Fixed Point Theory, Volume 7, No. 1, 2006, 83-89
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

ON THE FIXED POINT PROPERTY FOR ONE-RELATOR GROUPS

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Abstract. We say that a group A has the fixed point property (FPP for short) if, whenever A acts on a tree X without inversions, A fixes at least one vertex of X. In this note we prove that subgroups of HNN groups, satisfying (FPP) are contained in conjugates of the base. As application we show that if $G = \langle t, b, c, \ldots ; r\rangle$ is a one-relator group, r is cyclically reduced, and if H is a subgroup of G such that H has the (FPP), then H is contained in a conjugate of a subgroup G_0 of G, where G_0 is a one-relator group whose defining relator has shorter length than r.

Key Words and Phrases: Groups acting on trees without inversions, fixed point property, HMM groups, one-relator groups.
2000 Mathematics Subject Classification: 20F05, 20E07, 20E08.

1. Introduction

In [2], Serre introduced the concept of groups acting on trees without inversions and proved that ([2], Proposition 21, page 59) if H is a subgroup of the free product $G_1 *_A G_2$ of the groups G_1 and G_2 with amalgamation subgroup A such that H has property (FPP), then H is contained in a conjugate of G_1 or in a conjugate of G_2. In this paper we generalize the above result to one-relator groups.

We begin by giving preliminary definitions. By a graph X we understand a pair of disjoint sets $V(X)$ and $E(X)$ with $V(X)$ non-empty, together with a mapping $E(X) \to V(X) \times V(X)$, $y \to (o(y), t(y))$, and a mapping $E(X) \to E(X)$, $y \to \overline{y}$, satisfying the conditions that $\overline{y} \neq y$, $\overline{\overline{y}} = y$, and $o(\overline{y}) = t(y)$, for all $y \in E(X)$. If $x \in E(X)$, $o(x) = t(x)$, then x is called a loop. There are obvious definitions of subgraphs, trees, morphisms of graphs and,

$Aut(X)$, the set of all automorphisms of the graph X which is a group under the composition of morphisms of graphs. For more details see [2]. We say that a group G acts on a graph X without inversions, if there is a group homomorphism $\phi : G \to Aut(X)$. If $x \in X$ (vertex or edge) and $g \in G$, we write $g(x)$ for $(\phi(g))(x)$. If $y \in E(X)$ and $g \in G$, then $g(o(y)) = o(g(y))$, $g(t(y)) = t(g(y))$, $g(\overline{y}) = \overline{g(y)}$, and $g(y) \neq \overline{y}$ for all $g \in G$ and all $y \in E(X)$ may occur. That is, G acts without inversions on X.

We have the following notations related to the action of the group G on the graph X.

(1) If $x \in X$ (vertex or edge), we define $G_x = \{g \in G : \ g(x) = x\}$ the stabilizer of x which is a subgroup of G. It is clear that if $y \in E(X)$, and $u \in \{o(y), t(y)\}$, then $G_{\overline{y}} = G_y$, and $G_y \leq G_u$. Moreover, if H is a subgroup of G, then $H_x = H \cap G_x$.

(2) We define X^G to be the set of all elements of X fixed by all elements of G. It is clear that $X^G = \{x \in X : \ G_x = G\}$. It is noted that [2, page 58] if X is a tree and $X^G \neq \emptyset$, then X^G is a subtree of X, and the subgroup H of G has (FPP) if and only if $X^H \neq \emptyset$. We end this section with the following proposition.

Proposition 1.1. *Subgroups of groups acting on trees without inversions satisfying (FPP) are contained in vertex stabilizers.*

Proof. Let G be a group acting on a tree X wityhout inversions and H be a subgroup of G satisfying (FPP). The action of H on X implies that there exists a vertex v of X such that $H = H_v = H \cap G_v$. This implies that H is contained in G_v. This completes the proof.

