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POSITIVE SOLUTIONS OF SINGULAR SUBLINEAR SECOND-ORDER THREE-POINT BOUNDARY VALUE PROBLEMS

RUYUN MA* and DONAL O'REGAN**

*Department of Mathematics
Northwest Normal University
Lanzhou 730070, P R China

**Department of Mathematics
National University of Ireland
Galway, Ireland
E-mail: donal.oregan@nuigalway.ie

Abstract. We give some necessary and sufficient conditions for the existence of C or C^1 positive solutions of the singular boundary value problem

$$x''(t) + p(t)x^{\lambda}(t) = 0, \qquad t \in (0,1)$$
$$x(0) = 0, \ x(1) = \alpha x(\eta)$$

where $\eta \in (0,1)$, $\alpha \in (0,1]$ and $\lambda \in (0,1)$ are given, $p : (0,1) \to [0,\infty)$ can be singular at both ends $t = 0$ and $t = 1$. The main tool is the method of lower and upper solutions for singular three-point boundary value problems.

Key Words and Phrases: Singular boundary value problem, Existence, Schauder fixed point theorem, Green's function, Lower and upper solution.
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1. Introduction

Singular nonlinear two-point boundary value problems have been extensively studied in the literature, see [1, 2, 3, 4-5, 6]. Also the existence and multiplicity of solutions of non-singular multi-point boundary value problems

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have been studied by many authors, see [7, 8, 9, 10] and the references therein. However for singular multi-point boundary value problems, the research has proceeded very slowly. To the best of our knowledge, only [11] developed the monotone iterative technique for a class of singular three-point boundary value problems.

In [6], Zhang considered the existence of positive solutions of the singular boundary value problem

$$x''(t) + p(t)x^{\lambda}(t) = 0, \qquad t \in (0,1) \tag{1.1}$$

$$x(0) = 0, \quad x(1) = 0 \tag{1.2}$$

under the assumption

(A) $p \in C(0,1)$, $p(t) \geq 0$ on $(0,1)$, and $\lambda \in (0,1)$.

He proved the following

Theorem A Suppose (A) holds. Then (1.1),(1.2) has $C[0,1]$ positive solutions if and only if

$$0 < \int_0^1 t(1-t)p(t)dt < \infty. \tag{1.3}$$

Theorem B Suppose (A) holds. Then (1.1),(1.2) has $C^1[0,1]$ positive solutions if and only if

$$0 < \int_0^1 t^{\lambda}(1-t)^{\lambda}p(t)dt < \infty. \tag{1.4}$$

Of course a natural question is what would happen if (1.2) is replaced by the three-point boundary condition

$$x(0) = 0, \quad x(1) = \alpha x(\eta) \tag{1.5}$$

where α and η are given constants satisfying

(H) $\alpha \in (0,1]$, $\eta \in (0,1)$.

In our discussion, by a $C[0,1]$ *solution* of (1.1),(1.5) we mean a function $x(t) \in C[0,1] \cap C^2(0,1)$ which satisfies (1.5) as well as Eq. (1.1) on $(0,1)$. If in addition there is a solution $x(t) \in C^1[0,1]$, i.e. $x'(0+)$ and $x'(1-)$ both exist, we call it a $C^1[0,1]$ *solution*. We say a solution $x(t)$ is a *positive solution* if $x(t) > 0$ for $t \in (0,1)$.

The main results of this paper are as follows

Theorem 1.1 Suppose that (A) and (H) hold. Then (1.1),(1.5) has $C[0,1]$ positive solutions if and only if

$$0 < \int_0^1 t(1-t)p(t)dt < \infty. \tag{1.6}$$

Theorem 1.2 Suppose that (A) and (H) hold. Then (1.1),(1.5) has $C^1[0,1]$ positive solutions if and only if

$$0 < \int_0^1 t^\lambda p(t)dt < \infty. \tag{1.7}$$

To prove our main results, we develop the method of lower and upper solutions for singular three-point boundary value problems in Section 2.

