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SOME SURJECTIVITY CONDITIONS FOR NONLINEAR ACCRETIVE TYPE SINGLE-VALUED OPERATORS WITH A CLOSED RANGE IN BANACH SPACES

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Abstract. Our aim in this paper is to establish some new surjectivity conditions and study the existence of solutions of equation $Tx = f$ for operators T in a Banach space X that satisfy a general type of accretive condition.

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1. Introduction

The normalized duality mapping J from a general Banach space X into 2^{X^*} is given by

$$J(x) = \left\{ j \in X^* : \operatorname{Re} \langle x, j \rangle = \|x\|^2 = \|j\|^2 \right\}, \tag{1.1}$$

where X^* denotes the dual space of X and $\langle ., . \rangle$ the generalized duality pairing.

An operator T with domain $D(T)$ and range $R(T)$ in X is said to be $\varphi-$*generalized strongly accretive* if there exists a $k \in [0, 1]$ and a strictly increasing function $\varphi : [0, \infty) \to [0.\infty)$ with $\varphi(0) = 0$ such that for all $x, y \in D(T)$, there exists a $j(x - y) \in J(x - y)$ satisfying:

$$\operatorname{Re} \langle Tx - Ty, j(x - y) \rangle \geq \varphi (\|x - y\|) \|x - y\| - (1 - k) \|x - y\|^2 \tag{1.2}$$

Remark 1. It is interesting to note that the class of operators satisfying (1.2) includes the class of $\varphi-$*strongly accretive operators* [3] corresponding to

$k = 1$, that is, T satisfies the condition:

$$\operatorname{Re}\langle Tx - Ty, j(x-y)\rangle \geq \varphi(\|x-y\|)\,\|x-y\|$$

for all $x, y \in X$ and φ is as specified above.

The following examples justify the above remark.

Example 1. Let $X = \mathbb{R}$ with the usual norm $|.|$. Define $T : [0,\infty) \to [0,\infty)$ by

$$Tx = x^2 - \frac{1}{2}x + \frac{1}{16}$$

Then T is $\varphi-$*generalized strongly accretive* operator with $\varphi(s) = s^2$ and $k = \dfrac{1}{2}$ such that for every $f \in [0,\infty)$ the equation $Tx = f$ has at least one solution. In particular,T has two fixed points (namely, $\dfrac{3-2\sqrt{2}}{4}$ and $\dfrac{3+2\sqrt{2}}{4}$). However, it can be easily verified that T is not $\varphi-$*strongly accretive*.

Thus the class of $\varphi-$*strongly accretive* operators is a proper subset of the class of $\varphi-$*generalized strongly accretive* operators.

Example 2. Let $X = \mathbb{R}$ with the usual norm $|.|$ and define $T : [0,\infty) \to [0,\infty)$ by

$$Tx = x^2 - \frac{1}{2}x + 2$$

Then it can be easily verified that T is $\varphi-$*generalized strongly accretive* with $\varphi(s) = s^2$ and $k = \dfrac{1}{2}$ but T is not $\varphi-$*strongly accretive*. In addition, the equation $Tx = x$ has no real solution for any $x \in [0,\infty)$. Thus, in general, the equation $Tx = f$ $(f \in X)$ does not have a solution in X.

Remark 2. (i) For any non-negative real number λ with $\lambda \geq 1-k$, the operator $T + \lambda I$ (where I denotes the identity operator on X) is a bijection when T is continuous [4, Lemma 2.3].

Moreover, the relationship between the new class of operators and the class of $\varphi-$*strongly pseudo-contractive operators* considered by Liu and Kang [4] is that for any operator T satisfying (1.2), –T is $\varphi-$*strongly pseudo-contractive* and by [4, Lemma 2.2], $T + I$ has a unique zero in a real Banach space X when T is continuous.

(ii) Recall that an operator T with domain $D(T)$ and range $R(T)$ in a normed space X is $\varphi-$*expansive* (see [2]) if

$$\varphi(\|x-y\|) \leq \|Tx - Ty\| \tag{1.3}$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0.\infty)$ is a strictly increasing function with $\varphi(0) = 0$.

It is known (see [2]) that $\varphi-$*expansive operators* are invertible but their inverse operators need not be defined on the whole space X.