2. HNN groups

HNN groups have appeared in [1, page 179]. In this paper we associate trees in which HNN groups act without inversions, and then find the structure of subgroups of HNN groups satisfying (FPP). Then we use such result to form (FPP) for one-relator groups. Now we use the terminology and notation of [1]. Let G be a group, and I be an indexed set. Let $\{A_i : \ i \in I\}$ and $\{B_i : \ i \in I\}$ be families of subgroups of G. For each $i \in I$, let $\phi_i : A_i \to B_i$ be an onto isomorphism. The group G^* with the presentation

$$G^* = \langle G, t_i | relG, \ t_i A_i t_i^{-1} = B_i, \ i \in I\rangle$$

is called an HNN group with base G and associated pair (A_i, B_i), $i \in I$ of subgroups of G, where $\langle G|relG\rangle$ stands for any presentation of G and, $t_iA_it_i^{-1} = B_i$ stands for the set of relations $t_iw(a)t_i^{-1} = w(\phi_i(a))$, where $w(a)$ and $w(\phi_i(a))$ are words in the generating symbols of the presentation of G of values a and $\phi_i(a)$ respectively, where a runs over the generators of A_i.

For each $i \in I$, let

$$e_i = \pm 1, \quad C_i^{e_i} = \begin{cases} A_i & \text{if} \quad e_i = -1 \\ B_i & \text{if} \quad e_i = 1 \end{cases}$$

and, $\phi_i^{e_i} : C_i^{-e_i} \to C_i^{e_i}$ be the ismorphisms defined above.

Therefore, if $a \in C_i^{-e_i}$, then $\phi_i^{e_i}(a) \in C_i^{e_i}$.

It is proved in [1, page 182] that G is embedded in G^* (the embedding theorem for HNN groups) and every element g of G^*, $g \neq 1$ (Britton's lemma for HNN groups) can be written as the value of a reduced word w of G^*. That is, $g = w = g_0t_1^{e_1}g_1t_2^{e_2}\dots t_n^eg_n$, where $g_0, g_i \in G$, $e_i = \pm 1$, for $i = 1, \dots, n$ such that w contains no subword of the form $t_iat_i^{-1}$, $a \in C_i^{-e_i}$.

Note. From above it is clear that $t_i^{e_i}a = \phi_i^{e_i}(a)t_i^{e_i}$, $a \in C_i^{-e_i}$.

Britton's lemma for HNN groups implies the following proposition. The proof is clear.

Proposition 2.1. *Every nontrivial element of G^* is the value of a reduced word of G^*. Moreover, if $g \in G^*$, $g \neq 1$ is the value of the reduced words $f_0s_1^{k_1}f_1s_2^{k_2}\dots s_m^{k_m}$ and $g_0t_1^{e_1}g_1t_2^{e_2}\dots t_n^eg_n$, then $m = n$, $k_i = e_i$, $s_i = t_i$, and there exist unique elements $a_j \in C_j^{e_j}$, $j = 1, \dots, n$ such that $f_0 = g_0\phi_1^{e_1}(a_1^{-1})$, $f_i = a_ig_i\phi_{i+1}^{e_{i+1}}(a_{i+1}^{-1})$, $i = 1, \dots, n-1$, and $f_n = a_ng_n$.*

The following lemma is essential for the proof of the main result of this paper.

Lemma 2.2. *A group is an HNN group if and only if there exists a tree on which the group acts without inversions and is transitive on the set of the vertices such that the stabilizer of any vertex is a conjugate of the base and the stabilizer of any edge is a conjugate of an associative subgroup of the base.*

Proof. If a group G acts on a tree X without inversions such that G is transitive on $V(X)$, then G has exactly one vertex orbit. Then the quotient graph X/G is a loop, and Corollary 2 of [2, page 55] implies that G is an HNN group.

Conversely, let $G^* = \langle G, t_i | relG,\ t_i A_i t_i^{-1} = B_i,\ i \in I\rangle$ be the HNN group defined above. We need to find a tree X (the standard tree associated with G^*) on which G^* acts without inversions such that G^* is transitive on $V(X)$ and for every vertex v of X and every edge x of X, the stabilizer G^*_v of v is a conjugate of G and the stabilizer G^*_x of x is a conjugate of A_i, $i \in I$. The proof of the second part of lemma follows from the following propositions.

Proposition 2.3. *Let* $V = \{gG :\ g \in G^*\}$, $E = \{(gC_i^{e_i}, t_i^{e_i}) :\ i \in I,\ g \in G^*\}$, *and* $X = V \cup E$. *Then X forms a graph.*

Proof. Let $V(X) = V$, and $E(X) = E$.