We will use the classical Banach spaces $C[0,1]$, $C^k[0,1]$, $L^1[0,1]$ and $L^\infty[0,1]$. We denote by $AC[a,b]$ the space of all absolute continuous functions on $[a,b]$, and denote

$$AC^k[a,b] = \{u \in C^k[a,b] \mid u^{(k)} \in AC[a,b]\}.$$

Clearly $AC^0[a,b] = AC[a,b]$. Let

$$AC_{\text{loc}}(0,1) = \{u \mid u|_{[c,d]} \in AC[c,d] \text{ for every compact interval } [c,d] \subset (0,1)\},$$

$$L^1_{\text{loc}}(0,1) = \{u \mid u|_{[c,d]} \in L^1[c,d] \text{ for every compact interval } [c,d] \subset (0,1)\}.$$

Let E be the Banach space

$$E = \{y \in L^1_{\text{loc}}(0,1) \mid t(1-t)y(t) \in L^1[0,1]\}$$

equipped with the norm

$$||y||_E = \int_0^1 t(1-t)|y(t)|dt.$$

Lemma 1.1([2, Lemma 2.1]) Suppose that $\phi \in E$. Then
(i) $\int_0^t s\phi(s)ds, \int_t^1 (1-s)\phi(s)ds \in L^1[0,1]$;
(ii) $\lim_{t\to 0} t\int_t^1 (1-s)\phi(s)ds = 0,\ \lim_{t\to 1}(1-t)\int_0^t s\phi(s)ds = 0.$

2. Method of Lower and Upper Solutions

Consider the three-point boundary value problem

$$x'' + f(t,x) = 0, \qquad t \in (0,1) \tag{2.1}$$

$$x(0) = a, \ x(1) - \alpha x(\eta) = b \tag{2.2}$$

where $f : D \to \mathbb{R}$ is a continuous function with $D \subset (0,1) \times \mathbb{R}$, $\eta, \alpha, a, b \in \mathbb{R}$ are given constants satisfying $\eta \in (0,1)$ and $\alpha \in (0, \frac{1}{\eta})$.

Let $v \in C[0,1] \cap C^2(0,1)$ satisfying the following conditions: $(t, v(t)) \in D$, for all $t \in (0,1)$, and

$$\begin{aligned} &v''(t) + f(t, v(t)) \geq 0, \qquad t \in (0,1) \\ &v(0) \leq a, \quad v(1) - \alpha v(\eta) \leq b. \end{aligned} \tag{2.3}$$

In this case, we say $v(t)$ is a *lower solution* for problem (2.1),(2.2). The definition of an *upper solution* $w(t)$ for problem (2.1),(2.2) is given in a completely similar way, just reversing the above inequalities. Also, if $v, \ w \in C[0,1]$ are such that $v(t) \leq w(t), \ t \in [0,1]$, we define the set

$$D_v^w := \{(t,z) \in (0,1) \times \mathbb{R} : v(t) \leq z \leq w(t)\}.$$

The following result can be regarded as a generalization of [3, Theorem 1].

Theorem 2.1. Let

$$0 < \alpha < \frac{1}{\eta}$$

and let v, w be, respectively, a lower solution and an upper solution for problem (2.1),(2.2) such that

(a_1) $v(t) \leq w(t)$ for all $t \in [0,1]$

and suppose that

(a_2) $D_v^w \subset D$.

Assume also that there is a function $h \in C((0,1), \mathbb{R}^+)$ such that

(a_3) $|f(t,z)| \leq h(t)$ for all $(t,z) \in D_v^w$ and

(a_4) $\int_0^1 s(1-s)h(s)ds < \infty$.

Then problem (2.1),(2.2) has at least one solution $\tilde{x} \in C[0,1] \cap C^2(0,1)$ such that

$$\alpha(t) \leq \tilde{x}(t) \leq \beta(t), \qquad \text{for all } t \in (0,1). \tag{2.4}$$

Proof. Define an auxiliary function

$$f^*(t,x) := \begin{cases} f(t,v(t)), & x < v(t) \\ f(t,x), & v(t) \leq x \leq w(t) \\ f(t,w(t)), & x > w(t). \end{cases} \tag{2.5}$$

By (a_2) and the definition of f^* it can be easily checked that $f^* : (0,1) \times \mathbb{R} \to \mathbb{R}$ is continuous and, by (a_3), it satisfies

$$|f^*(t,x)| \leq h(t), \quad \text{for all } (t,x) \in (0,1) \times \mathbb{R}. \tag{2.6}$$

Consider now the problem

$$x'' + f^*(t,x) = 0, \qquad t \in (0,1) \tag{2.7}$$

$$x(0) = a, \; x(1) - \alpha x(\eta) = b. \tag{2.8}$$

We claim that if $x(t)$ is any solution of (2.7),(2.8), then

$$v(t) \leq x(t) \leq w(t), \qquad t \in [0,1]. \tag{2.9}$$

and hence $x(t)$ is a solution of (2.1),(2.2) which satisfies condition (2.4).

In fact, suppose on the contrary that there exists $t^* \in (0,1)$ such that $x(t^*) < v(t^*)$. By continuity, we know that one of the following cases must occur.