Notice that $\varphi-$*strongly accretive operators* are $\varphi-$*expansive* since they satisfy the condition

$$\varphi\left(\|x-y\|\right)\|x-y\| \leq \|Tx-Ty\|\,\|x-y\|\,. \tag{1.4}$$

Our aim in this paper is to establish some new surjectivity conditions for operators of type (1.2) with $k \in [0, 1)$ and study the existence of solutions of equation $Tx = f$ for operators T. This is done, in particular, via some fixed point conditions for $\varphi-$*expansive operators* that we establish in this paper. For a detailed account of iterative approximations of fixed point and solution of operator equation $Tx = f$, we refer to Berinde [1].

2. Main results

Now onward, $\mathbb{N}$ will denote the set of natural numbers while $R(T)$ will denote the range of an operator T.

The following lemma gives us sufficient conditions for an operator of type (1.2) with $k \in (0, 1)$ to have a zero.

Lemma 2.1. *Let X be a general Banach space and $T : X \to X$ be an operator of type* (1.2) *with $\varphi(t) > t$ for all t sufficiently large and for $0 < k < 1$. Assume that the following conditions are satisfied.*

(2.1.1) $\overline{R(T)} = R(T)$.

(2.1.2) *$T + \lambda I$ is one to one for every $\lambda > 0$ sufficiently small.*

(2.1.3) *There exists $r > 0 : \|T0\| < r \leq \liminf\limits_{\|x\|\to\infty} \left[\varphi(\|x\|) - \|x\|\right]$.*

Then T has a zero in X.

Proof. Suppose $\mu = r - \|T0\| > 0$ and let $p \in B_\mu(0)$ be arbitrary, where $B_\mu(0) = \{x \in X : \|x\| < \mu\}$. Now consider the equation

$$Tx + \frac{x}{n} = p \text{ for all } n \in \mathbb{N}.$$

By condition (2.1.2), $x_n = (T + \frac{1}{n}I)^{-1}p$ exists for all sufficiently large n. Hence

$$\frac{1}{n}\|x_n\| = \|Tx_n - p\| \geq \|Tx_n - T0\| - \|T0 - p\|\,. \tag{2.1}$$

But from (1.2) we deduce that

$$\varphi(\|x-y\|) \le (1-k)\,\|\,x-y\| + \|Tx-Ty\|\,.$$

Hence (2.1) becomes

$$\frac{1}{n}\,\|x_n\| \ge \varphi(\|x_n\|) - (1-k)\varphi(\|x_n\|) - \|T0\| - \|p\|$$

from which we deduce that

$$\varphi(\|x_n\|) - (1-k+\frac{1}{n})\,\|x_n\| \le \|T0\| + \|p\|\,.$$

Since k is fixed, we may take $n \in \mathbb{N}$ sufficiently large so that $(1-k+\frac{1}{n}) < 1$. Therefore from the above inequality we obtain

$$\varphi(\|x_n\|) - \|x_n\| \le \varphi(\|x_n\|) - (1-k+\frac{1}{n})\,\|x_n\| \le \|T0\| + \|p\|\,. \tag{2.2}$$

We now show that the sequence $\{x_n\}$ is bounded. Suppose this is not the case. Without loss of generality we may assume that $\lim_{n\to\infty} \|x_n\| = \infty$. Then by condition (2.1.3) we have that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$r - \epsilon < \varphi(\|x_n\|) - \|x_n\| \text{ for all } n \ge N$$

and by (2.2) this implies that

$$r - \epsilon < \|T0\| + \|p\|\,.$$

Since $\epsilon > 0$ is arbitrary, the above inequality gives us

$$r \le \|T0\| + \|p\|\,. \tag{2.3}$$

Since $p \in B_\mu(0)$ and $\mu = r - \|T0\|$, from (2.3) we immediately obtain that

$$r \le \|T0\| + \|p\| < \|T0\| + \mu = r,$$

a contradiction. Hence the sequence $\{x_n\}$ is bounded. Therefore from (2.1) we deduce that

$$\lim_{n\to\infty} \|Tx_n - p\| = 0$$

and by condition (2.1.1) we have that $p \in R(T)$. Therefore $B_\mu(0) \subset R(T)$. Consequently, there exists a point $x_0 \in X$ such that $Tx_0 = 0$.□

