SOME SURJECTIVITY CONDITIONS FOR NONLINEAR ACCRETIVE TYPE SINGLE-VALUED OPERATORS WITH A CLOSED RANGE IN BANACH SPACES

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Abstract. Our aim in this paper is to establish some new surjectivity conditions and study the existence of solutions of equation $Tx = f$ for operators $T$ in a Banach space $X$ that satisfy a general type of accretive condition.

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1. INTRODUCTION

The normalized duality mapping $J$ from a general Banach space $X$ into $2^{X^*}$ is given by

$$J(x) = \left\{ j \in X^* : \Re \langle x, j \rangle = \|x\|^2 = \|j\|^2 \right\},$$

where $X^*$ denotes the dual space of $X$ and $\langle ., . \rangle$ the generalized duality pairing.

An operator $T$ with domain $D(T)$ and range $R(T)$ in $X$ is said to be $\varphi-$generalized strongly accretive if there exists a $k \in [0, 1]$ and a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that for all $x, y \in D(T)$, there exists a $j(x - y) \in J(x - y)$ satisfying:

$$\Re \langle Tx - Ty, j(x - y) \rangle \geq \varphi(\|x - y\|) \|x - y\| - (1 - k) \|x - y\|^2$$

Remark 1. It is interesting to note that the class of operators satisfying (1.2) includes the class of $\varphi-$strongly accretive operators [3] corresponding to...
$k = 1$, that is, $T$ satisfies the condition:

$$\Re \langle Tx - Ty, j(x - y) \rangle \geq \varphi (\|x - y\|) \|x - y\|$$

for all $x, y \in X$ and $\varphi$ is as specified above.

The following examples justify the above remark.

**Example 1.** Let $X = \mathbb{R}$ with the usual norm $\| \cdot \|$. Define $T : [0, \infty) \rightarrow [0, \infty)$ by

$$Tx = x^2 - \frac{1}{2}x + \frac{1}{16}$$

Then $T$ is $\varphi$-generalized strongly accretive operator with $\varphi(s) = s^2$ and $k = \frac{1}{2}$ such that for every $f \in [0, \infty)$ the equation $Tx = f$ has at least one solution. In particular, $T$ has two fixed points (namely, $3 - 2\sqrt{2}$ and $3 + 2\sqrt{2}$). However, it can be easily verified that $T$ is not $\varphi$-strongly accretive.

Thus the class of $\varphi$-strongly accretive operators is a proper subset of the class of $\varphi$-generalized strongly accretive operators.

**Example 2.** Let $X = \mathbb{R}$ with the usual norm $\| \cdot \|$ and define $T : [0, \infty) \rightarrow [0, \infty)$ by

$$Tx = x^2 - \frac{1}{2}x + 2$$

Then it can be easily verified that $T$ is $\varphi$-generalized strongly accretive with $\varphi(s) = s^2$ and $k = \frac{1}{2}$ but $T$ is not $\varphi$-strongly accretive. In addition, the equation $Tx = x$ has no real solution for any $x \in [0, \infty)$. Thus, in general, the equation $Tx = f$ ($f \in X$) does not have a solution in $X$.

**Remark 2.** (i) For any non-negative real number $\lambda$ with $\lambda \geq 1 - k$, the operator $T + \lambda I$ (where $I$ denotes the identity operator on $X$) is a bijection when $T$ is continuous [4, Lemma 2.3].

Moreover, the relationship between the new class of operators and the class of $\varphi$-strongly pseudo-contractive operators considered by Liu and Kang [4] is that for any operator $T$ satisfying (1.2), $-T$ is $\varphi$-strongly pseudo-contractive and by [4, Lemma 2.2], $T + I$ has a unique zero in a real Banach space $X$ when $T$ is continuous.

(ii) Recall that an operator $T$ with domain $D(T)$ and range $R(T)$ in a normed space $X$ is $\varphi$-expansive (see [2]) if

$$\varphi (\|x - y\|) \leq \|Tx - Ty\|$$

(1.3)
for all \( x, y \in X \), where \( \varphi : [0, \infty) \to [0, \infty) \) is a strictly increasing function with \( \varphi(0) = 0 \).

It is known (see [2]) that \( \varphi - \text{expansive operators} \) are invertible but their inverse operators need not be defined on the whole space \( X \).