Case I There exists a maximal interval $(r,s) \subset (0,1]$ such that $t^* \in (r, \; s)$, and

$$x(r) = v(r), \; x(s) = v(s), \; x(t) < v(t) \; \text{ for all } t \in (r,s).$$

Case II There exists $r \in (0,1)$ such that $t^* \in (r, \; 1]$, and

$$x(r) = v(r), \; x(t) < v(t) \; \text{ for all } t \in (r,1]. \tag{2.10}$$

In Case I, we have that $f^*(t,x(t)) = f(t,v(t))$ for $t \in (r,s)$, and accordingly

$$x''(t) + f(t,v(t)) = 0, \qquad t \in (r,s).$$

On the other hand, as v is a lower solution for (2.1), (2.2), we also have

$$v''(t) + f(t,v(t)) \geq 0, \qquad t \in (r,s)$$

$$v(0) \le a, \ v(1) - \alpha v(\eta) \le b.$$

Then, setting

$$z(t) = v(t) - x(t)$$

we obtain

$$z(r) = z(s) = 0, \ z'' \ge 0 \quad \text{for all } t \in (r, s).$$

Since the graph of z on (r, s) is concave up, we conclude that

$$z(t) \le 0, \quad \text{for all } t \in (r, s),$$

that is $v(t) \le x(t)$ for all $t \in (r, s)$, a contradiction with the assumption $v(t^*) > x(t^*)$. Thus Case I cannot occur.

In case II, we must have the following three subcases:

Subcase (i). $r \in (\eta, 1)$;

Subcase (ii). $r = \eta$;

Subcase (iii). $r \in (0, \eta]$.

If $r \in (\eta, 1)$, then we know from $z(1) = v(1) - x(1) > 0$ and $z(1) \le \alpha z(\eta)$ that $z(\eta) > 0$. Thus there exists a maximum open interval $(r_1, s_1) \subset (0, r)$ such that

$$x(r_1) = v(r_1), \ x(s_1) = v(s_1), \ x(t) < v(t) \ \text{ for all } t \in (r_1, s_1). \tag{2.11}$$

By the same method used in the proof of Case I, we can get a desired contradiction, and accordingly, Subcase (i) cannot occur.

If $r = \eta$, then $z(\eta) = 0$. Now we have from $z(1) \le \alpha z(\eta)$ that $z(1) \le 0$, which contradicts (2.10). So Subcase (ii) cannot occur.

If $r \in (0, \eta)$, then $f^*(t, x(t)) = f(t, v(t))$ for $t \in (r, 1]$, and accordingly

$$x''(t) + f(t, v(t)) = 0, \qquad t \in (r, 1].$$

On the other hand, as v is a lower solution for (2.1), (2.2), we also have

$$v''(t) + f(t, v(t)) \ge 0, \qquad t \in (0, 1)$$

and

$$v(0) \le a, \ v(1) - \alpha v(\eta) \le b.$$

Moreover

$$z''(t) \geq 0, \qquad t \in (r, 1], \tag{2.12}$$

$$z(r) = 0, \quad z(1) \leq \alpha z(\eta). \tag{2.13}$$

Now $z(1) > 0$ and the fact $z(1) \leq \alpha z(\eta)$ yields

$$z(\eta) > 0 \tag{2.14}$$

and this together with the assumption that $\alpha \in (0, \frac{1}{\eta})$ yields

$$\frac{z(1)}{1} \leq \alpha z(\eta) < \frac{z(\eta)}{\eta}. \tag{2.15}$$

This is impossible since (2.12) implies that the graph of z is concave up on $(r, 1]$. (Note if $s = \frac{\eta - r}{1 - r}$ then $z(\eta) = z((1-s) \cdot r + s \cdot 1) \leq \frac{\eta - r}{1 - r} z(1) \leq \eta z(1)$). Thus subcase (iii) cannot occur.

To sum up, we have shown that $x(t) \geq v(t)$ for all $t \in [0, 1]$. The same argument, with obvious changes, works to show that $x(t) \leq w(t)$ in $[0, 1]$.

Then, the claim is verified and now we must prove that problem (2.7), (2.8) has at least one solution.