Theorem 2.2. *Let X be a reflexive Banach space and $T : X \to X$ be an operator that is weakly sequentially continuous and satisfies condition* (1.2) *with $k = 0$ and $\varphi(t) > t$ for all t sufficiently large. Assume that*

(2.2.1) *For any $f \in X$, there exists $r > 0$ such that*

$$\|T0\| + \|f\| < r \leq \liminf_{\|x\|\to\infty}[\varphi(\|x\|) - \|x\| \,.$$

(2.2.2) $\overline{R(T+\lambda I)} = R(T+\lambda I)$ *and* $(T+\lambda I)$ *is one to one for every* $\lambda > 0$ *sufficiently small.*

Then the equation $Tx = f$ *has at least one solution for each* $f \in X$.

Proof. For any $f \in X$ set $S = I - T + f$. Then S satisfies the condition

$$\mathrm{Re}\,\langle Sx - Sy,\ j(x-y)\rangle \leq 2\,\|x-y\|^2 - \varphi(\|x-y\|)\,\|x-y\|$$

for all $x, y \in X$.

To show that S has a fixed point we consider the approximation

$$S_n = \frac{n+1}{n} I - S$$

which is an operator of type (1.2) with $k = \dfrac{1}{n}$ for every $n \in \mathbb{N}$ since

$$\mathrm{Re}\,\langle S_n x - S_n y,\ j(x-y)\rangle \geq \varphi(\|x-y\|)\,\|x-y\| - (1-\frac{1}{n})\,\|x-y\|^2\,. \tag{2.4}$$

But

$$\|S_n 0\| = \|S0\| = \|f - T0\| \leq \|f\| + \|T0\| \tag{2.5}$$

Hence by(2.2.1) above, the condition (2.1.3) of Lemma 2.1 is satisfied. Since $S_n = (T + \frac{1}{n}I) - f$, it is clear by the first part of the condition (2.2.2) that the range of S_n is closed.

Moreover, for any $m, n \in \mathbb{N}$ arbitrarily large we have

$$\frac{1}{m} I + S_n = \frac{m+n}{m.n} I + T - f.$$

Hence for any $x, y \in X$ if

$$\frac{1}{m} x + S_n x = \frac{1}{m} y + S_n y,$$

then we obtain

$$\frac{m+n}{m.n} x + Tx = \frac{m+n}{m.n} y + Ty.$$

Therefore by the second part of the condition (2.2.2), $\frac{1}{m}I + S_n$ is one to one. Since the conditions of Lemma 2.1 are satisfied for n sufficiently large, there exists $x_n \in X$ such that $S_n x_n = 0$.

Now we show that the sequence $\{x_n\}$ so obtained is bounded. Suppose the contrary and without loss of generality assume that

$$\lim_{n\to\infty} \|x_n\| = \infty$$

Then by (2.4) we get

$$\varphi(\|x_n\|)\,\|x_n\| - (1-\frac{1}{n})\,\|x_n\|^2 \leq \|S_n0\|\,.\,\|x_n\|$$

which implies that

$$\varphi(\|x_n\|)\,\|x_n\| - \|x_n\|^2 \leq \|S_n0\|\,.\,\|x_n\|\,.$$

Therefore we immediately obtain that

$$\varphi(\|x_n\|) - \|x_n\| \leq \|S_n0\|\,.$$

Now using (2.2.1) along with the above inequality we have that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$r - \epsilon < \varphi(\|x_n\|) - \|x_n\| \leq \|S_n0\| = \|S0\|\,.$$

Since $\epsilon > 0$ is arbitrary, we have

$$r \leq \|S0\|\,.$$

Therefore by (2.5) and (2.2.1) we arrive at the contradiction

$$r \leq \|S0\| < r.$$

Consequently, the sequence $\{x_n\}$ is bounded.

