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THE BORSUK-ULAM THEOREM FOR QUASI-RULED FREDHOLM MAPS

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Dedicated to the memory of Karl Kalik

Abstract. We present an analog of the Borsuk-Ulam theorem for quasi-ruled Fredholm maps between Banach spaces. The result is applied to prove global solvability for nonlinear Riemann-Hilbert problems with Lipschitz continuous boundary conditions.

Key Words and Phrases: quasi-ruled Fredholm map, Sobolev space, Borsuk-Ulam property.
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1. Introduction

Among all nonlinear mappings, those defined on function spaces by nonlinear pseudodifferential operators play a special rôle in mathematical physics. Every boundary value problem for a nonlinear elliptic equation (or system of) may be transformed into the language of infinite-dimensional geometry. This program requires the development of infinite-dimensional geometry and topology for its use to solve the geometrical problem obtained. For instance, to find the pre-image of a point under a nonlinear map we need a degree theory for maps on a Banach space or manifold analogously to the degree theory for finite-dimensional maps. Recall that in the finite-dimensional case, a degree is defined in the class of continuous maps, which has the following basic properties:

Let Ω be a bounded domain in $\mathbb{R}^n$ and let $f : \overline{\Omega} \to \mathbb{R}^n$ be continuous with $0 \notin f(\partial\Omega)$. Then

1. $\deg_B(f, \Omega, 0) \neq 0$ implies that $f(x) = 0$ for some $x \in \Omega$;
2. if $f_t(x)$ is a homotopy with $0 \notin f_t(\partial\Omega)$ for $t \in [0, 1]$, then $\deg_B(f_t, \Omega, 0)$ is independent of t;
3. $\deg_B(Id, \Omega, 0) = 1$ (if $0 \in \Omega$).

The Borsuk-Ulam theorem implies that an odd map f has a nonzero degree – therefore $f(x) = 0$ has a solution. More precisely:

The Borsuk-Ulam theorem [10] Let Ω be a symmetric bounded open set in $\mathbb{R}^n$ containing the origin. Suppose that $f : \overline{\Omega} \to \mathbb{R}^n$ is continuous with $0 \notin f(\partial\Omega)$ and satisfies $f(-x) = -f(x)$. Then $\deg_B(f, \Omega, 0)$ is odd.

Unfortunately, the degree theory does not admit a direct generalization to infinite-dimensional maps, as the following theorem shows. Let H be an arbitrary infinite-dimensional (real) Hilbert space and $B_1 = \{x | \; \|x\| \leq 1\}$ in H.

Proposition 1.1. *There exists a $C^\infty-$ diffeomorphism $h : B_1 \to H\backslash\{0\}$ with $0 \notin h(\partial B_1)$, such that one cannot define a degree* $\deg(h, B_1, 0)$ *with the properties* 1.-3.

PROOF. The proof is based on the construction of a C^∞-diffeomorphism $h : B_1 \to H\backslash\{0\}$, which admits a linear homotopy to the identity map Id. Assume for a moment that such a map h can be constructed. If, for such a map, $\deg(h, B_1, 0)$ could be defined with the properties 1.-3., then we would have:

$$\deg(h, B_1, 0) \quad = \quad \deg(Id, B_1, 0) \neq 0,$$

and, as a consequence, the equation $h(x) = 0$ had at least one solution in B_1. But this is a contradiction, since $h : B_1 \to H\backslash\{0\}$. The construction of such a diffeomorphism h is due to Bessega [1]. □

This example shows that the maps which are admissible in degree theory are singled out among the general continuous ones by special additional geometrical properties allowing the definition of degree and other topological invariants. At present there are various degree theories generalizing the classical Leray-Schauder degree [3, 8, 10]. However, none of the degree theories mentioned above are adequate for a nonlinear map, which is defined by nonlinear pseudo-differential operators. Many of the available degree theories contain a rather "narrow" class of nonlinear maps and consequently have "narrow" domains of application. Other theories are based on some "secondary" property of maps, the so-called properness. We will derive this property from more fundamental geometrical properties of quasi-ruled Fredholm maps, defined in Definition 2.2 below.

With the notion of quasi-ruled Fredholm maps, our main result is a new version of the Borsuk-Ulam theorem for a class of maps between Banach spaces. The examples in the next section show that this Borsuk-Ulam theorem applies to a large class of maps.

Theorem 1.2. *Let X, Y be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ respectively. Let Φ be a monotonically increasing function on $[0,\infty)$ with $\lim_{\xi\to\infty}\Phi(\xi) = \infty$. Let $\mathfrak{S}$ be the set of quasi-ruled Fredholm maps $A : X \to Y$ satisfying $\|x\|_X \leq \Phi(\|Ax\|_Y)$. Then, for $A \in \mathfrak{S}$, a degree d can be defined satisfying the requirements* 1.-3. *(with Ω a bounded domain in X). Moreover, if $A(x) = -A(-x)$ and Ω is a symmetric neighborhood of $0 \in X$, then $d(A)$ is odd.*

2. Definitions and examples

Let X and Y be real Banach spaces and $\pi_\nu : X \to X_\nu$ be a linear map of X to the ν-dimensional space $X_\nu \subset X$. We denote by X^ν_α the inverse image of the point $\alpha \in X_\nu$ under this map; this will be a closed plane of codimension ν in X, and for different α, the planes are parallel. Based on the work by A. Šnirelman [8] (see also [3] and the references therein) we introduce the following

Definition 2.1. *Let A be a continuous map of the bounded domain $\Omega \subset X$ to Y. We call A Fredholm-ruled, in short F-R map, if*

1. *there are linear maps $\pi_\nu : X \to X_\nu$;*
2. *restricted to each plane $X^\nu_\alpha (\alpha \in X_\nu)$ passing through Ω, the map A is affine, i.e.*

$$A^\nu_\alpha \quad = \quad A\big|_{X^\nu_\alpha} \in \mathit{Aff}\,(X^\nu_\alpha, Y),$$

   and the operator family A^ν_α depends continuously on $\alpha \in X_\nu$;

3. *codim* $A^\nu_\alpha(X^\nu_\alpha) = \nu$ *for all* $\alpha \in X_\nu$, *i.e. the image of each plane* X^ν_α *under the affine map* $A^\nu_\alpha = A^\nu\big|_{X^\nu_\alpha}$ *is closed in* Y *and has there codimension* ν, *i.e. the same as* X^ν_α *in* X.

This definition is illustrated in Figure 1. Note that a Fredholm-ruled map is affine in all coordinates with the exception of finitely many.

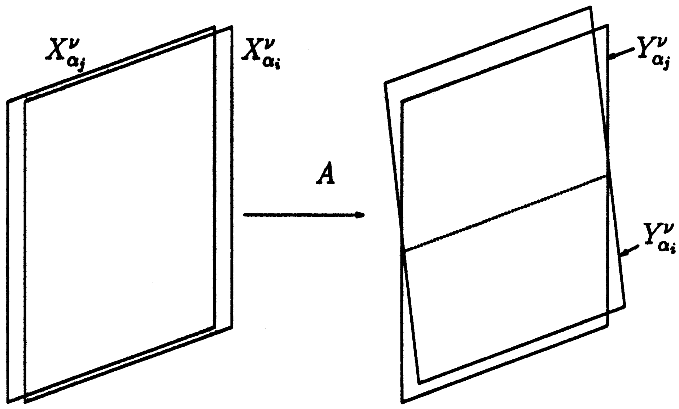

FIGURE 1. A maps the family of parallel codimensional ν-planes $\{X^\nu_\alpha\}$ to codimensional ν-planes $\{Y^\nu_\alpha\}$. For different α, the corresponding planes $\{Y^\nu_\alpha\}$ may mutually intersect.