Let $G(t, s)$ be the Green's function of the second-order linear boundary value problem

$$-u''(t) = 0, \qquad t \in (0, 1) \tag{2.16}$$

$$u(0) = u(1) = 0 \tag{2.17}$$

which can be explicitly given by

$$G(t, s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.18}$$

For each $y \in C[0, 1]$, we define

$$\begin{aligned} (Ty)(t) = & \psi(t) + \int_0^1 G(t, s) f^*(s, y(s)) ds \\ & + \frac{\alpha t}{1 - \alpha \eta} \int_0^1 G(\eta, s) f^*(s, y(s)) ds \end{aligned} \tag{2.19}$$

where

$$\psi(t) := a + \frac{b + a\alpha - a}{1 - \eta \alpha} t. \tag{2.20}$$

Now since

$$\begin{aligned}
&\ |\textstyle\int_0^1 G(t,s)f^*(s,y(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)f^*(s,y(s))ds| \\
&\le \textstyle\int_0^1 G(t,s)|f^*(s,y(s))|ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)|f^*(s,y(s))|ds \\
&\le \textstyle\int_0^t (1-t)s|h(s)|ds + \int_t^1 (1-s)t|h(s)|ds \\
&\quad + \textstyle\frac{\alpha}{1-\alpha\eta}\big[\int_0^\eta (1-\eta)s|h(s)|ds + \int_\eta^1 (1-s)\eta|h(s)|ds\big] \\
&\le \textstyle\big(1+\frac{\alpha}{1-\alpha\eta}\big)\int_0^1 s(1-s)|h(s)|ds < \infty
\end{aligned}$$

we know from (a_4) that $(Ty) : [0,1] \to \mathbb{R}$ is well-defined.

The existence of solutions of (2.7), (2.8) follows now from the Schauder fixed point theorem provided we can check each of the following steps:

Step 1. $T : C[0,1] \to C[0,1]$ is well-defined.

Step 2. $(Ty)''(t) + f^*(t, y(t)) = 0, \qquad t \in (0,1)$.

Step 3. $(Ty)(0) = a,\ (Ty)(1) - \alpha(Ty)(\eta) = b$.

Step 4. $T(C[0,1])$ is a relatively compact subset of $C[0,1]$.

Step 1. For $y(t) \in C[0,1]$, we have from (2.5) and (a_3) and (a_4) that $t(1-t)f^*(t,y(t)) \in L^1[0,1]$. So for each $r \in (0,1)$, $tf^*(t,y(t)) \in L^1[0,r]$ and $(1-t)f^*(t,y(t)) \in L^1[r,1]$. Thus $(Ty)(t) \in AC_{\text{loc}}(0,1)$ since

$$\begin{aligned}
(Ty)(t) = &\ \textstyle\psi(t) + \int_0^t (1-t)sf^*(t,y(t))ds + \int_t^1 (1-s)tf^*(s,y(s))ds \\
&+ \textstyle\frac{\alpha t}{1-\alpha\eta}\big[\int_0^\eta (1-\eta)sf^*(s,y(s))ds + \int_\eta^1 (1-s)\eta f^*(s,y(s))ds\big].
\end{aligned} \tag{2.21}$$

Recall that $f^* : (0,1)\times\mathbb{R} \to \mathbb{R}$ is continuous, so we have $(Ty)(t) \in C^1(0,1) \cap AC_{\text{loc}}(0,1)$. Moreover

$$\begin{aligned}
(Ty)'(t) = \psi'(t) - &\int_0^t sf^*(s,y(s))ds \\
&+ \int_t^1 (1-s)f^*(s,y(s))ds + D_y, \quad t \in (0,1)
\end{aligned} \tag{2.22}$$

where

$$D_y := \frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)f^*(s,y(s))ds.$$

Set

$$D := \sup\{|D_y| \mid y \in C[0,1]\}.$$

Now since

$$\begin{aligned}
&\int_0^1 |(Ty)'(t)|dt \\
&= |\psi'(t)| + \int_0^1 | - \int_0^t sf^*(t,y(t))ds + \int_t^1 (1-s)f^*(t,y(t))ds + D_y|dt \\
&\le |\psi'(t)| + \int_0^1 \int_0^t s|f^*(t,y(t))|dsdt + \int_0^1 \int_t^1 (1-s)|f^*(t,y(t))|dsdt + D \\
&\le |\psi'(t)| + \int_0^1 \int_s^1 s|h(s)|dtds + \int_0^1 \int_0^s (1-s)|h(s)|dtds + D \\
&= |\psi'(t)| + 2\int_0^1 s(1-s)|h(s)|ds + D < \infty
\end{aligned}$$

we have $Ty \in AC[0,1]$.