Now from the boundedness of $\{x_n\}$ and the fact that

$$S_n x_n = 0 = \frac{n+1}{n} x_n - Sx_n$$

we deduce that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Therefore the sequence $\{x_n - Sx_n\}$ converges weakly to 0. Since X is reflexive and $\{x_n\}$ is bounded, $\{x_n\}$ has a weakly convergent subsequence say, $\{x_{n_k}\}$. Further, notice that S is weakly sequentially continuous (since $S = I - T + f$), there exists $x_0 \in X$ such that $x_0 = Sx_0$ implying that $Tx_0 = f$. Hence x_0 is a solution of the equation $Tx = f$.□

Using a dual type of relationship with $\varphi-$ *pseudocontractive operators*, Liu and Kang [4] recently proved that for any continuous $\varphi-$*strongly accretive* *operators* $T : X \to X$ on an arbitrary Banach space, the equation $Tx = f$ has a unique solution. In what follows, we are interested in using a fixed point theorem for $\varphi-$*expansive operators* T and Remark 2(ii) in solving the equation $Tx = f$, where T satisfies the condition (1.2) for $k = 0$.

First we have the following:

Lemma 2.3. *Let X be a general Banach space and $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$ and that its inverse function φ^{-1} is upper semi-continuous. Suppose that $T : X \to X$ is a $\varphi-$expansive operator such that there exists $x_0 \in X$ and that the series $\sum\limits_{n\in\mathbb{N}} (\varphi^{-1})^n(\|x_0 - Tx_0\|$ converges. If the range $R(T)$ of T is closed, then T has at least one fixed point in X.*

Proof. Set $y_0 = Tx_0$ and consider the approximation process

$$y_0 \in R(T),\ y_{n+1} = T^{-1}y_n(n \geq 0). \tag{2.6}$$

We shall show that the sequence $\{y_n\}$ converges. First, we show by induction that

$$\|y_{n+1} - y_n\| \leq (\varphi^{-1})^n(\|x_0 - Tx_0\| .$$

Notice from (1.3) that

$$\varphi\left(\left\|T^{-1}Tx - T^{-1}Ty\right\|\right) \leq \|Tx - Ty\|$$

and since φ is invertible, we have

$$\left(\left\|T^{-1}Tx - T^{-1}Ty\right\|\right) \leq \varphi^{-1}(\|Tx - Ty\|) \text{ for all } x, y \in X.$$

For $n = 1$, we have

$$\begin{aligned} \|y_2 - y_1\| &= \left\|T^{-1}y_1 - T^{-1}y_0\right\| \leq \varphi^{-1}(\|y_1 - y_0\|) \\ &= \varphi^{-1}(\|x_0 - Tx_0\| . \end{aligned}$$

Therefore (2.6) is true for $n = 1$. Let it be true for $n = k, i.e.$

$$\|y_{k+1} - y_k\| \leq (\varphi^{-1})^k(\|x_0 - Tx_0\|)$$

Then for $n = k + 1$, we have

$$\begin{aligned} \|y_{k+2} - y_{k+1}\| &= \left\|T^{-1}y_{k+1} - T^{-1}y_k\right\| \leq (\varphi^{-1})^k(\|y_{k+1} - y_k\|) \\ &\leq (\varphi^{-1})(\varphi^{-1})^k(\|x_0 - Tx_0\|) = (\varphi^{-1})^{k+1}(\|x_0 - Tx_0\|) \end{aligned}$$

and the result holds. Hence by induction (2.6) holds for all n.

Now, for any $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} \|y_m - y_n\| &\leq \|y_m - y_{m-1}\| + \|y_{m-1} - y_{m-2}\| + ... + \|y_{n+1} - y_n\| \\ &\leq (\varphi^{-1})^{m-1}(\|x_0 - Tx_0\|) + (\varphi^{-1})^{m-2}(\|x_0 - Tx_0\|) + ... + \\ &+ (\varphi^{-1})^{n}(\|x_0 - Tx_0\| . \end{aligned}$$

By setting $S_n(\|x_0 - Tx_0\|) = \sum_{k=1}^{n} (\varphi^{-1})^k(\|x_0 - Tx_0\|$ in the above inequality we obtain

$$\|y_m - y_n\| \leq S_{m-1}(\|x_0 - Tx_0\|) - S_{n-1}(\|x_0 - Tx_0\|).$$

Since the series $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|x_0 - Tx_0\|$ converges, the sequence $\{S_n(\|x_0 - Tx_0\|)\}$ is a Cauchy sequence in X. Hence the above inequality implies that $\{y_n\}$ is a Cauchy sequence in X. Therefore it converges to some $y \in R(T)$ as $R(T)$ is closed. Hence we have

$$\begin{aligned} \left\|y - T^{-1}y\right\| &\leq \|y - y_n\| + \left\|y_n - T^{-1}y\right\| \\ &\leq \|y - y_n\| + \left\|T^{-1}y_{n-1} - T^{-1}y\right\| \\ &\leq \|y - y_n\| + \varphi^{-1}(\|y_{n-1} - y\|). \end{aligned}$$