In the following, the planes X^ν_α will be called *fibers* and two F-R maps will be considered to be different if they have different fibers, *even if they coincide as maps*. Thus we shall denote a F-R map A by the symbol A^ν, where ν is the codimension of the fiber. Let $\tilde{A}$ be an affine map defined on the closed plane $\tilde{X} \subset X$ mapping $\tilde{X}$ to Y. Then we set

$$\|\tilde{A}\| \;=\; \inf\{C|\; \|\tilde{A}x\|_Y \le C(1+\|x\|_X)\forall x \in \tilde{X}\}.$$

The notion $\|\tilde{A}^{-1}\|$ has the traditional meaning if $\tilde{A}$ is invertible.

Definition 2.2. *A continuous map* $A : X \to Y$ *is called quasi-ruled Fredholm if there exists a sequence of F-R maps* A^{ν_k} *with* $\nu_k \to \infty$ *as* $k \to \infty$ *such that*

1. $\lim_{k\to\infty} A^{\nu_k} = A$ *uniformly on every bounded domain* $\Omega \subset X$; *and*

2. *$\|A_\alpha^{\nu_k}\| < C(\Omega)$, $\|(A_\alpha^{\nu_k})^{-1}\| \le C(\Omega)$ for $k \ge k_0(\Omega)$, if $\alpha \in \pi_{\nu_k}(\Omega)$, where $C(\Omega)$ is independent of $k \ge k_0(\Omega)$.*

We denote the class of quasi-ruled Fredholm maps $A : X \to Y$ by F-QR (X, Y). In this section we consider just two specific examples of F-QR maps. More complex and other important examples of F-QR maps in connection with nonlinear boundary value problems will be considered further below.

Lemma 2.3. *Let $X = Y = H^s(S^1)$ be the Sobolev space of real functions $u(\tau)$ on a circle, where $0 \le \tau < 2\pi$ and $2 \le s \in \mathbb{N}$ (natural number); $f(\tau, u)$ is a smooth real function of $u \in \mathbb{R}$ and τ with $f'_u(\tau, u) \neq 0$ for all (τ, u). Then the Nemytzki operator*

$$A : u(\tau) \mapsto f(\tau, u(\tau))$$

defines a F-QR *map of X to Y for $s \ge 2$.*

Proof. Recall that $H^s(S^1)$ consists of functions $u(\tau)$ with the norm

$$\|u\|_s^2 = \int_0^{2\pi} \sum_{l=0}^{s} |\frac{d^l u}{d\tau^l}(\tau)|^2 d\tau.$$

It is not difficult to see that A is a continuous map of X to Y. The F-R approximations of $A : u(\tau) \mapsto f(\tau, u(\tau))$ are chosen by

$$\begin{aligned} A^{\nu_k} u &= f\left(\tau_0, \sum_{|m| \le p_k} u_m e^{im\tau_0}\right) + \int_{\tau_0}^{\tau} f'_\tau\left(\sigma, \sum_{|m|\le p_k} u_m e^{im\sigma}\right) d\sigma + \\ &\quad + \int_{\tau_0}^{\tau} f'_u\left(\sigma, \sum_{|m|\le p_k} u_m e^{im\sigma}\right) \cdot u'(\sigma) d\sigma, \end{aligned}$$

where $\nu_k = 2p_k + 1$ and $u_m = \frac{1}{2\pi}\int_0^{2\pi} u(\tau) e^{-im\tau} d\tau$ with $u_{-m} = \bar{u}_m$. In this case

$$X_\alpha^{\nu_k} = \left\{ u(\tau) = \sum_{m=-\infty}^{+\infty} u_m e^{im\tau} \mid u_m = \alpha_m \quad \text{for } |m| \le p_k \right\},$$

with $A^{\nu_k}|_{X_\alpha^{\nu_k}} \in$ Aff $(X_\alpha^{\nu_k}, Y)$. Obviously, $codim_{H^s(S^1)} X_\alpha^{\nu_k} = \nu_k$. □

Let

$$H_0 u(\tau) = \frac{1}{2\pi} p.v. \int_0^{2\pi} u(\sigma) ctg \frac{\tau - \sigma}{2} d\sigma$$

(where p.v. stands for the Cauchy principal value) be the Hilbert transform.

Lemma 2.4. *Let $X = Y = H^s(S^1), 0 \le \tau < 2\pi, 2 \le s \in \mathbb{N}$ and let $f(\tau, u, v)$ be a smooth real function of $\tau \in [0, 2\pi)$ and $u, v \in \mathbb{R}$ with $grad_{u,v} f \neq 0$ for all (τ, u, v). Moreover we assume that $wind[f'_u + i f'_v] := \frac{1}{2\pi}\int_0^{2\pi} darg(f'_u + i f'_v) = 0$ for all $\tau \mapsto (u(\tau), v(\tau)) \in H^s(S^1) \oplus H^s(S^1)$. Then the operator*

$$Au(\tau) = f(\tau, u(\tau), H_0 u(\tau)) \tag{2.1}$$

defines a F-QR map of X to Y.

Proof. As was shown in [3, 4, 8, 9], A defines a continuous map of $H^s(S^1)$ into itself. As before we set $u = \sum_{m \in \mathbb{R}} u_m e^{im\tau}$ with $u_m = \frac{1}{2\pi}\int_0^{2\pi} u(\tau) e^{-im\tau} d\tau$ for $u \in H^s(S^1)$. It is well-known that H_0 is a pseudo-differential operator (ψDO) of order zero with the symbol i sgn ξ, i.e.

$$(H_0 u)_m \quad = \quad i \operatorname{sgn} m \cdot u_m .$$

One can easily check that F-R approximations of the operator A defined by (2.1) take the form of

$$\begin{aligned} A_u^{\nu k} \quad = \quad & f(\tau_0, \sum_{|m| \le p_k} u_m e^{im\tau_0}, \sum_{|m| \le p_k} i \operatorname{sgn}(m) \cdot u_m e^{im\tau_0}) \\ & + \int_{\tau_0}^{\tau} f'_\tau(\sigma, \sum_{|m| \le p_k} i \operatorname{sgn}(m) \cdot u_m e^{im\sigma}) d\sigma \\ & + \int_{\tau_0}^{\tau} f'_u \left(\sigma, \sum_{|m| \le p_k} u_m e^{im\sigma}, \sum_{|m| \le p_k} i \operatorname{sgn}(m) u_m e^{im\sigma} \right) u'(\sigma) d\sigma \\ & + \int_{\tau_0}^{\tau} f'_v \left(\sigma, \sum_{|m| \le p_k} u_m e^{im\sigma}, \sum_{|m| \le p_k} i \operatorname{sgn}(m) u_m e^{im\sigma} \right) H_0 u'(\sigma) d\sigma \end{aligned}$$

where $\nu_k = 2p_k + 1$. □

3. Degree for quasi-ruled Fredholm maps

We now present a new approach to define a degree for F-QR (X, Y) which has advantages over that of [3, 6, 8] as we shall see below. Our approach is based on the concept of subdivisions of a special category of bundles introduced below.

Throughout this paper we assume that for $A \in$ F-QR (X, Y) the following a priori estimate is satisfied:

$$\|x\|_X \leq \Phi(\|Ax\|_Y) \tag{3.2}$$

where Φ is a positive monotonically increasing function on $[0, \infty)$ with $\Phi(\xi) \to \infty$ as $\xi \to \infty$.