Step 2. Now (2.22) together with the fact that $sf^*(t,y(t)) \in L^1[0,r]$ and $(1-s)f^*(t,y(t)) \in L^1[r,1]$ for each $r \in (0,1)$ implies that $(Ty)'(t) \in AC_{\mathrm{loc}}(0,1)$. Using the fact that $f^* : (0,1)\times\mathbb{R} \to \mathbb{R}$ is continuous, we can easily conclude that $(Ty)'(t) \in C^1(0,1) \cap AC_{\mathrm{loc}}(0,1)$, and accordingly

$$(Ty)''(t) = -f^*(t,y(t)), \qquad t \in (0,1). \tag{2.23}$$

Step 3. By Step 1, $Ty \in C[0,1]$. Thus we have from (2.20) and Lemma 1.1 (ii) that

$$\begin{aligned}
&(Ty)(0) \\
&= \psi(0) + \lim_{t\to 0}(Ty)(t) \\
&= \psi(0) + \lim_{t\to 0}\int_0^t (1-t)sf^*(s,y(s))ds + \lim_{t\to 0}\int_t^1 (1-s)tf^*(s,y(s))ds \\
&\quad + \tfrac{0}{1-\alpha\eta}\,\alpha\int_0^1 G(\eta,s)f^*(s,y(s))ds \\
&= a.
\end{aligned}$$

Again, applying (2.20) and the fact $Ty \in C[0,1]$, we have that

$$\begin{aligned}
&(Ty)(1) \\
&= \psi(1) + \lim_{t\to 1}(Ty)(t) \\
&= \psi(1) + \lim_{t\to 1}\int_0^t (1-t)sf^*(s,y(s))ds + \lim_{t\to 1}\int_t^1 (1-s)tf^*(s,y(s))ds \\
&\quad + \tfrac{1}{1-\alpha\eta}\,\alpha\int_0^1 G(\eta,s)f^*(s,y(s))ds.
\end{aligned}$$

Applying Lemma 1.1 (ii) and using the fact $(1-s)y(s) \in L^1[0,1]$, we conclude that

$$(Ty)(1) = \psi(1) + \frac{1}{1-\alpha\eta}\alpha\int_0^1 G(\eta,s)f^*(s,y(s))ds. \tag{2.24}$$

Similarly

$$(Ty)(\eta) = \psi(\eta) + \int_0^1 G(\eta,s)f^*(s,y(s))ds + \frac{\alpha\eta}{1-\alpha\eta}\int_0^1 G(\eta,s)f^*(s,y(s))ds. \tag{2.25}$$

This together with (2.24) implies that $(Ty)(1) - \alpha(Ty)(\eta) = b$.

Step 4. By (2.6) and (a_4), we have that

$$|f^*(t, y(t))| \le h(t), \qquad \text{for all } y \in C[0,1].$$

This together with (2.20) implies

$$\begin{aligned} |Ty(t)| \le & |\psi(t)| + \int_0^1 G(t,s)h(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)h(s)ds \\ \le & ||\psi||_{C[0,1]} + \Big(1 + \frac{\alpha}{1-\alpha\eta}\Big)\int_0^1 s(1-s)h(s)ds \end{aligned} \tag{2.26}$$

which means that $T(C[0,1])$ is a bounded subset of $C[0,1]$.

Let $t \in (0,1)$. Then we have from (2.22) and (2.6) that

$$\begin{aligned} |(Ty)'(t)| &\le |\psi'(t)| + \int_0^t s|f^*(s,y(s))|ds + \int_t^1 (1-s)|f^*(t,y(t))|ds + D \\ &\le |\psi'(t)| + \int_0^t sh(s)ds + \int_t^1 (1-s)h(s)ds + D \\ &:= \gamma(t). \end{aligned} \tag{2.27}$$

By Lemma 1.1 (i), $\int_0^\tau sh(s)ds$ and $\int_\tau^1 (1-s)h(s)ds \in L^1[0,1]$, and accordingly $\gamma \in L^1[0,1]$. Thus $T(C[0,1])$ is equi-continuous on $[0,1]$ since for all $t_1, t_2 \in [0,1]$ with $t_1 > t_2$, we have that

$$|Ty(t_1) - Ty(t_2)| = |\int_{t_2}^{t_1} (Ty)'(\tau)d\tau| \le \int_{t_2}^{t_1} \gamma(\tau)d\tau$$

for $y \in C[0,1]$.

Now by Arzela-Ascoli theorem, $T(C[0,1])$ is relatively compact.

3. Proofs of The Main Results

In this section, we will apply the method of lower and upper solutions developed in Section 2 to prove Theorem 1.1 and 1.2.