By the upper semi-continuity of φ^{-1}, the above inequality implies that

$$\left\|y - T^{-1}y\right\| \leq \limsup_{n\to\infty}(\|y - y_n\|) + \varphi^{-1}(\limsup_{n\to\infty}(\|y_{n-1} - y\|). = 0,$$

proving that $y = T^{-1}y$, that is, y is a fixed point of T^{-1}.

Let $x \in X$ be such that $y = Tx$. Then $y = T^{-1}y$ implies that $Tx = x$ and the lemma is established.□

We notice that the condition that $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|x_0 - Tx_0\|$ is convergent is quite reasonable as all the linear operators defined at the origin satisfy it trivially.

Theorem 2.4. *Let X be a general Banach space and $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$ and whose inverse function φ^{-1} is upper semi-continuous. Let $T : X \to X$ be an operator satisfying condition* (1.2) *with $k \in (0, 1)$ such that $I + T$ has a closed range. If for any $f \in X$, there exists $x_0 \in X$ such that the series $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|Tx_0 - f\|$ converges, then the equation $Tx = f$ has at least one solution in X.*

Proof. Suppose $S = I + T - f$ for $f \in X$ fixed. Then S is $\varphi-$ strongly accretive as it satisfies the condition

$$\operatorname{Re}\langle Sx - Sy,\ j(x-y)\rangle \geq \varphi(\|x-y\|)\,\|x-y\| + k\,\|x-y\|^2 .$$

Hence S is $\varphi-$expansive (and strongly accretive).

Moreover, for any $x \in X$ we have $Sx - x = Tx - f$. Therefore by the given hypothesis of the theorem, there exists $x_0 \in X$ such that the series $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|Tx_0 - f\|$ converges. Therefore the series $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|Sx_0 - x_0\|$ also converges. Further, since $I + T$ has a closed range, it follows that S also has a closed range. Therefore by Lemma 2.3, S has at least one fixed point say, x_1. Then

$$x_1 = Sx_1 = x_1 + Tx_1 - f,$$

proving that $Tx_1 = f$ for some $x_1 \in X$. Hence the equation $Tx = f$ has a solution in X.□

Remark 3. If the operator T in the above theorem is continuous, then the hypothesis that $I + T$ has a closed range can be dispensed with in view of Remark 2.

Theorem 2.5. L*et X be a general Banach space and $\varphi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\varphi(0) = 0$ such that*

$$\limsup_{t\to\infty} \frac{\varphi(t)}{t} = \infty$$

and that its inverse function φ^{-1} is upper semi-continuous. Assume that the operator $T : X \to X$ satisfies condition (1.2) with $k = 0$ and the operators $T + \frac{n+1}{n} I$ have a closed range in X for every $n \in N$. If for every $f \in X$, there exists $x_0 \in X$ such that the series $\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|Tx_0 - f\| + \|x_0\|)$ converges, then the equation $Tx = f$ has at least one solution in X.

Proof. For any $n \in \mathbb{N}$, set $T_n = T + \frac{1}{n} I$. Then for all $x, y \in X$ and for $j(x-y) \in J(x-y)$ we have

$$\operatorname{Re}\langle T_n x - T_n y,\ j(x-y)\rangle \geq \varphi(\|x-y\|)\,\|x-y\| - (1 - \frac{1}{n})\,\|x-y\|^2 .$$

Thus T_n satisfies condition (1.2) with $k = \frac{1}{n}$. Now for any $f \in X$ fixed, let $S_n = I + T_n - f$. Then we have

$$\operatorname{Re}\langle S_n x - S_n\, y,\ j(x-y)\rangle \geq \varphi(\|x-y\|)\,\|x-y\| + \frac{1}{n}\,\|x-y\|^2\,. \tag{2.7}$$