In what follows, we shall significantly make use of the following property.

Proposition 3.1. *Let $A \in$ F-QR (X, Y). Then A maps bounded sets in X to bounded sets in Y.*

Proof. Let Ω be a bounded domain in X and $A^\nu : X \to Y$ be a sequence of F-R maps as in Definition 2.2. First we prove that every A^ν is a bounded map. Let

$$B = \pi_\nu(\Omega) := \{\alpha \in X_\nu | \pi_\nu^{-1}(\alpha) \cap \overline{\Omega} \neq \emptyset\}. \tag{3.3}$$

Clearly, B is a bounded region in the finite-dimensional space X_ν.

Let $\{\alpha_1, \ldots, \alpha_N\}$ be a δ-net of the compact set B. Since $A^\nu_{\alpha_i} = A\Big|_{X^\nu_{\alpha_i}} \in$ Aff $(X^\nu_{\alpha_i}, Y)$, $i = 1, \ldots, N$, we have that there exists $M > 0$ such that

$$\|A^\nu_{\alpha_i}(x)\| \quad \leq \quad M$$

for all $x \in X^\nu_{\alpha_i} \cap \Omega$, $i = 1, \ldots, N$. Note that A^ν_α is in Aff (X^ν_α, Y) and depends continuously on α. Hence, we may use the Heine-Borel arguments; i.e. to every $\varepsilon > 0$, there exists $\delta > 0$, so that for all $x_1 \in X^\nu_{\alpha_1} \cap \Omega, x_2 \in X^\nu_{\alpha_2} \cap \Omega$, satisfying $\|x_1 - x_2\| < \delta$ it follows that $\|A^\nu_{\alpha_1} - A^\nu_{\alpha_2}\| < \varepsilon$. As a result, for all $\alpha \in B, x \in X^\nu_\alpha \cap \Omega$, there exists α_i and $x' \in X^\nu_{\alpha_i}$ with

$$\|A^\nu_\alpha(x)\| \leq \|A^\nu_{\alpha_i}(x')\| + \|A^\nu_\alpha(x) - A^\nu_{\alpha_i}(x')\| \leq M + \varepsilon.$$

So, A^ν is a bounded map. In consequence, for all $x \in \Omega$, we obtain

$$\|A(x)\| \leq \|A^\nu(x)\| + \|A^\nu(x) - A(x)\| \leq M + 2\varepsilon.$$

So, A is a bounded map, too. $\square$

Lemma 3.2. *Let $A : \Omega \subset X \to Y$ be a F-R map in the Banach spaces X and Y. Then there exists a sequence of maps $A^{\nu_k} : \Omega \to Y$, bundles*

$$\zeta_{\nu_k} = \Big\{ \bigcup_{\alpha \in X_{\nu_k}} X^{\nu_k}_\alpha, \pi_{\nu_k}, X_{\nu_k} \Big\} \quad \textit{and} \quad \eta_{\nu_k} = \Big\{ \bigcup_{\beta \in Y_{\nu_k}} Y^{\nu_k}_\beta, p_{\nu_k}, Y_{\nu_k} \Big\},$$

with $\mathrm{codim}_X X^{\nu_k}_\alpha = \nu_k$ and $\mathrm{codim}_Y Y^{\nu_k}_\beta = \nu_k$, such that

1) $\lim_{k\to\infty} A^{\nu_k} = A$ *uniformly in* Ω *and*

2) A^{ν_k} *is an affine bundle morphism between the bundles* ζ_{ν_k} *and* η_{ν_k} *with* $\bar{\Omega} \subset \bigcup_{\alpha\in X_{\nu_k}} X_\alpha^{\nu_k}, A(\bar{\Omega}) \subset \bigcup_{\beta\in Y_{\nu_k}} Y_\beta^{\nu_k}$. *(For definitions see* [11]*.)*

Proof. Let $\{\bigcup_{\alpha\in X_\nu} X_\alpha^\nu, \pi, X_\nu\}$ be a bundle, which corresponds to the F-R map A. Then to the bundle there belongs a family of parallel planes $\{X_\alpha^\nu\}$. It is obvious that for $\nu'' \geq \{\nu\}$ there holds $\bigcup_{\alpha''\in X_{\nu''}} X_{\alpha''}^{\nu''} \supset \bar{\Omega}$. Let

$$U_\nu \quad = \quad \{\alpha \in X_\nu | X_\alpha^\nu \cap \bar{\Omega} \neq \emptyset\}.$$

Note that U_ν is bounded. Let $Y_\alpha^\nu = A(X_\alpha^\nu)$. Since $A_\alpha^\nu = A\big|_{X_\alpha^\nu}$ depends continuously on $\alpha \in X_\nu$, so does the family of planes $\{Y_\alpha^\nu\}$ in Y, with $\operatorname{codim}_Y Y_\alpha^\nu = \nu''$.

Let $\{\alpha_1, \ldots, \alpha_N\}$ be a δ-net of U_ν so that any plane Y_α^ν with $\alpha \in U_\nu$ lies in an ε-neighborhood of some $Y_{\alpha_i}^\nu$, $\{i = 1, \ldots, N\}$ restricted to a sufficiently large ball $B(0, R)$ in Y. Hence, for all Y_α^ν there exists $Y_{\alpha_i}^\nu$, $i = 1, \ldots, N$, such that

$$\sin(Y_\alpha^\nu, Y_{\alpha_i}^\nu) \quad < \quad \varepsilon$$

where, by definition,

$$\sin(Y_\alpha^\nu, Y_{\alpha_i}^\nu) \quad := \quad \sup_{x,x_i}\{\|x - x_i\|; \big| \, x \in {}'Y_\alpha^\nu \cap B_1, x_i \in {}'Y_{\alpha_i}^\nu \cap B_1\},$$

with $B_1 = \{x;\ \|x\| \leq 1\}$ and with ${}'Y_\alpha^\nu$, ${}'Y_{\alpha_i}^\nu$ denoting the parallel shifts of Y_α^ν, $Y_{\alpha_i}^\nu$ to the origin in Y. Let

$$Y^m \quad = \quad \bigcap_{i=1}^N {}'Y_{\alpha_i}^\nu.$$

It is clear that Y^m is a subspace of Y. We decompose each $Y_{\alpha_i}^\nu$, $i = 1, \ldots, N$ into planes with codimension m which are parallel to Y^m. By orthogonal projection we decompose each Y_α^ν into a family $\{Y_{\alpha,\beta}^m\}$ with $(\alpha, \beta) \in U_\nu \times R^{m-\nu}$ which satisfies the following conditions:

1) for every α and every pair β_1, β_2 we have $Y_{\alpha,\beta_1}^m \parallel Y_{\alpha,\beta_2}^m$

2) $\{Y_{\alpha,\beta}^m\}$ is an $(N\varepsilon)$-parallel family.

The last fact means that for all (α_1, β_1) and (α_2, β_2), we have

$$\sin(Y_{\alpha_1,\beta_1}^m, Y_{\alpha_2,\beta_2}^m) \leq \sin(Y_{\alpha_1}^\nu, Y_{\alpha_2}^\nu) < \varepsilon.$$

Since A^ν_α is an affine isomorphism for every α, it follows that each X^ν_α can be decomposed (by the use of $(A^\nu_\alpha)^{-1}$) into parallel sub-planes $X^m_{\alpha,\beta} = (A^\nu_\alpha)^{-1}(Y^m_{\alpha,\beta})$ with codimension m, see Figure 2.