Proof of Theorem 1.1 *Necessary.* Let $x \in C[0,1] \cap C^2(0,1)$ be a positive solution of (1.1),(1.5). Then the boundary condition $x(1) = \alpha x(\eta)$ implies

$$x(1) > 0. \tag{3.1}$$

From (1.5) and assumption (H), there exists $t_0 \in (\eta, 1)$ such that $x'(t_0) = 0$. So

$$\int_{t_0}^{t} p(s)x^{\lambda}(s)ds = -\int_{t_0}^{t} x''(s)ds = -x'(t), \quad t \in (t_0, 1). \tag{3.2}$$

Multiplying both sides by $x^{-\lambda}(t)$ and then integrating on $[t_0, 1]$, we have that

$$0 \leq \int_{t_0}^{1} x^{-\lambda}(t) \int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt = \frac{1}{1-\lambda}\big(x^{1-\lambda}(t_0) - x^{1-\lambda}(1)\big) < \infty.$$

Since $x''(t) \leq 0$, we have that $x'(t) \leq 0$ on $[t_0, 1)$. So $x(t)$ is nonincreasing on $[t_0, 1)$. This implies

$$\int_{t_0}^{t} p(s)ds \leq x^{-\lambda}(t) \int_{t_0}^{t} p(s)x^{\lambda}(s)ds, \quad t \in [t_0, 1).$$

Thus,

$$0 \leq \int_{t_0}^{1} (1-s)p(s)ds = \int_{t_0}^{1}\int_{t_0}^{t} p(s)dsdt \leq \int_{t_0}^{1} x^{-\lambda}(t) \int_{t_0}^{t} p(s)x^{\lambda}(s)dsdt < \infty. \tag{3.3}$$

On the other hand, since $x'(t_0) = 0$,

$$\int_{t}^{t_0} p(s)x^{\lambda}(s)ds = -\int_{t}^{t_0} x''(s)ds = x'(t), \quad t \in (0, t_0). \tag{3.4}$$

Multiplying both sides of (3.4) by $x^{-\lambda}(t)$ and then integrating on $[0, t_0]$, we know from the assumption $\lambda \in (0, 1)$ that

$$\begin{aligned} 0 \leq& \int_{0}^{t_0} x^{-\lambda}(t) \int_{t}^{t_0} p(s)x^{\lambda}(s)dsdt \\ =& \frac{1}{1-\lambda}\big(x^{1-\lambda}(t_0) - x^{1-\lambda}(0)\big) = \frac{1}{1-\lambda} x^{1-\lambda}(t_0) < \infty. \end{aligned}$$

Since $x(t)$ is nondecreasing on $(0, t_0)$, we have that

$$\int_{t}^{t_0} p(s)ds \leq x^{-\lambda}(t) \int_{t}^{t_0} p(s)x^{\lambda}(s)ds, \quad t \in (0, t_0].$$

Thus

$$0 \leq \int_{0}^{t_0} sp(s)ds = \int_{0}^{t_0}\int_{t}^{t_0} p(s)dsdt \leq \int_{0}^{t_0} x^{-\lambda}(t) \int_{t}^{t_0} p(s)x^{\lambda}(s)dsdt < \infty. \tag{3.5}$$

Combining (3.3) with (3.5), we conclude the second inequality in (1.6). We must have that $\int_0^1 t(1-t)p(t)dt > 0$, for otherwise $p(t) \equiv 0$ on $(0, 1)$, which means that the related problem (1.1),(1.5) has only the trivial solution $u \equiv 0$.

Sufficiency. Suppose that (1.6) holds. Let

$$q_1(t) = \int_0^1 G(t,s)s^\lambda p(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)s^\lambda p(s)ds \tag{3.6}$$

$$q_2(t) = \int_0^1 G(t,s)p(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)p(s)ds. \tag{3.7}$$

By similar arguments used in Step 1-3 of the proof of Theorem 2.1, we can conclude that $q_1, q_2 \in C[0,1]\cap C^2(0,1)$, and

$$q_i(0) = 0, \quad q_i(1) = \alpha q_i(\eta), \quad i = 1, 2. \tag{3.8}$$

Moreover, for $t \in (0,1)$

$$\begin{aligned} 0 &< \Big(\frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)s^\lambda p(s)ds\Big)t \\ &\le q_1(t) < q_2(t) \le \Big(1+\frac{\alpha}{1-\alpha\eta}\Big)\int_0^1 s(1-s)p(s)ds \end{aligned} \tag{3.9}$$

$$q_1''(t) = -t^\lambda p(t), \qquad t\in(0,1) \tag{3.10}$$

and

$$q_2''(t) = -p(t), \qquad t\in(0,1). \tag{3.11}$$

Let $l_1,\ l_2$ be constants such that

$$(l_1)^{1-\lambda} = \big[\frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)s^\lambda p(s)ds\big]^\lambda \tag{3.12}$$

and

$$(l_2)^{1-\lambda} = \big[\big(1+\frac{\alpha}{1-\alpha\eta}\big)\int_0^1 s(1-s)p(s)ds\big]^\lambda \tag{3.13}$$