Notice that \( \varphi - \text{strongly accretive operators} \) are \( \varphi - \text{expansive} \) since they satisfy the condition
\[
\varphi (\|x - y\|) \|x - y\| \leq \|Tx - Ty\| \|x - y\|. \tag{1.4}
\]

Our aim in this paper is to establish some new surjectivity conditions for operators of type (1.2) with \( k \in (0, 1) \) and study the existence of solutions of equation \( Tx = f \) for operators \( T \). This is done, in particular, via some fixed point conditions for \( \varphi - \text{expansive operators} \) that we establish in this paper. For a detailed account of iterative approximations of fixed point and solution of operator equation \( Tx = f \), we refer to Berinde [1].

2. Main results

Now onward, \( \mathbb{N} \) will denote the set of natural numbers while \( R(T) \) will denote the range of an operator \( T \).

The following lemma gives us sufficient conditions for an operator of type (1.2) with \( k \in (0, 1) \) to have a zero.

**Lemma 2.1.** Let \( X \) be a general Banach space and \( T : X \to X \) be an operator of type (1.2) with \( \varphi(t) > t \) for all \( t \) sufficiently large and for \( 0 < k < 1 \). Assume that the following conditions are satisfied.

(2.1.1) \( R(T) = R(T) \).

(2.1.2) \( T + \lambda I \) is one to one for every \( \lambda > 0 \) sufficiently small.

(2.1.3) There exists \( r > 0 : \|T0\| < r \leq \lim inf_{\|x\| \to \infty} [\varphi(\|x\|) - \|x\|] \).

Then \( T \) has a zero in \( X \).

**Proof.** Suppose \( \mu = r - \|T0\| > 0 \) and let \( p \in B_{\mu}(0) \) be arbitrary, where \( B_{\mu}(0) = \{ x \in X : \|x\| < \mu \} \). Now consider the equation
\[
Tx + \frac{x}{n} = p \quad \text{for all} \quad n \in \mathbb{N}.
\]

By condition (2.1.2), \( x_n = (T + \frac{1}{n}I)^{-1}p \) exists for all sufficiently large \( n \). Hence
\[
\frac{1}{n} \|x_n\| = \|Tx_n - p\| \geq \|Tx_n - T0\| - \|T0 - p\|. \tag{2.1}
\]
But from (1.2) we deduce that
\[ \varphi(\|x - y\|) \leq (1 - k) \|x - y\| + \|Tx - Ty\|. \]

Hence (2.1) becomes
\[ \frac{1}{n} \|x_n\| \geq \varphi(\|x_n\|) - (1 - k) \varphi(\|x_n\|) - \|T0\| - \|p\| \]
from which we deduce that
\[ \varphi(\|x_n\|) - (1 - k + \frac{1}{n}) \|x_n\| \leq \|T0\| + \|p\|. \]\n
Since \( k \) is fixed, we may take \( n \in \mathbb{N} \) sufficiently large so that \((1 - k + \frac{1}{n}) < 1\). Therefore from the above inequality we obtain
\[ \varphi(\|x_n\|) - \|x_n\| \leq \varphi(\|x_n\|) - (1 - k + \frac{1}{n}) \|x_n\| \leq \|T0\| + \|p\|. \]

We now show that the sequence \( \{x_n\} \) is bounded. Suppose this is not the case. Without loss of generality we may assume that \( \lim_{n \to \infty} \|x_n\| = \infty \). Then by condition (2.3) we have that for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that
\[ r - \epsilon < \varphi(\|x_n\|) - \|x_n\| \]
for all \( n \geq N \) and by (2.2) this implies that
\[ r - \epsilon < \|T0\| + \|p\|. \]

Since \( \epsilon > 0 \) is arbitrary, the above inequality gives us
\[ r \leq \|T0\| + \|p\|. \] \tag{2.3}

Since \( p \in B_{\mu}(0) \) and \( \mu = r - \|T0\| \), from (2.3) we immediately obtain that
\[ r \leq \|T0\| + \|p\| < \|T0\| + \mu = r, \]
a contradiction. Hence the sequence \( \{x_n\} \) is bounded. Therefore from (2.1) we deduce that
\[ \lim_{n \to \infty} \|Tx_n - p\| = 0 \]
and by condition (2.1.1) we have that \( p \in R(T) \). Therefore \( B_{\mu}(0) \subset R(T) \). Consequently, there exists a point \( x_0 \in X \) such that \( Tx_0 = 0 \). \( \square \)