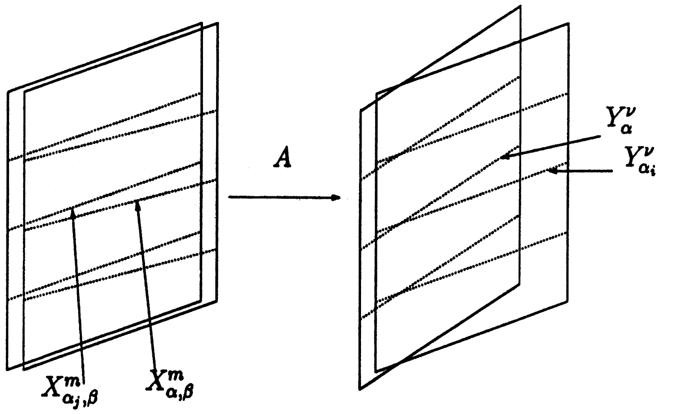

FIGURE 2. The codimension-ν-planes $\{Y^\nu_\alpha\}$ are subdivided into codimension-m-planes $\{Y^m_{\alpha,\beta}\}$, with $m > \nu$, so that any two planes $Y^m_{\alpha_i,\beta}$ and $Y^m_{\alpha,\beta}$ are $(N\varepsilon)$-parallel. The bundle $\{X^m_{\alpha,\beta}\}$ of codimension-m-planes in X is defined by $X^m_{\alpha,\beta} = A^{-1}(Y^m_{\alpha,\beta})$.

Since $\{Y^m_{\alpha,\beta}\}$ is $(N\varepsilon)$-parallel, it follows that there exists a plane $Y_m \subset Y$ (for sufficiently small ε) with dim $Y_m = m$, such that Y_m is transversal to all $Y^m_{\alpha,\beta}$ (in short $Y_m \pitchfork Y^m_{\alpha,\beta}$) if Y_m is orthogonal to some $Y^m_{\alpha_0,\beta_0}$. Hence, each plane $Y^m_{\alpha,\beta}$ will intersect Y_m only at exactly one point $\gamma = \gamma(\alpha,\beta)$. The family $\{Y^m_{\alpha,\beta}\}$ depends continuously on (α,β), hence, $\gamma = \gamma(\alpha,\beta)$ is a continuous function.

Let Y^m_γ be a plane which passes through the intersection point $\gamma = \gamma(\alpha,\beta)$ and is parallel to some $Y^m_{\alpha_0,\beta_0}$. In this way we obtain a family of parallel planes $\{Y^m_\gamma\}$. According to the above construction, these approximate the family $\{Y^m_{\alpha,\beta}\}$ in the ball $B(0,R_2)$ in Y. Since $A(\Omega)$ is bounded due to Proposition 3.1, we can choose R_2 at the beginning of our proof sufficiently large such that $B(0,R_2) \supset A(\Omega)$. Let $\tilde{A} : \bigcup X^m_{\alpha,\beta} \to \bigcup Y^m_\gamma$ be a map defined in the following way: to every $x \in X^m_{\alpha,\beta}$ there corresponds a point $z \in Y^m_\gamma$ which is the intersection of the plane $Y^m_{\gamma(\alpha,\beta)}$ with $Y_m(y)$, where $Y_m(y)$ is the plane which passes through the point $y = Ax$ and is parallel to Y_m i.e. $\tilde{A}(x) = \{Y_m + A(x)\} \overline{\pitchfork}\, Y_{\gamma_{\alpha,\beta}}$. (See Figure 3.)

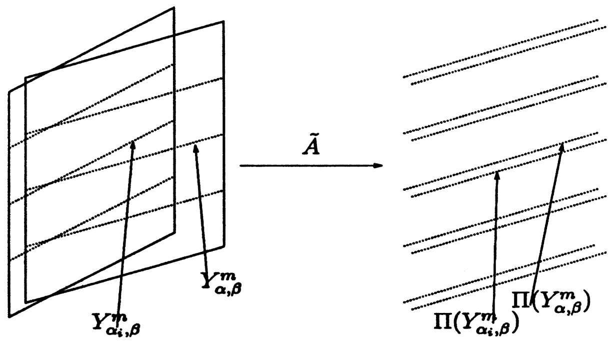

FIGURE 3. Π maps the planes $\{Y^m_{\alpha,\beta}\}$ to a collection of parallel planes. So $\tilde{A} = \Pi \circ A$ maps the bundle $\{X^m_{\alpha,\beta}\}$ to a bundle of parallel planes.

Thus we obtain a continuous map $\tilde{A} : \cup\, X^m_{\alpha,\beta} \to \cup\, Y^m_{\gamma}$, which is sufficiently close to A in Ω due to the construction. Moreover, $\tilde{A}$ is an affine invertible map from the space $X^m_{\alpha,\beta}$ to the image $Y^m_{\gamma(\alpha,\beta)}$. This completes the proof of Lemma 3.2. $\square$

Now we are in the position to define a degree for $A \in$ F-QR (X, Y). We assume that the a priori estimate (3.2) holds. Then it follows from (3.2) that all solutions of $Ax = y$ lie in the ball

$$B_R \;=\; \{x \mid ||x|| \le R\}, R = \Phi(||y||). \tag{3.4}$$

Let A^{ν_k} be a sequence of maps due to Lemma 3.2, which approximate of A uniformly in B_R (in fact A^{ν_k} approximates A uniformly in each bounded domain). Note that A^{ν_k} is an affine bundle morphism between $(\cup_\alpha X^{\nu_k}_\alpha,\, \pi_{\nu_k} X_{\nu_k})$ and $\left(\cup_\beta Y^{\nu_k}_\beta, P_{\nu_k}, Y_{\nu_k}\right)$, where $\operatorname{codim}_X X^{\nu_k}_\alpha =\ \operatorname{codim}_Y Y^{\nu_k}_\beta$, for all $\alpha \in \pi_{\nu_k}(B_R)$, $\beta \in Y^{\nu_k}_\beta$, respectively. First we define a concept of degree for A^{ν_k}. To this end consider

$$A^{\nu_k}(x) \quad = \quad y, \quad \text{for} \;\; y \in Y. \tag{3.5}$$

We shall look for solutions of (3.5) in B_{R_0} with $R_0 = \Phi(||y|| + 3\delta)$, where δ is given by

$$\sup_{x \in B_{R_0}} \|A^{\nu_k}(x) - A(x)\| \quad < \quad \delta. \tag{3.6}$$

Our goal is to reduce problem (3.5) to the finite-dimensional case. This will be done in the following way: Let π_{ν_k} be the projection of X onto the fibers of A^{ν_k} and $\tilde{B}^{R_0} := \pi_{\nu_k}(B_{R_0})$. For each $\alpha \in \tilde{B}^{R_0}$ we define the plane $Y_\alpha^{\nu_k} = A^{\nu_k}(X_\alpha^{\alpha_{\nu_k}})$ with codim $Y_\alpha^{\nu_k} = \nu_k$. Let us consider the factor space $Y/Y_\alpha^{\nu_k} = Y_{\nu_k}^*$ with $\dim Y_{\nu_k}^* = \nu_k$. Note that $Y_{\nu_k}^*$ does not depend on $\alpha \in X_{\nu_k}$ due to Lemma 3.2.