and let

$$v(t) = l_1q_1(t), \quad w(t) = l_2q_2(t), \qquad t\in[0,1]. \tag{3.14}$$

Then $v, w \in C[0,1]\cap C^2(0,1)$, $0 < v(t) < w(t)$ for $t\in(0,1)$, and

$$v''(t) + p(t)v^\lambda(t) = l_1^\lambda t^\lambda p(t)\big[\frac{q_1^\lambda(t)}{t^\lambda} - l_1^{1-\lambda}\big] \ge 0 \tag{3.15}$$

$$v(0) = 0, \quad v(1) = \alpha v(\eta) \tag{3.16}$$

$$w''(t) + p(t)w^\lambda(t) = l_2^\lambda t^\lambda p(t)[q_2^\lambda(t) - l_2^{1-\lambda}] \le 0 \tag{3.17}$$

$$w(0) = 0, \quad w(1) = \alpha w(\eta). \tag{3.18}$$

Therefore v and w are lower and upper solutions of (1.1), (1.5), respectively. By Theorem 2.1, (1.1), (1.5) has a solution $\tilde{x}(t) \in C[0,1] \cap C^2(0,1)$ satisfying

$$v(t) \le \tilde{x}(t) \le w(t), \qquad t \in (0,1).$$

Proof of Theorem 1.2. *Necessary.* Let $x \in C^1[0,1] \cap C^2(0,1)$ be a positive solution of (1.1),(1.5). Then the graph of x is concave down on $(0,1)$. This together with the fact $x(1) = \alpha x(\eta) > 0$ implies that there exists a constant $c > 0$, such that

$$x(t) \ge ct, \qquad t \in [0,1].$$

(Note that $x(t) = x\big((1-t)\cdot 0 + t\cdot 1\big) \ge (1-t)x(0) + tx(1) = x(1)t$). Hence

$$\begin{aligned} 0 &\le \int_0^1 t^\lambda p(t)dt \le c^{-\lambda}\int_0^1 p(t)x^\lambda(t)dt \\ &= -c^{-\lambda}\int_0^1 x''(t)dt \;= c^{-\lambda}[x'(0) - x'(1)] < \infty. \end{aligned}$$

Sufficient. Suppose that (1.7) holds. Let

$$q(t) = \int_0^1 G(t,s)(s^\lambda p(s))ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)(s^\lambda p(s))ds. \tag{3.19}$$

Then $q \in C^1[0,1]$ since $s^\lambda p(s) \in L^1[0,1]$. By similar arguments used in Step 1-3 of the proof of Theorem 2.1, we can conclude that $q \in C^1[0,1] \cap C^2(0,1)$, and

$$\begin{aligned} &q''(t) = -t^\lambda p(t), \qquad t \in (0,1) \\ &q(0) = 0, \quad q(1) = \alpha q(\eta). \end{aligned} \tag{3.20}$$

Moreover, for $t \in (0,1)$

$$\begin{aligned} 0 <& \Big(\frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)(s^\lambda p(s))ds\Big)t \le q(t) \\ &\le \Big[\int_0^1 (1-s)(s^\lambda p(s))ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)(s^\lambda p(s))ds\Big]t \end{aligned} \tag{3.21}$$

since

$$\begin{aligned} \int_0^1 G(t,s)(s^\lambda p(s))ds &= \int_0^t (1-t)s(s^\lambda p(s))ds + \int_t^1 t(1-s)(s^\lambda p(s))ds \\ &\le \int_0^t (1-s)t(s^\lambda p(s))ds + \int_t^1 t(1-s)(s^\lambda p(s))ds. \end{aligned}$$

Let $l_3,\ l_4$ be constants such that

$$(l_3)^{1-\lambda} = \big[\frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)(s^\lambda p(s))ds\big]^\lambda \tag{3.22}$$

and

$$(l_4)^{1-\lambda} = \big[\int_0^1 (1-s)(s^\lambda p(s))ds + \frac{\alpha}{1-\alpha\eta}\int_0^1 G(\eta,s)(s^\lambda p(s))ds\big]^\lambda \tag{3.23}$$

and let

$$v(t) = l_3 q(t), \quad w(t) = l_4 q(t), \qquad t\in[0,1]. \tag{3.24}$$

Then $v, w \in C^1[0,1]\cap C^2(0,1)$, $0 < v(t) < w(t)$ for $t\in(0,1)$, and

$$v''(t) + p(t)v^\lambda(t) = l_3^\lambda t^\lambda p(t)\big[\frac{q^\lambda(t)}{t^\lambda} - l_3^{1-\lambda}\big] \geq 0 \tag{3.25}$$