For the edge $y_i = (gC_i^{e_i}, t_i^{e_i})$ define $o(y_i) = gG$, $t(y_i) = gt_i^{e_i}G$, and $\overline{y}_i = (gt_i^{e_i}C_i^{-e_i}, t_i^{-e_i})$. Then $o(\overline{y}_i) = gt_i^{e_i}G = t(y_i)$, $t(\overline{y}_i) = gt_i^{e_i}t_i^{-e_i}G = gG = o(y_i)$, and $\overline{\overline{y}}_i = \overline{(gt_i^{e_i}C_i^{-e_i}, t_i^{-e_i})} = (gt_i^{e_i}t_i^{-e_i}C_i^{e_i}, t_i^{e_i}) = (gC_i^{e_i}, t_i^{e_i}) = y_i$.

From the above we see that X forms a graph. This completes the proof.

Proposition 2.4. *Let g be an element of* G^*, $g \neq 1$. *Let* $g = g_0 t_1^{e_1} g_1 t_2^{e_2} \ldots t_n^{e} g_n$ *be a reduced word of* G^*, *and for each* i, $1 \leq i \leq n$, *let* y_i *be the edge* $(g_0 t_1^{e_1} g_1 t_2^{e_2} \ldots t_{i-1}^{e_{i-1}} g_{i-1} C_i^{e_i}, t_i^{e_i})$. *Then* $y_1, \ldots, y_n$ *is a reduced path in X joining the vertices G and gG.*

Proof. First we show that $o(y_1) = G$, $t(y_i) = o(y_{i+1})$, and $t(y_n) = gG$.

Now $o(y_1) = o(g_0 C_1^{e_1}, t_1^{e_1}) = g_0 G = G$, because $g_0 \in G$,

$$t(y_i) = t(g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} C_i^{e_i}, t_i^{e_i}) = g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} t_i^{e_i} G$$

$$= g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} t_i^{e_i} g_i G, \text{ because } g_i \in G$$

$$= o(g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} t_i^{e_i} C_{i+1}^{e_{i+1}}, t_{i+1}^{e_{i+1}}) = o(y_{i+1}),$$

and

$$t(y_n) = t(g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{n-1}^{e_{n-1}} C_n^{e_{n_i}}, t_n^{e_n}) = g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_n^{e_n} g_n G = gG.$$

This implies that $y_1, \ldots, y_n$ is a path in X joining the vertices G and gG.

Now we show that the path $y_1, \ldots, y_n$ is reduced.

If for i, $1 \leq i \leq n-1$ we have $y_{i+1} = \overline{y}_i$, then

$$(g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e-1} g_{i-1} t_i^{e_i} g_i C_{i+1}^{e_{i+1}}, t_{i+1}^{e_{i+1}})$$

$$= (g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} t_i^{e_1} C_i^{-e_i}, t_i^{-e_i}).$$

This implies that $t_{i+1}^{e_{i+1}} = t_i^{-e_i}$, $C_{i+1}^{e_{i+1}} = C_i^{-e_i}$, and $g_i \in C_i^{-e_i}$. Then g contains the subword $t_i^{e_i} g_i t_i^{-e_i}$, $g_i \in C_i^{-e_i}$. This contradicts the assumption that g is reduced. Hence the path $y_1, \ldots, y_n$ is a reduced path in X joining the vertices G and gG. This completes the proof.

Remark. $y_1, \ldots, y_n$ is called the reduced path induced by the word $g_0 t_1^{e_1} g_1 t_2^{e_2} \ldots t_n^e g_n$.

