ such that:

(a) $S^n \to 0$, as $n \to \infty$

(b)

$$\begin{pmatrix} \|A_1(x)\|_C \\ \cdots \\ \|A_p(x)\|_C \end{pmatrix} \leq S \cdot \begin{pmatrix} \|x_1\|_C \\ \cdots \\ \|x_p\|_C \end{pmatrix}, \text{ for each } x \in C(\overline{\Omega}, \mathbb{R}^p), \text{ with } x|_D = 0.$$

Then A is a WPO.

Proof. The approach is similar to that of Theorem 10.1. with the mention that one apply Perov's fixed point theorem (see [44]) instead of Banach contraction principle. □

Example 10.1. $X = \mathbb{R}, \overline{\Omega} = [0, a] \times [0, b] \in \mathbb{R}^2$ and $D = ([0, a] \times \{0\} \cup (\{0\} \times [0, b])$. Consider on $C(\overline{\Omega})$ the Bielecki norm $\|x\|_B := \max_{t\in\overline{\Omega}}(|x(t)| \cdot e^{-\tau t})$. Let $A : C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by:

$$A(x)(t_1, t_2) := x(t_1, 0) + x(0, t_2) - x(0, 0)$$
$$+ \int_0^{t_1}\int_0^{t_2} K(t_1, t_2, s_1, s_2) \cdot x(s_1, s_2)ds_1ds_2,$$

where $K \in C(\overline{\Omega} \times \overline{\Omega})$. We may remark that $\|A(x)\|_B \leq M_K \cdot \tau^{-1}\|x\|$, for all $x \in C(\overline{\Omega})$ with $x|_D = 0$, where $M_K > 0$ is such that $|K(t, s) \leq M_K$, on $\overline{\Omega} \times \overline{\Omega}$. Taking $\tau > 0$ such that $M_K \cdot \tau^{-1} < 1$ we are in the conditions of Theorem 10.1. and so the conclusion follows. □

Remark 10.3. For other results concerning this open question see I. A. Rus [44], Agratini-Rus [1], [2], I. A. Rus-A. Petruşel-G. Petruşel [48] (pp. 45-49, V. Dincuţă B[1], pp. 169, V. Mureşan R[5] pp. 263, M. A. Şerban B[1] pp. 211, I. A. Rus R[1] pp. 274, etc.)

11. Converses of the fixed point principles in K-metric spaces

The considerations in section 2 give rise to the following problems:

Open question 2A. *Let X be a nonempty set and $A : X \to X$ be such that $F_A = F_{A^n} = \{x^*\}$, for each $n \in \mathbb{N}^*$.*

Construct a K-metric d on X such that $A : (X, d) \to (X, d)$ is a contraction.

For a better understanding of this problem (as well as the following one) we present:

Example 11.1. Let (A, ψ) be a Schröder pair. Consider the K-metric d_ψ in Lemma 2.1. In this case $A : (X, d_\psi) \to (X, d_\psi)$ is a Q-contraction. Hence, the problem is to construct an operator $\psi : X \to K$ such that (A, ψ) to be a Schröder pair.

Example 11.2. Let (X, d) be a complete K-metric space, $Q; K \to K$ and $A; X \to X$. Suppose that:

(i) Q is bijective;

(ii) $Q^n \to 0$, as $n \to +\infty$;

(iii) there exists $n_0 \in \mathbb{N}^*$ such that A^{n_0} is a Q^{n_0}-contraction.

Then

$$\rho(x,y) := d(x,y) + S^{-1}d(A(x),A(y)) + \cdots + S^{1-n_0}d(A^{n_0-1}(x), A^{n_0-1}(y))$$

is a K-metric equivalent with d and $A : (X,\rho) \to (X,\rho)$ is a Q-contraction.

References: V. G. Angelov [3], [4], C. Bessaga [7], P. Hitzler and A. K. Seda [21], J. Jachymski [25], V. I. Opoizev [38], I. Rosenholtz [42], J. S. Wong [53].

Open question 2B. *Let X be a nonempty set and $A : X \to X$ be such that $F_A = F_{A^n} \neq \emptyset$, for each $n \in \mathbb{N}^*$.*

Construct a K-metric d on X such that:

(i) $A : (X,d) \to (X,d)$ is orbitally continuous

(ii) there exists a linear positive operator $S : K \to K$, with $S^n \to 0$, as $n \to \infty$ such that $d(A^2(x), A(x)) \leq S \cdot d(A(x), x)$, for each $x \in X$.

References: T. L. Hicks and B. E. Rhoades [20], T. T. Hsieh and K. K. Tan [22], I. A. Rus [45], [44].

Open question 2C. *Let (X,d) be a compact K-metric space and $A : X \to X$ be a continuous operator such that $\bigcap\limits_{n\in\mathbb{N}} A^n(X) = \{x^*\}$.*

Construct a K-metric ρ on X such that:

(i) d and ρ are topologically equivalent

(ii) $A : (X,d) \to (X,d)$ is a contraction.

References: L. Janos [27], [29], I Rosenholtz [42].

Open question 2D. *Let (X,d) be a compact K-metric space and $A : X \to X$ be a continuous operator such that $\bigcap\limits_{n\in\mathbb{N}} A^n(X) = F_A$.*

Construct a K-metric ρ on X such that:

(i) d and ρ are topologically equivalent

(ii) there exists a linear positive operator $S : K \to K$, with $S^n \to 0$, as $n \to \infty$ such that $d(A^2(x), A(x)) \leq S \cdot d(A(x), x)$, for each $x \in X$.

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