We set

$$E_{\nu_k} = \left\{(\alpha, \beta) | \alpha \in \pi_{\nu_k}(B_{R_0}), \beta \in Y_{\nu_k}^*\right\}.$$

A topology is here introduced by means of the neighborhood system

$$W(U, V) = \{(\alpha, \beta); \;\; \alpha \in \pi_{\nu_k}(U), \beta \in \tilde{P}_{\nu_k}(V)\}$$

where U, V are neighborhoods in $\tilde{B}^{R_0}$ in Y and $\tilde{P}_{\nu_k} : Y \to Y_{\nu_k}^*$ is the projection to the factor space $Y_{\nu_k}^*$. The basis of the bundle is $\pi_{\nu_k}(\tilde{B}^{R_0})$, the total space of the bundle is E_{ν_k}. The projection is $P_{\nu_k} : E_{\nu_k} \to \pi_\nu(\tilde{B}^{R_0})$, given by

$$P_{\nu_k}(\alpha, \beta) = \alpha. \tag{3.7}$$

The structure of the locally trivial bundle is introduced as follows. For each $\alpha_0 \in \tilde{B}^{R_0}$ we shall consider some ν_k-dimensional plane Z_{ν_k}, contained in Y and transversal to $Y_{\alpha_0}^{\nu_k} := A^{\nu_k}(X_{\alpha_0}^{\nu_k})$. Then the plane Z_{ν_k} is transversal to all planes $Y_\alpha^{\nu_k}$ for $\alpha \in \tilde{U}$ where $\tilde{U}$ is some neighborhood of α_0. For each $\alpha \in \tilde{U}$ we define an isomorphism $J_{\alpha, Z_{\nu_k}, \tilde{U}} : Z_{\nu_k} \to Y_{\nu_k}^*$, namely, to every $z \in Z_{\nu_k}$ there corresponds a fiber $Y_\alpha^{\nu_k}(z)$ which is parallel to $Y_\alpha^{\nu_k}$ and which contains the point z and defines an element of the factor space $Y_{\nu_k}^*$. Note that $J_{\alpha, Z_{\nu_k}, \tilde{U}}(z)$ is an affine isomorphism, according to the transversality to Z_{ν_k} and $Y_\alpha^{\nu_k}$. Let us define the map $\tilde{U} \times Z_{\nu_k} \to P_{\nu_k}^{-1}(\tilde{U})$ which is given by the trivialization in the following way:

$$(\alpha, z) \quad \to \quad (\alpha, J_{\alpha, Z_\nu, \tilde{U}}(z)) \quad \text{for} \;\; \alpha \in \tilde{U}, z \in Z_{\nu_k}.$$

This trivialization depends on $\alpha, Z_{\nu_k}, \tilde{U}$. It is easy to see that different trivialization are compatible with each other and give in E_{ν_k} the structure of the affine bundle, that is, if $\tilde{U}$ and $\tilde{U}'$ are two neighborhoods in the basis of E_{ν_k}, containing the points α_0, α_0' respectively, Z_{ν_k} and Z'_{ν_k} are two planes transversal to $Y_\alpha^{\nu_k}$ and $Y_{\alpha'}^{\nu_k}$, for $\alpha \in \tilde{U}$ and $\alpha' \in \tilde{U}'$ and $\alpha_1 \in \tilde{U} \cap \tilde{U}'$, then

$$J^{-1}_{\alpha_1, Z_{\nu_k}, \tilde{U}} \circ J_{\alpha_1, Z_{\nu_k}, \tilde{U}'} : Z_{\nu_k} \to Z'_{\nu_k}$$

is an affine map, depending on $\alpha_1 \in \tilde{U} \cap \tilde{U}'$ continuously. Since $\tilde{B}^{R_0}$ is contractible it follows that the bundle E_{ν_k} is trivial. We define the following two sections of E_{ν_k}:

$$\begin{aligned} s^0_{\nu_k}(\alpha) &= Y_\alpha^{\nu_k} \cap Z_{\nu_k} , \\ s^1_{\nu_k}(\alpha) &= \{Y_\alpha^{\nu_k} + y\} \cap Z_{\nu_k} . \end{aligned}$$

Note that $Y_\alpha^{\nu_k}$ and that $\tilde{Y}_\alpha^{\nu_k}(y) := \{y + Y_\alpha^{\nu_k}\}$ are considered as element of the factor space $Y/Y_\alpha^{\nu_k} \simeq Y^*_{\nu_k}$.

Now we are in the position to define the degree for $A \in$ F-QR(X, Y) in the presence of the a priori estimate (3.2). By (3.2), all solutions of $Ax = y$ lie in the ball $B_R = \{x \mid ||x|| \le \Phi(||y||) = R\}$. Let A^{μ_k} be an F-R map which approximates A uniformly in the ball B_R. Let A^{ν_k} be a sequence of maps as in Lemma 3.2.

We now consider the equation

$$A^{\nu_k}(x) \quad = \quad y \quad \text{with given } \ y \in Y. \tag{3.8}$$

We shall seek solutions of (3.8) in B_{R_0}, with $R_0 = \Phi(||y|| + 3\delta)$. Now, the reduction of problem (3.8) to the finite-dimensional case can be done in the following way.

Proposition 3.3. *For sufficiently large k, finding solutions of $A^{\nu_k}(x) = y$ in B_{R_0} is equivalent to finding solutions of the equation*

$$s^0_{\nu_k}(\alpha) \quad = \quad s^1_{\nu_k}(\alpha)$$

in $\pi_{\nu_k}(B_{R_0})$. Moreover, $s^0(\alpha) \neq s^1(\alpha)$ for $\alpha \in \partial\pi_{\nu_k}(B_{R_0})$.

Proof Let $x \in B_{R_0}$ and $A^{\nu_k}(x) = y$ with $\pi_{\nu_k}(x) = \alpha$. Then the plane $\tilde{Y}_\alpha^{\nu_k}(y)$ coincides with $Y_\alpha^{\nu_k}$, so $s^0_{\nu_k}(\alpha) = s^1_{\nu_k}(\alpha)$. Conversely, let $s^0_{\nu_k}(\alpha) = s^1_{\nu_k}(\alpha)$. Hence $y \in Y_\alpha^{\nu_k}$, and so $A_\alpha^{\nu_k}(x) = y$ for some $x \in X$. The point x, however, might lie outside the ball B_{R_0}. We will show that this can not happen. Indeed, from definition of $A \in$ F-QR(X, Y) and the construction of the maps A^{ν_k} it follows that there exists $C > 0$ such that $||A^{\nu_k}_\alpha||, ||(A^{\nu_k}_\alpha)^{-1}|| < 2C$ for $\alpha \in \pi_{\nu_k}(B_{R_0})$ and for sufficiently large k. Therefore, each pre-image of the point y under the affine map $A^{\nu_k}_\alpha$ lies in the ball B_{C_1}, where $C_1 = C(1 + ||y||)$. Recall that we have to show that $x \in B_{R_0}$, where $A^{\nu_k}_\alpha(x) = y$. If not so, we would have $R_0 \leq ||x|| < C_1$. But then

$$||A^{\nu_k}(x)|| \geq ||Ax|| - \sup_{x \in B_{C_1}} ||Ax - A^{\nu_k}x|| \geq ||y|| + 3\delta - \delta = ||y|| + 2\delta$$

because of (3.2), so that x could not be the pre-image of y. Hence, $||x|| < R_0$, which completes the proof. $\square$

Thus the equation $A^{\nu_k}(x) = y$ is reduced to the finite-dimensional case. We now define

$$d(A^{\nu_k}) \quad = \quad \deg_B(s^0(\alpha) - s^1(\alpha), \tilde{B}^{R_0}, 0), \tag{3.9}$$

where $\deg_B$ is the Brouwer degree. Note that $d(A^{\nu_k})$ is an integer depending on the orientation in $\pi_\nu(B_{R_0})$ or in Z_ν. In fact, we have no reason to prefer one orientation in $\pi_\nu(B_{R_0})$ or in Z_ν over another, since $GL(X, Y)$ is connected, in general.