Proposition 2.5. *Reduced words of G^* of the same value induce the same reduced path.*

Proof. Let w_1 and w_2 be two reduced words of G^* of the same value. Let $w_1 = g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_n^{e_n} g_n$. Then by Proposition 2.1, $w_2 = f_0 t_1^{e_1} f_1 t_2^{e_2} f_2 \ldots t_n^{e_n} f_n$, where $f_0 = g_0 \phi_1^{e_1}(a_1^{-1})$, $f_i = a_i g_i \phi_{i+1}^{e_{i+1}}(a_{i+1}^{-1})$, $i = 1, \ldots, n-1$, and $f_n = a_n g_n$, $a_j \in C_j^{e_j}$, $j = 1, \ldots, n$. Then $y_1, \ldots, y_n$ is the reduced path induced by w_1, and $x_1, \ldots, x_n$ is the reduced path induced by w_2, where y_i is the edge $(g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 \ldots t_{i-1}^{e_{i-1}} g_{i-1} C_i^{e_i}, t_i^{e_i})$, and x_i is the edge $(f_0 t_1^{e_1} f_1 t_2^{e_2} f_2 \ldots t_{i-1}^{e_{i-1}} f_{i-1} C_i^{e_i}, t_i^{e_i})$, $1 \le i \le n$. Now we show that $x_i = y_i$. Now

$$x_i = (f_0 t_1^{e_1} f_1 t_2^{e_2} f_2 \ldots t_{i-1}^{e_{i-1}} f_{i-1} C_i^{e_i}, t_i^{e_i})$$

$$= (g_0 \phi_1^{e_1}(a_1^{-1}) t_1^{e_1} a_1 g_1 \phi_2^{e_2}(a_2^{-1}) t_2^{e_2} a_2 g_2 \phi_3^{e_3}(a_3^{-1}) \ldots t_{i-1}^{e_{i-1}} a_i g_i \phi_i^{e_i}(a_i^{-1}) C_i^{e_i}, t_i^{e_i})$$

$$= (g_0 t_1^{e_1} a_1^{-1} a_1 g_1 t_2^{e_2} a_2^{-1} a_2 g_2 t_3^{e_3} a_3^{-1} \ldots t_{i-1}^{e_{i-1}} a_i g_i \phi_i^{e_i}(a_i^{-1}) C_i^{e_i}, t_i^{e_i}), \text{ see note above}$$

$$= (g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 t_3^{e_3} \ldots t_{i-1}^{e_{i-1}} a_i g_i \phi_i^{e_i}(a_i^{-1}) C_i^{e_i}, t_i^{e_i})$$

$$= (g_0 t_1^{e_1} g_1 t_2^{e_2} g_2 t_3^{e_3} \ldots t_{i-1}^{e_{i-1}} g_i C_i^{e_i}, t_i^{e_i}), \text{ because } \phi_i e_i(a_i^{-1}) \in C_i^{e_i}$$

$$= y_i.$$

This completes the proof.

Proposition 2.6. *Every reduced path in X joining the vertices $u = G$, and $v = gG$, $g \neq 1$ is induced by a reduced word of G^* of value g.*

Proof. Let $z_1, z_2, \ldots, z_n$ be a reduced path in X joining u and v. Then $z_i = (h_i C_i^{e_i}, t_i^{e_i})$, $h_i \in G^*$, $i = 1, \ldots, n$.

Now $o(z_1) = h_1 G = G$, $t(z_i) = o(z_{i+1})$, and $t(z_n) = gG$ imply that $h_1 \in G$, $h_i t_i^{e_i} G = h_{i+1} G$, and $h_n t_n^{e_n} G = gG$. Then $h_1 = g_0$, $h_{i+1} = h_i t_i^{e_i} g_i$, and $g = h_n t_n^{e_n} g_n$, where $g_j \in G$ for $j = 1, \ldots, n-1$. Therefore g is the value of the word $w = g_0 t_1^{e_1} g_1 t_2^{e_2} \ldots t_n^e g_n$. If w is not reduced, then for some i, $i = 1, \ldots, n-1$ we have $g_i \in C_i^{e_i}$, and $t_{i+1}^{e_{i+1}} = t_i^{-e_i}$. Then $(h_{i+1} C_{i+1}^{e_{i+1}}, t_{i+1}^{e_{i+1}}) = (h_i t_i^{e_i} g_i C_i^{-e_i}, t_i^{-e_i}) = (h_i t_i^{e_i} C_i^{-e_i}, t_i^{-e_i})$. This implies that $z_{i+1} = \overline{z}_i$. This contradicts the assumption that path $z_1, z_2, \ldots, z_n$ be a reduced. This completes the proof.