**Theorem 2.2.** Let \( X \) be a reflexive Banach space and \( T : X \to X \) be an operator that is weakly sequentially continuous and satisfies condition (1.2) with \( k = 0 \) and \( \varphi(t) > t \) for all \( t \) sufficiently large. Assume that
For any \( f \in X \), there exists \( r > 0 \) such that
\[
\|T0\| + \|f\| < r \leq \liminf_{\|x\| \to \infty} \phi(\|x\|) - \|x\|.
\]

(2.2.2) \( R(T + \lambda I) = R(T + \lambda I) \) and \( (T + \lambda I) \) is one to one for every \( \lambda > 0 \) sufficiently small.

Then the equation \( Tx = f \) has at least one solution for each \( f \in X \).

Proof. For any \( f \in X \) set \( S = I - T + f \). Then \( S \) satisfies the condition
\[
\text{Re} \langle Sx - Sy, j(x - y) \rangle \leq 2 \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|.
\]
for all \( x, y \in X \).

To show that \( S \) has a fixed point we consider the approximation
\[
S_n = \frac{n + 1}{n} I - S
\]
which is an operator of type (1.2) with \( k = \frac{1}{n} \) for every \( n \in \mathbb{N} \) since
\[
\text{Re} \langle S_n x - S_n y, j(x - y) \rangle \geq \phi(\|x - y\|) \|x - y\| - (1 - \frac{1}{n}) \|x - y\|^2.
\]
(2.4)
But
\[
\|S_n 0\| = \|S 0\| = \|f - T0\| \leq \|f\| + \|T0\|
\]
(2.5)
Hence by (2.2.1) above, the condition (2.1.3) of Lemma 2.1 is satisfied. Since \( S_n = (T + \frac{1}{n} I) - f \), it is clear by the first part of the condition (2.2.2) that the range of \( S_n \) is closed.

Moreover, for any \( m, n \in \mathbb{N} \) arbitrarily large we have
\[
\frac{1}{m} I + S_n = \frac{m + n}{m, n} I + T - f.
\]
Hence for any \( x, y \in X \) if
\[
\frac{1}{m} x + S_n x = \frac{1}{m} y + S_n y,
\]
then we obtain
\[
\frac{m + n}{m, n} x + Tx = \frac{m + n}{m, n} y + Ty.
\]
Therefore by the second part of the condition (2.2.2), \( \frac{1}{m} I + S_n \) is one to one. Since the conditions of Lemma 2.1 are satisfied for \( n \) sufficiently large, there exists \( x_n \in X \) such that \( S_n x_n = 0 \).
Now we show that the sequence \( \{x_n\} \) so obtained is bounded. Suppose the contrary and without loss of generality assume that
\[
\lim_{n \to \infty} \|x_n\| = \infty
\]
Then by (2.4) we get
\[
\varphi(\|x_n\|) \|x_n\| - (1 - \frac{1}{n}) \|x_n\|^2 \leq \|S_n 0\| \cdot \|x_n\|
\]
which implies that
\[
\varphi(\|x_n\|) \|x_n\| - \|x_n\|^2 \leq \|S_n 0\| \cdot \|x_n\|
\]
Therefore we immediately obtain that
\[
\varphi(\|x_n\|) - \|x_n\| \leq \|S_n 0\|.
\]
Now using (2.2.1) along with the above inequality we have that for any \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that
\[
r - \epsilon < \varphi(\|x_n\|) - \|x_n\| \leq \|S_n 0\| = \|S0\|.
\]
Since \( \epsilon > 0 \) is arbitrary, we have
\[
r \leq \|S0\|.
\]
Therefore by (2.5) and (2.2.1) we arrive at the contradiction
\[
r \leq \|S0\| < r.
\]
Consequently, the sequence \( \{x_n\} \) is bounded.
Now from the boundedness of \( \{x_n\} \) and the fact that
\[
S_n x_n = 0 = \frac{n+1}{n} x_n - S x_n
\]
we deduce that
\[
\lim_{n \to \infty} \|x_n - S x_n\| = 0.
\]
Therefore the sequence \( \{x_n - S x_n\} \) converges weakly to 0. Since \( X \) is reflexive and \( \{x_n\} \) is bounded, \( \{x_n\} \) has a weakly convergent subsequence say, \( \{x_{n_k}\} \). Further, notice that \( S \) is weakly sequentially continuous (since \( S = I - T + f \)), there exists \( x_0 \in X \) such that \( x_0 = S x_0 \) implying that \( T x_0 = f \). Hence \( x_0 \) is a solution of the equation \( T x = f \). \( \Box \)
Using a dual type of relationship with \( \varphi - \text{pseudocontractive operators} \), Liu and Kang [4] recently proved that for any continuous \( \varphi - \text{strongly accretive} \)
operators $T : X \to X$ on an arbitrary Banach space, the equation $Tx = f$ has a unique solution. In what follows, we are interested in using a fixed point theorem for $\varphi$-expansive operators $T$ and Remark 2(ii) in solving the equation $Tx = f$, where $T$ satisfies the condition (1.2) for $k = 0$.