Now let $\nu_k \to \infty$ and $A^{\nu_k} \to A$. We will show that there exists $k_0 > 0$ such that

$$|d(A^{\nu_k})| \quad = \quad \text{const}, \tag{3.10}$$

for $k \geq k_0$. Assuming (3.10), we can define

Definition 3.4. $d(A) = \lim_{k\to\infty} |d(A^{\nu_k})|$

The next three lemmata will imply (3.10).

The maps A^{μ_k} and A^{ν_k} are considered equal if and only if $\mu_k = \nu_k$, the decomposition of X into parallel planes $X^{\mu_k}_\alpha$ coincides with the decomposition $X^{\nu_k}_\beta$ and $A^{\mu_k}(x) = A^{\nu_k}(x)$ for all $x \in X$. Note that two F-R maps are considered different if they have different fibers, even if they coincide as maps.

Lemma 3.5. *Let the F-R-maps $A^{\mu_{k+1}}$ which approximate A be such that each fiber $X^{\mu_{k+1}}_{\tilde{\alpha}}$ is contained in some fiber $X^{\mu_k}_\alpha$ of the F-R map A^{μ_k}. Moreover,* *we suppose that* $A^{\mu_{k+1}}(x) = A^{\mu_k}(x)$ *in* $B_{C_1}, C_1 = C(1 + ||y||)$. *Then* $d(A^{\mu_k}) = d(A^{\mu_{k+1}})$.

PROOF. Without loss of generality one can assume that $\mu_{k+1} = \mu_k + 1$. Then $\alpha = (\alpha_1, \ldots, \alpha_{\nu_k})$ and $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_{\nu_k}, \alpha_{\nu_k+1})$. Let A^{ν_k} and A^{ν_k+1} be the maps due to our construction in Lemma 3.2, that is, they are F-R maps which approximate A^{μ_k} and A^{μ_k+1}, respectively, in some sufficiently large ball B_R in X. To prove that $d(A^{\mu_k}) = d(A^{\mu_k+1})$ it suffices to show $d(A^{\nu_k}) = d(A^{\nu_k+1})$.

Let us recall that corresponding locally trivial bundles E_{ν_k} and E_{ν_k+1} for A^{ν_k} and A^{ν_k+1}, respectively, satisfy the following conditions:

$$E_{\nu_k+j} = (\cup_{\alpha \in X_{\nu_k+j}} X_\alpha^{\nu_k+j}, \pi_{\nu_k}, \cup_{\beta \in Y_{\nu_k+j}} Y_\beta^{\nu_k+j}) \quad \text{for } j = 0, 1\,,$$

where $\alpha = (\alpha_1, \ldots, \alpha_{\nu_k+j})$ are the coordinates in the original basis and $(\beta_1, \ldots, \beta_{\nu_k+j})$ are those in the fiber. By assumption, each plane $X_{\tilde{\alpha}}^{\nu_k+1}$, with $\tilde{\alpha} = (\alpha_1, ..., \alpha_{\nu_k+1})$, lies in some plane $X_\alpha^{\nu_k}$ where $\alpha = (\alpha_1, ..., \alpha_{\nu_k})$ and so,

$$Y^{\nu_k+1}_{\alpha_1 \ldots \alpha_{\nu_k+1}} = A^{\nu_k+1}_{\tilde{\alpha}}(X^{\nu_k+1}_{\tilde{\alpha}}) \quad \subset \quad Y^{\nu_k}_{\alpha_1 \ldots \alpha_{\nu_k}} = A^{\nu_k}_\alpha(X^{\nu_k}_\alpha).$$

Hence each plane $Y^{\nu_k+1}_{\tilde{\alpha},\tilde{\beta}}$, is parallel to $Y^{\nu_k}_\alpha$ and lies in some plane $Y^{\nu_k}_{\alpha,\beta}$ parallel to $Y^{\nu_k}_\alpha$. As a coordinate $\vec{\beta}_j = (\beta_1, ..., \beta_{\nu_{k+j}})$ we take intersections of the parallel families $Y/Y^{\nu_k}_\alpha$ and $(Y/Y^{\nu_k+1}_{\tilde{\alpha}})$ with the transversal planes Z_{ν_k} and Z_{ν_k+1}, respectively. Let $\tilde{s}^0(\tilde{\alpha})$ and $\tilde{s}^1(\tilde{\alpha})$ be sections of the bundles E_{ν_k+1}. Then it is not difficult to see that $\tilde{s}^1(\tilde{\alpha}) = const = \beta^*$, since the planes $\{Y^{\nu_k+1}_{\tilde{\alpha}}\}$ are parallel to each other for different $\tilde{\alpha}$. On the other hand, one can see from $Y^{\nu_k+1}_{\tilde{\alpha}} \subset Y^{\nu_k}_\alpha$ and the fact that $A^{\nu_k+1}_{\tilde{\alpha}}$ is an affine isomorphism, that

$$\tilde{s}^0(\tilde{\alpha}) \quad = \quad (\tilde{s}^0_1(\alpha), \ldots, \tilde{s}^0_{\nu_k}(\alpha), a(\alpha) \cdot \alpha_{\nu_k+1} + b(\alpha))$$

with continuous $a(\alpha)$ and $b(\alpha)$ and $a(\alpha) \neq 0$. Let $\tilde{s}(\tilde{\alpha}) = \tilde{s}^0(\tilde{\alpha}) - \beta_*$ and $\bar{s}_*(\tilde{\alpha}) = (\bar{s}^0_1(\alpha), \ldots, \bar{s}^0_{\nu_k}(\alpha), \bar{a}(\alpha)\alpha_{\nu_k+1} + \bar{b}(\alpha)) - \beta_*$ be a smooth approximation of $\tilde{s}(\tilde{\alpha})$. Then it is clear that for each point $\alpha^{(k)}$ with $\bar{s}(\alpha^{(k)}) = 0$, there exists a unique point $\tilde{\alpha}^{(k)} = (\alpha^{(k)}, \alpha_{\nu_k+1})$ at which $\bar{s}_*(\tilde{\alpha}^{(k)}) = 0$. Hence

$$\frac{\partial \bar{s}_*}{\partial \tilde{\alpha}}\Big|_{\alpha = \tilde{\alpha}^{(k)}} \quad = \quad \begin{pmatrix} \frac{\partial \bar{s}^0(\alpha^{(k)})}{\partial \alpha} & \vdots \\ \underbrace{0 \ldots 0}_{\nu_k} & \bar{a}(\alpha^{(k)}) \end{pmatrix},$$

and so,

$$\det(\frac{\partial \bar{s}_*}{\partial \tilde{\alpha}})\Big|_{\alpha = \tilde{\alpha}^{(k)}} \quad = \quad \bar{a}(\alpha^{(k)}) \cdot \det(\frac{\partial \bar{s}}{\partial \alpha})\,.$$

Hence

$$|d(A^{\nu_{k+1}})| = |\sum sgn \det(\frac{\partial \bar{s}_*(\tilde{\alpha}^{(k)})}{\partial \tilde{\alpha}})| = |\sum sgn \det(\frac{\partial \bar{s}(\alpha^{(k)})}{\partial \alpha})| = |d(A^{\nu_k})|$$

since $a(\alpha) \neq 0$ for all $\alpha \in \pi_{\nu_k}(B_{R_0})$. This proves the lemma, since $d(A^{\nu_k})$ is defined only up to the sign. □