Proposition 2.7. *X is a tree.*

Proof. It is clear that X contains no loops. For, if $y = (gC_i^{e_i}, t_i^{e_i}) = \overline{y} = (gt_i^{e_i}C_i^{-e_i}, t_i^{-e_i})$, then $t_i^{e_i} = t_i^{-e_i}$. This leads a contradiction. Let u and v be two vertices of X. Now we show that there is exactly one reduce path in X joining u and v. Clearly, $u = aG$ and $v = bG$, where a and b are two elements of G^*. If $u = v$, the case is clear. Let $u \neq v$ and $g = a^{-1}b$. Propositions 2.4, 2.5 and 2.6 imply that there is a unique reduced path $y_1, \ldots, y_n$ in X joining the vertices G and gG. Then $a(y_1), \ldots, a(y_n)$ is the unique reduced path in X joining the vertices u and v. This completes the proof.

Proposition 2.8. *G^* acts on X without inversions such that G^* is transitive on $V(X)$, and if v is the vertex aG, and y is the edge $y = (gC_i^{e_i}, t_i^{e_i})$, then $G_v^* = aGa^{-1}$, and $G_y^* = gC_i^{e_i}g^{-1}$.*

Proof. Let $g' \in G^*$. Then $g'(v) = g'aG$, and $g'(y) = (g'gC_i^{e_i}, t_i^{e_i})$. The action of G^* on the vertices of X is transitive because for any two vertices aG and bG of X we have $ba^{-1}(aG) = bG$. That is, the element ba^{-1} of G^* maps the vertex aG to the vertex bG. Then there is exactly one G^* vertex orbit. G^* acts on X without inversions, because if $g'(y) = \overline{y}$, then $(g'hC_i^{e_i}, t_i^{e_i}) = (ht_i^{e_i}C_i^{-e_i}, t_i^{-e_i})$. Then $t_i^{e_i} = t_i^{-e_i}$, and this is a contradiction. If $g(v) = v$, then $gaG = aG$. This implies $g \in G$. So $G_v^* = aGa^{-1}$. Similarly we can show that $G_y^* = gC_i^{e_i}g^{-1}$. This completes the proof.

Remark. The tree X constructed above will be called the standard tree of the HNN group G^*.

The main result of this section is the following theorem.

Theorem 2.9. *Subgroups of HNN groups with property (FPP) are contained in conjugates of the base.*

Proof. Let H be a subgroup of the HNN group $G^* = \langle G, t_i | relG,\ t_iA_it_i^{-1} = B_i,\ i \in I\rangle$ with property (FPP). Then the action of G^* on the standard tree X of G^* implies that H acts on X without inversions. As H has property (FPP), Propositions 1.1 and 2.8 imply that there exist a vertex $v = aG$ of X such that $H \subseteq G_v^* = aGa^{-1}$. This completes the proof.

By taking G in the HNN group $G^* = \langle G, t_i | relG,\ t_iA_i^{-1} = B_i,\ i \in I\rangle$ to be trivial yields that G^* is a free group generated by t_i, $i \in I$. This leads the following corollary.

Corollary 2.10. *Subgroups of free groups with property (FPP) are trivial.*

3. Application

As an application of Theorem 2.9, we get the following theorem.

Theorem 3.1. *Subgroups of one-relator groups with property (FPP) are contained in conjugates of one-relator subgroups of shorter relators.*

Proof. Let $G = \langle X|r \rangle$ be a one-relator presentation group generated by the set X, and of one relator r, where r is a cyclically reduced, and contains at least two different letters from X. Let K be a subgroup of G with property (FPP). By Theorem 5.1 of [1, pages 198, 294], G can be embedded in an HNN group $\langle H, t|rel(H), tUt^{-1} = V \rangle$, where U and V are isomorphic free groups, and H is a one-relator group, $H \cong \langle X'|r' \rangle$ where r' is cyclically reduced, and r' is shorter than r. Then Theorem 2.9 implies that K is contained in a conjugate of H. This completes the proof.

Acknowledgement. The author would like to thank the referee for his sincere evaluation and constructive comments which improved the paper considerably.

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Received 09.02.2006; Revised 15.04.2006