$$v(0)=0, \quad v(1) = \alpha v(\eta) \tag{3.26}$$

$$w''(t) + p(t)w^\lambda(t) = l_4^\lambda t^\lambda p(t)[\frac{q^\lambda(t)}{t^\lambda} - l_4^{1-\lambda}] \leq 0 \tag{3.27}$$

$$w(0)=0, \quad w(1) = \alpha w(\eta). \tag{3.28}$$

Therefore v and w are lower and upper solutions of (1.1), (1.5), respectively. By Theorem 2.1, (1.1), (1.5) has a solution $\bar{x}(t)\in C[0,1]\cap C^2(0,1)$ satisfying

$$v(t)\leq \bar{x}(t)\leq w(t), \qquad t\in(0,1).$$

From (3.21) we have

$$p(t)\bar{x}^\lambda(t) \leq p(t)w^\lambda(t) = p(t)[l_4 q(t)]^\lambda \leq l_4 t^\lambda p(t).$$

That is,

$$|\bar{x}''(t)| \leq l_4 t^\lambda p(t), \quad t\in(0,1).$$

This together with (1.7) implies that $\bar{x}''$ is absolutely integrable on $[0,1]$. So both $\bar{x}'(0+)$ and $\bar{x}'(1-)$ exist, i.e. $\bar{x}\in C^1[0,1]$. This completes the proof.

Remark 3.1 We remark that Theorem 2.1 is established under the condition

$$\alpha \in \big(0,\ \frac{1}{\eta}\big). \tag{3.29}$$

Also several results on the existence of positive solutions for nonsingular three-point boundary value problems have been established under (3.29), see [8-9, 10]. However Theorem 1.1 and 1.2 give no information on the interesting question as to what happens to (1.1), (1.5) when

$$\alpha \in \big(1, \ \frac{1}{\eta}\big). \tag{3.30}$$

4. Uniqueness

The uniqueness of the positive solutions of (1.1), (1.5) appears somewhat difficult to study. In this section we can only show that (1.7) implies that (1.1), (1.5) cannot have two $C^1[0,1]$ positive solutions. However we give no information about the uniqueness of $C[0,1]$ positive solutions of (1.1), (1.5) under condition (1.7).

Theorem 4.1 Suppose that (A) and (H) hold. Assume that

$$0 < \int_0^1 t^\lambda p(t)dt < \infty. \tag{4.1}$$

Then (1.1),(1.5) cannot have two $C^1[0,1]$ positive solutions.

Proof. Let x_1, x_2 be two $C^1[0,1]$ positive solutions of (1.1), (1.5). Define

$$\Lambda = \{\mu \in (0,\infty) \mid x_1(t) - \mu x_2(t) \geq 0 \text{ on } [0,1]\}. \tag{4.2}$$

Then Λ is well-defined since x_1, $x_2 \in C^1[0,1] \cap C^2(0,1)$. Clearly $\Lambda \neq \emptyset$. Let

$$\tau = \sup \Lambda \tag{4.3}$$

and we claim that

$$\tau \geq 1.$$

Suppose on the contrary that $\tau < 1$. We have from (4.2) that

$$x_1(t) - \tau x_2(t) \geq 0, \qquad t \in [0,1].$$

We note that under condition (4.1) and (A) and (H), any $C[0,1]$ solution u of (1.1),(1.5) satisfies

$$u(t) = \int_0^1 G(t,s)p(s)u^\lambda(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)p(s)u^\lambda(s)ds, \quad t \in [0,1]$$

so, for $t \in [0,1]$

$$\begin{aligned} x_1(t) &= \int_0^1 G(t,s)p(s)x_1^\lambda(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)p(s)x_1^\lambda(s)ds \\ &\geq \int_0^1 G(t,s)p(s)(\tau x_2)^\lambda(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)p(s)(\tau x_2)^\lambda(s)ds \\ &= \tau^\lambda \int_0^1 G(t,s)p(s)x_2^\lambda(s)ds + \frac{\alpha t}{1-\alpha\eta}\int_0^1 G(\eta,s)p(s)x_2^\lambda(s)ds \\ &= \tau^\lambda x_2(t) \end{aligned}$$

which contradicts (4.3) since $\tau^\lambda > \tau$. Therefore $\tau \geq 1$, and

$$x_1(t) \geq \tau x_2(t) \geq x_2(t), \qquad t \in [0,1].$$

In the same way we obtain

$$x_2(t) \geq x_1(t), \qquad t \in [0,1].$$

Thus

$$x_1(t) \equiv x_2(t), \qquad t \in [0,1].$$

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