Lemma 3.6. *Let $A_t^{\mu_k}$ be a series of Fredholm F-R maps defined in B_{C_1} and continuous for $t \in [0,1]$. Let $\sup_{x \in B_{C_1}} ||Ax - A_t^{\mu_k}x|| < \delta$ for all $t \in [0,1]$ where δ is defined as in (3.5). Then $d(A_t^{\mu_k}) = const$.*

Proof. Due to our construction, let $A_t^{\nu_k}(\nu_k \geq \mu_k)$ be an approximation of $A_t^{\mu_k}$ in some sufficiently large ball. Then $d(A_t^{\nu_k})$ is continuous in t and equals only just integer values. □

Lemma 3.7. *Let A^{μ_k} be a sequence of F-R maps which converges to A. Then there exists $k_0 > 0$ such that $d(A^{\mu_p}) = d(A^{\mu_q})$ for $p \geq q \geq k_0$.*

Proof. Without loss of generality one can assume that the decomposition of X into planes $X_\alpha^{\mu_p}$ is contained in the decomposition of X into planes $X_\beta^{\mu_q}$. Let $\tilde{A}^{\mu_p}$ be the map A^{μ_q} considered as an F-R map with fibers $X_\alpha^{\mu_p}$. As follows from Lemma 3.6, we have $d(\tilde{A}^{\mu_p}) = d(A^{\mu_q})$. Let us consider the family of F-R maps

$$A_t^{\mu_p} = (1-t)A^{\mu_p} + t\tilde{A}^{\mu_p}.$$

Due to the homotopy invariance property in Lemma 3.6, we have

$$d(A^{\mu_p}) = d(\tilde{A}^{\mu_p}).$$

Hence,

$$d(A^{\mu_p}) = d(A^{\mu_q}),$$

which proves the lemma. □

This justifies Definition 3.4.

Before proving the Borsuk-Ulam property for $A \in$ F-QR(X, Y) we still need the following propositions. Their proofs are more or less standard and based on both, compactness and homotopy invariance for F-R approximations. We omit the proofs.

Proposition 3.8. *(See e.g.* [2, Proposition 4.5]*.) Let* $A \in$ F-QR(X, Y) *and suppose an a priori estimate* $||x||_X \leq \Phi(||Ax||_Y)$ *as in* (3.2)*. In addition assume that* $d(A) \neq 0$*. Then* $Ax = y$ *has a solution for every* $y \in Y$*.*

Proposition 3.9. *Let* $A_t \in$ F-QR(X, Y) *depending continuously on* $t \in [0, 1]$*. Assume moreover that an a priori estimate* $||x||_X \leq \Phi(||A_t x||_Y)$ *holds as in* (3.2)*. Then* $d(A_t) = const.$

After these preparations we finally can prove the Borsuk-Ulam property for $A \in$ F-QR(X, Y). Indeed, let all the assumptions of Theorem 1.2 be satisfied. Let $A^{\mu_k} : X \to Y$ be a sequence of F-R maps due to Definition 2.2. Because $A(-x) = -Ax$, one can choose $A^{\mu_k} : X \to Y$ in such a way that $A^{\mu_k}(-x) = -A^{\mu_k}(x)$. Indeed, if B^{μ_k} is some sequence of F-R maps approximating A, then choose $A^{\mu_k}(x) = \frac{1}{2}\left(-B^{\mu_k}(-x) + B^{\mu_k}(x)\right)$ which implies $A^{\mu_k}(-x) = -A^{\mu_k}(x)$. As a result of our construction, we obtain also that $A^{\nu_k}(-x) = -A^{\nu_k}(x)$. Thus we have that the map $s^0_{\nu_k} : X_{\nu_k} \to Z_{\nu_k}$ defined by (3.7), is also odd. Hence, $d(A^{\nu_k})$ is odd.

As a result of Definition 3.4, we obtain that $d(A)$ is also odd which proves Theorem 1.2.

4. Nonlinear Riemann-Hilbert problems with nonsmooth and noncompact boundary data

This section is devoted to the global existence theorem of solutions of nonlinear Riemann-Hilbert problems, when the boundary condition is nonsmooth, i.e. only Lipschitz continuous and going off to infinity (see below). Let us consider the nonlinear Riemann-Hilbert problem

$$v(e^{i\tau}) \quad = \quad f(\tau, u(e^{i\tau})), \quad 0 \leq \tau < 2\pi, \tag{4.1}$$

where $f : [0, 2\pi) \times \mathbb{R} \to \mathbb{R}$ is a given real-valued function and the unknown function $w(z) = u(z) + iv(z)$ is holomorphic in $G_1 = \{z| \ |z| < 1\}$ and continuous on the closure $\bar{G}_1$. Write $\gamma_\tau = \{(u, v)|v - f(\tau, u) = 0\}$ for every fixed τ. The family of curves γ_τ is called the boundary data for the $(RHP)_1$. We emphasize that most of the existing results concerning the global existence of solutions are restricted to the case when γ_τ is at least C^1 and depends "smoothly" on τ (see [3, 8, 9] and references therein). The first global existence result for nonsmooth boundary conditions goes back to M.A. Efendiev in [2] for compact boundary data (see also [5, 7]). Here we present an existence theorem for solutions of the nonlinear Riemann-Hilbert problem (4.1) for noncompact data which is based on the Borsuk-Ulam property of a quasiruled Fredholm mapping. We impose the following restrictions on the boundary conditions

$$\gamma_\tau \quad = \quad \{(u,v)|v - f(\tau,u) = 0\}\,. \tag{4.2}$$

Conditions 4.1. *Let the function $f(\tau,u)$ satisfy:*

a) $f(\tau + 2\pi, u) = f(\tau, u)$ *for all* $(\tau, u) \in \mathbb{R} \times \mathbb{R}$
b) $f(\tau, -u) = -f(\tau, u)$
c) $f(\tau, u)$ *is Lipschitz with respect to both arguments* (τ, u).

Let $C^\alpha(S^1)$ be the standard Hölder space of periodic functions with some $\alpha \in (0,1)$ and $\mathfrak{M}$ be a set of Hölder continuous solutions of the nonlinear $(RHP)_1$ for some $\alpha \in (0,1)$.

Our main result is the following theorem.

Theorem 4.2. *Let the family of curves γ_τ defined by* (4.1) *satisfy Conditions* 4.1. *Then $\mathfrak{M} \neq \phi$. Moreover, if in addition to Conditions* 4.1, *supp $f \in (-R, R)$ for all (τ, u) then either*

a) $\mathfrak{M} = \{const|\ |const| \geq R\}$
or
b) *for all non constant $w \in \mathfrak{M}$ there holds $||w(z)||_{L^\infty} \leq R$.*

Proof. Case 1: We consider first the case, when supp $f \subset (-R, R)$ and $f(\tau, u)$ satisfies Conditions 4.1. Let $f_\varepsilon(\tau, u)$ be a smooth approximation of f, such that

a) supp $f_\varepsilon \subset [-R, R]$
b) $f_\varepsilon(\tau, -u) = -f_\varepsilon(\tau, u)$
c) $|\frac{\partial f_\varepsilon}{\partial \tau}| \leq M$ and $|\frac{\partial f_\varepsilon}{\partial u}| \leq M$ for all (τ, u)

with M independent of ε. The existence of such approximations f_ε is evident since f can be approximated by the use of a sequence of mollifiers.