First we have the following:

**Lemma 2.3.** Let $X$ be a general Banach space and $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$ and that its inverse function $\varphi^{-1}$ is upper semi-continuous. Suppose that $T : X \to X$ is a $\varphi$-expansive operator such that there exists $x_0 \in X$ and that the series $\sum_{n \in \mathbb{N}} (\varphi^{-1})^n(\|x_0 - Tx_0\|)$ converges. If the range $R(T)$ of $T$ is closed, then $T$ has at least one fixed point in $X$.

**Proof.** Set $y_0 = Tx_0$ and consider the approximation process

$$y_0 \in R(T), \quad y_{n+1} = T^{-1}y_n (n \geq 0). \quad (2.6)$$

We shall show that the sequence $\{y_n\}$ converges. First, we show by induction that

$$\|y_{n+1} - y_n\| \leq (\varphi^{-1})^n(\|x_0 - Tx_0\|).$$

Notice from (1.3) that

$$\varphi\left(\|T^{-1}Tx - T^{-1}Ty\|\right) \leq \|Tx - Ty\|$$

and since $\varphi$ is invertible, we have

$$\left(\|T^{-1}Tx - T^{-1}Ty\|\right) \leq \varphi^{-1}(\|Tx - Ty\|)$$

for all $x, y \in X$.

For $n = 1$, we have

$$\|y_2 - y_1\| = \|T^{-1}y_1 - T^{-1}y_0\| \leq \varphi^{-1}(\|y_1 - y_0\|)$$

$$= \varphi^{-1}(\|x_0 - Tx_0\|).$$

Therefore (2.6) is true for $n = 1$. Let it be true for $n = k$, i.e.

$$\|y_{k+1} - y_k\| \leq (\varphi^{-1})^k(\|x_0 - Tx_0\|)$$

Then for $n = k + 1$, we have

$$\|y_{k+2} - y_{k+1}\| = \|T^{-1}y_{k+1} - T^{-1}y_k\| \leq (\varphi^{-1})^k(\|y_{k+1} - y_k\|)$$

$$\leq (\varphi^{-1})(\varphi^{-1})^k(\|x_0 - Tx_0\|) = (\varphi^{-1})^{k+1}(\|x_0 - Tx_0\|)$$

and the result holds. Hence by induction (2.6) holds for all $n$. 
Now, for any $n, m \in \mathbb{N}$ with $n < m$, we have
\[
\|y_m - y_n\| \leq \|y_m - y_{m-1}\| + \|y_{m-1} - y_{m-2}\| + \ldots + \|y_{n+1} - y_n\|
\leq (\varphi^{-1})^{m-1}(\|x_0 - Tx_0\|) + (\varphi^{-1})^{m-2}(\|x_0 - Tx_0\|) + \ldots + (\varphi^{-1})^{n}(\|x_0 - Tx_0\|).
\]

By setting $S_n(\|x_0 - Tx_0\|) = \sum_{k=1}^{n}(\varphi^{-1})^k(\|x_0 - Tx_0\|)$ in the above inequality we obtain
\[
\|y_m - y_n\| \leq S_{m-1}(\|x_0 - Tx_0\|) - S_{n-1}(\|x_0 - Tx_0\|).
\]