Therefore S_n is $\varphi-$ strongly accretive (and $\varphi-$expansive). Let $x_0 \in X$ be such that the series

$$\sum_{n\in\mathbb{N}} (\varphi^{-1})^n(\|Tx_0 - f\| + \|x_0\|)$$

converges. Now consider the expression

$$S_n x_0 - x_0 = Tx_0 - f + \frac{x_0}{n}.$$

Then

$$\|S_n x_0 - x_0\| \leq \|Tx_0 - f\| + \|x_0\| \tag{2.8}$$

Hence from (2.8) and the hypothesis of the theorem, we conclude that the series $\sum_{m\in\mathbb{N}} (\varphi^{-1})^m(\|Sx_0 - x_0\|$ converges. Since $T+(\frac{n+1}{n})I$ has a closed range in X for each $n \in \mathbb{N}$, S_n has a closed range in X. Therefore by Lemma 2.3, S_n has at least one fixed point for each $n \in \mathbb{N}$, say, x_n. That is

$$x_n = S_n x_n = x_n + T_n x_n - f$$

from which we deduce that

$$Tx_n + \frac{x_n}{n} - f = 0. \tag{2.9}$$

Also from

$$Tx_n + \frac{x_n}{n} - f - \frac{x_1}{n} + \frac{x_1}{n} = 0$$

we obtain that

$$\|Tx_n - f\| \leq \frac{\|x_n - x_1\|}{n} + \frac{\|x_1\|}{n}. \tag{2.10}$$

Now we show that the sequence $\{x_n\}$ so obtained is bounded. If not, we may assume without loss of generality that $\lim_{n\to\infty} \|x_n\| = \infty$ and since $x_n = x_n - x_1 + x_1$, we conclude that $\lim_{n\to\infty} \|x_n - x_1\| = \infty$. Hence from (2.10) we deduce that

$$\lim_{n\to\infty} \frac{\|Tx_n - f\|}{\|x_n - x_1\|} = 0.$$

But from (2.7) we have

$$\varphi(\|x_n - x_1\|)\,\|x_n - x_1\| \leq \operatorname{Re}\langle S_1 x_n - S_1 x_1,\ j(x_n - x_1)\rangle$$

$$\leq \operatorname{Re}\langle (x_n + T_1 x_n - f) - (x_1 + T_1 x_1 - f),\ j(x_n - x_1)\rangle$$
$$\leq \operatorname{Re}\langle 2(x_n - x_1) + Tx_n - Tx_1,\ j(x_n - x_1)\rangle$$
$$\leq 2\left\|x_n - x_1\right\|^2 + \left\|x_n - x_1\right\|.\left\|Tx_n - Tx_1\right\|$$
$$\leq 2\left\|x_n - x_1\right\|^2 + \left\|x_n - x_1\right\|.\left(\left\|Tx_n - f\right\| + \left\|f - Tx_1\right\|\right).$$

This implies that

$$\frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} \leq 2 + \frac{\|Tx_n - f\|}{\|x_n - x_1\|} + \frac{\|f - Tx_1\|}{\|x_n - x_1\|}.$$

Now taking the limit superior on both sides of the above inequality along with (2.10) and the fact that $\lim_{n\to\infty}\|x_n - x_1\| = \infty$ we get

$$\limsup_{n\to\infty}\frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} \leq 2,$$

a contradiction to the hypothesis

$$\limsup_{n\to\infty}\frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} = \infty.$$

Therefore from (2.9) we deduce that

$$\|Tx_n - f\| = \frac{\|x_n\|}{n}$$

and from which it follows that $f \in \overline{R(T)}$. Now since $\overline{R(T)} = R(T)$, the equation $Tx = f$ has at least one solution in X.□

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References

[1] A.G. Kartatos, *Mapping theorems involving compact perturbations and compact resolvents of nonlinear operators in Banach spaces*, J. Math. Anal. Appl., **80**(1981), 130-146.

[2] M.O. Osilike, *Iterative solution of nonlinear equations of the $\varphi-$strongly accretive type*, J. Math. Anal. Appl., **200**(1996), 259-271.

[3] Z. Liu and S.M. Kang, *Convergence theorems for $\varphi-$ strongly accretive and $\varphi-$ hemicontractive operators*, J. Math. Anal. Appl., **253**(2001), 35-49.

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