Since the series $\sum_{n \in \mathbb{N}}(\varphi^{-1})^n(\|x_0 - Tx_0\|)$ converges, the sequence $\{S_n(\|x_0 - Tx_0\|)\}$ is a Cauchy sequence in $X$. Hence the above inequality implies that $\{y_n\}$ is a Cauchy sequence in $X$. Therefore it converges to some $y \in R(T)$ as $R(T)$ is closed. Hence we have
\[
\|y - T^{-1}y\| \leq \|y - y_n\| + \|y_n - T^{-1}y\|
\leq \|y - y_n\| + \|T^{-1}y_{n-1} - T^{-1}y\|
\leq \|y - y_n\| + \varphi^{-1}(\|y_{n-1} - y\|).
\]

By the upper semi-continuity of $\varphi^{-1}$, the above inequality implies that
\[
\|y - T^{-1}y\| \leq \limsup_{n \to \infty}(\|y - y_n\|) + \varphi^{-1}(\limsup_{n \to \infty}(\|y_{n-1} - y\|)) = 0,
\]
proving that $y = T^{-1}y$, that is, $y$ is a fixed point of $T^{-1}$.

Let $x \in X$ be such that $y = Tx$. Then $y = T^{-1}y$ implies that $Tx = x$ and the lemma is established. $\square$

We notice that the condition that $\sum_{n \in \mathbb{N}}(\varphi^{-1})^n(\|x_0 - Tx_0\|)$ is convergent is quite reasonable as all the linear operators defined at the origin satisfy it trivially.

**Theorem 2.4.** Let $X$ be a general Banach space and $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$ and whose inverse function $\varphi^{-1}$ is upper semi-continuous. Let $T : X \to X$ be an operator satisfying condition (1.2) with $k \in (0, 1)$ such that $I + T$ has a closed range. If for any $f \in X$, there exists $x_0 \in X$ such that the series $\sum_{n \in \mathbb{N}}(\varphi^{-1})^n(\|Tx_0 - f\|)$ converges, then the equation $Tx = f$ has at least one solution in $X$. 
Proof. Suppose $S = I + T - f$ for $f \in X$ fixed. Then $S$ is $\varphi$-strongly accretive as it satisfies the condition

$$\text{Re} \langle Sx - Sy, j(x - y) \rangle \geq \varphi(||x - y||) ||x - y|| + k ||x - y||^2.$$ 

Hence $S$ is $\varphi$-expansive (and strongly accretive).

Moreover, for any $x \in X$ we have $Sx - x = Tx - f$. Therefore by the given hypothesis of the theorem, there exists $x_0 \in X$ such that the series $\sum_{n \in \mathbb{N}} (\varphi^{-1})^n(||Tx_0 - f||$ converges. Therefore the series $\sum_{n \in \mathbb{N}} (\varphi^{-1})^n(||Sx_0 - x_0||$ also converges. Further, since $I + T$ has a closed range, it follows that $S$ also has a closed range. Therefore by Lemma 2.3, $S$ has at least one fixed point say, $x_1$. Then

$$x_1 = Sx_1 = x_1 + Tx_1 - f,$$

proving that $Tx_1 = f$ for some $x_1 \in X$. Hence the equation $Tx = f$ has a solution in $X$. □

Remark 3. If the operator $T$ in the above theorem is continuous, then the hypothesis that $I + T$ has a closed range can be dispensed with in view of Remark 2.

Theorem 2.5. Let $X$ be a general Banach space and $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\varphi(0) = 0$ such that

$$\limsup_{t \to \infty} \frac{\varphi(t)}{t} = \infty$$

and that its inverse function $\varphi^{-1}$ is upper semi-continuous. Assume that the operator $T : X \to X$ satisfies condition (1.2) with $k = 0$ and the operators $T + \frac{n + 1}{n} I$ have a closed range in $X$ for every $n \in \mathbb{N}$. If for every $f \in X$, there exists $x_0 \in X$ such that the series $\sum_{n \in \mathbb{N}} (\varphi^{-1})^n(||Tx_0 - f|| + ||x_0||)$ converges, then the equation $Tx = f$ has at least one solution in $X$.

Proof. For any $n \in \mathbb{N}$, set $T_n = T + \frac{1}{n} J$. Then for all $x, y \in X$ and for $j(x - y) \in J(x - y)$ we have

$$\text{Re} \langle T_n x - T_n y, j(x - y) \rangle \geq \varphi(||x - y||) ||x - y|| - \left(1 - \frac{1}{n}\right) ||x - y||^2.$$
Thus $T_n$ satisfies condition (1.2) with $k = \frac{1}{n}$. Now for any $f \in X$ fixed, let $S_n = I + T_n - f$. Then we have

$$\operatorname{Re} \langle S_n x - S_n y, j(x - y) \rangle \geq \varphi(\|x - y\|) \|x - y\| + \frac{1}{n} \|x - y\|^2. \quad (2.7)$$

Therefore $S_n$ is $\varphi-$ strongly accretive (and $\varphi-$expansive). Let $x_0 \in X$ be such that the series

$$\sum_{n \in \mathbb{N}} (\varphi^{-1})^n (\|T x_0 - f\| + \|x_0\|)$$

converges. Now consider the expression

$$S_n x_0 - x_0 = T x_0 - f + \frac{x_0}{n}.$$ 

Then

$$\|S_n x_0 - x_0\| \leq \|T x_0 - f\| + \|x_0\| \quad (2.8)$$

Hence from (2.8) and the hypothesis of the theorem, we conclude that the series $\sum_{m \in \mathbb{N}} (\varphi^{-1})^m (\|S x_0 - x_0\|)$ converges. Since $T + (\frac{n+1}{n}) I$ has a closed range in $X$ for each $n \in \mathbb{N}$, $S_n$ has a closed range in $X$. Therefore by Lemma 2.3, $S_n$ has at least one fixed point for each $n \in \mathbb{N}$, say, $x_n$. That is

$$x_n = S_n x_n = x_n + T_n x_n - f$$

from which we deduce that

$$T x_n + \frac{x_n}{n} - f = 0. \quad (2.9)$$

Also from

$$T x_n + \frac{x_n}{n} - f - \frac{x_1}{n} + \frac{x_1}{n} = 0$$

we obtain that

$$\|T x_n - f\| \leq \|x_n - x_1\| + \|x_1\|. \quad (2.10)$$

Now we show that the sequence $\{x_n\}$ so obtained is bounded. If not, we may assume without loss of generality that $\lim_{n \to \infty} \|x_n\| = \infty$ and since $x_n = x_n - x_1 + x_1$, we conclude that $\lim_{n \to \infty} \|x_n - x_1\| = \infty$. Hence from (2.10) we deduce that

$$\lim_{n \to \infty} \|T x_n - f\| = 0.$$ 

But from (2.7) we have

$$\varphi(\|x_n - x_1\|) \|x_n - x_1\| \leq \operatorname{Re} \langle S_1 x_n - S_1 x_1, j(x_n - x_1) \rangle$$
\[
\begin{align*}
\leq & \Re \langle (x_n + T_1x_n - f) - (x_1 + T_1x_1 - f), j(x_n - x_1) \rangle \\
\leq & \Re \langle 2(x_n - x_1) + Tx_n - Tx_1, j(x_n - x_1) \rangle \\
\leq & 2 \|x_n - x_1\|^2 + \|x_n - x_1\| \cdot \|Tx_n - Tx_1\| \\
\leq & 2 \|x_n - x_1\|^2 + \|x_n - x_1\| \cdot (\|Tx_n - f\| + \|f - Tx_1\|).
\end{align*}
\]
This implies that
\[
\frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} \leq 2 + \frac{\|Tx_n - f\|}{\|x_n - x_1\|} + \frac{\|f - Tx_1\|}{\|x_n - x_1\|}.
\]
Now taking the limit superior on both sides of the above inequality along with (2.10) and the fact that \(\lim_{n \to \infty} \|x_n - x_1\| = \infty\) we get
\[
\limsup_{n \to \infty} \frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} \leq 2,
\]
a contradiction to the hypothesis
\[
\limsup_{n \to \infty} \frac{\varphi(\|x_n - x_1\|)}{\|x_n - x_1\|} = \infty.
\]
Therefore from (2.9) we deduce that
\[
\|Tx_n - f\| = \frac{\|x_n\|}{n}
\]
and from which it follows that \(f \in R(T)\). Now since \(\overline{R(T)} = R(T)\), the equation \(Tx = f\) has at least one solution in \(X\). \(\Box\)

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