Given $f_\varepsilon(\tau, u)$, we consider the nonlinear Riemann-Hilbert problem

$$v(e^{i\tau}) \quad = \quad f_\varepsilon(\tau, u(e^{i\tau})), \tag{4.3}$$

for the unknown holomorphic function $w(z) = u(z)+iv(z)$ in $G_1 = \{z|\ |z| < 1\}$ which is continuous. Note that (4.3) is equivalent to the following family of nonlinear singular integral equations

$$A_{\lambda,\varepsilon} u \quad = \quad 0, \tag{4.4}$$

where

$$(A_{\lambda,\varepsilon} u)(e^{i\tau}) \quad := \quad (Hu)(e^{i\tau}) + \lambda - f_\varepsilon(\tau, u(e^{i\tau})), \tag{4.5}$$

for every fixed $\lambda \in \mathbb{R}^1$ where H is the Hilbert transform introduced in Section 2.

We denote by $\mathfrak{M}_\varepsilon$ the set of solutions of this nonlinear $(RHP)_\varepsilon$. As a consequence of the main result in [3, Theorem3.1], we obtain either $\mathfrak{M}_\varepsilon = \{const|\ |const| \geq R\}$ or any nonconstant solution $w_\varepsilon(z) = u_\varepsilon(z) + iv_\varepsilon(z)$ of $(RHP)_\varepsilon$ satisfies an a priori estimate $||w_\varepsilon||_{H^s(S^1)} \leq C$ with $s \geq 2$ and where C does not depend on ε. Taking into account that $A_{\lambda,\varepsilon} \in$ F-QR$(H^s(S^1), H^s(S^1)$, for every fixed λ and ε, we conclude that the degree $d(A_{\lambda,\varepsilon})$ for $A_{\lambda,\varepsilon}$ is well-defined. Note that $A_{0,\varepsilon}$ is odd, hence $d(A_{0,\varepsilon}) \neq 0$ which yields existence of at least one solution $w_\varepsilon(z)$ of $(RHP)_\varepsilon$. Due to embedding results there holds $w_\varepsilon(e^{i\tau}) \in C^\alpha(S^1)$ for some $\alpha \in (0,1)$. It remains to use convergence arguments. Indeed, due to the a priori estimate $||w_\varepsilon||_{C^\alpha} \leq C$ (uniformly with respect to ε) and due to Montel's theorem, there exists a subsequence $w_{\varepsilon_j}(z)$ which converges to some $w_0(z) = u_0(z) + iv_0(z)$ uniformly on every compact subset $K \subset G_1$. Moreover, since w_ε are uniformly Hölder continuous on S^1, the subsequence $w_{\varepsilon_j}(z)$ is uniformly Hölder continuous on $\bar{G}_1$. Hence, $w_0(z)$ is also Hölder continuous on $\bar{G}_1$. We now show that $w_0(z) \in \mathfrak{M}$. Indeed, let

$$\gamma_\tau^\delta = \{w = u + iv|\ dist(w, \gamma_\tau) < \delta\}$$

It is obvious that $w_{\varepsilon_j}(e^{i\tau}) \in clos\, int\gamma_\tau^\delta$ for all $\delta > 0$, since $||w_{\varepsilon_j}||_{L^\infty} \leq R$. Hence

$$w_{\varepsilon_j}(re^{i\tau}) \in clos\ int\gamma_\tau^{2\delta} \tag{4.6}$$

for all $r_0 < r < 1$ and for some $0 < r_0 < 1$ since the family $\{w_{\varepsilon_j}(z)\}$ is uniformly Hölder continuous on $\bar{G}_1$. Then it follows from (4.6) that $w_0(re^{i\tau}) \in clos\ int\gamma_\tau^\delta$ for $r \in (r_0, 1)$. Sending $r \to 1$ we obtain $w_0(e^{i\tau}) \in clos\ int\gamma_\tau^\delta$ for every $\delta > 0$. Hence, $w_0(e^{i\tau}) \in \cap_{\delta>0} clos\ int\gamma_\tau^\delta = \gamma_\tau$. This proves Theorem 4.2 in the case supp $f \in (-R, R)$.

Remark 4.1. Our arguments are not valid for closed boundary data. For example, consider $\gamma_\tau \equiv \gamma_0 = \{|w| = 1\}$ and $w_n(z) = z^n$. In the above proof we essentially used the fact that $int\gamma_\tau^\delta$ is simply-connected, which is not the case, when γ_τ is defined by closed boundary data.

General case: Let the given function $f(\tau, u)$ satisfy Conditions 4.1. Note that in this case we cannot use the arguments of [3] and we need a different approach. As in the previous case, let $f_\varepsilon(\tau, u)$ be some smooth approximation of $f(\tau, u)$ which satisfies the following conditions:

1) $f_\varepsilon \in C^\infty(S^1 \times \mathbb{R})$,
2) $f_\varepsilon(\tau, -u) = -f_\varepsilon(\tau, u)$,
3) $|\frac{\partial f_\varepsilon}{\partial \tau}| \leq M$ and $|\frac{\partial f_\varepsilon}{\partial u}| \leq M$, for all $(\tau, u) \in (S^1 \times \mathbb{R})$ uniformly with respect to ε.

The existence of such approximations again is evident. We begin again with the nonlinear $(\widetilde{RHP})_\varepsilon$, that is

$$v(e^{i\tau}) = f_\varepsilon(\tau, u(e^{i\tau}))\ (\widetilde{RHP})_\varepsilon$$

for the unknown holomorphic function $w(z) = u(z) + iv(z)$ in G_1, continuous in $\bar{G}_1$. First of all we derive an a priori estimate for solutions of $(\widetilde{RHP})_\varepsilon$. Note that due to the smoothness of $f_\varepsilon(\tau, u)$, the solutions of $w^\varepsilon := u^\varepsilon + iv^\varepsilon$ of $(\widetilde{RHP})_\varepsilon$ (if they exist) are also smooth (see[3],[8],[9]). Therefore one can differentiate the boundary conditions $(\widetilde{RHP})_\varepsilon$ with respect to the angular coordinate $\tau \in [0, 2\pi)$. As a result we obtain

$$a_\varepsilon(e^{i\tau})ie^{i\tau}\frac{\partial u^\varepsilon}{\partial \tau} + b_\varepsilon(e^{i\tau})ie^{i\tau}\frac{\partial v^\varepsilon}{\partial \tau} = c_\varepsilon(e^{i\tau}) \tag{4.4}$$

where

$$a_\varepsilon(e^{i\tau}) := -\frac{\partial f_\varepsilon(\tau, u(e^{i\tau}))}{\partial u}\ b_\varepsilon(e^{i\tau}) := 1,\ \ c_\varepsilon(e^{i\tau}) := -ie^{i\tau}\frac{\partial f_\varepsilon}{\partial \tau}(\tau, u(e^{i\tau})).$$

Thus, for $\tilde{w}_\varepsilon(z) := iz\frac{dw_\varepsilon}{dz}$ we obtain a "quasi-linear RHP" of the form (4.4). Taking into account the estimate $||a_\varepsilon||_{L^\infty} + ||b_\varepsilon||_{L^\infty} + ||c_\varepsilon||_{L^\infty} \leq M$, we obtain that $\partial_\tau w^\varepsilon := \partial_\tau u^\varepsilon + i\partial_\tau v^\varepsilon$ can be extended holomorphically into G_1 and vanishes at the origin. Note that, due to Conditions 4.1 the functions $\varphi_\varepsilon := \arg(a_\varepsilon + b_\varepsilon)$ satisfy $||\varphi_\varepsilon||_{L^\infty} < \frac{\pi}{2p}$ for some $p > 1$ uniformly. Using explicit expressions for the solutions of the linear (RHP) (see [8, 9]) we obtain the uniform L_p-estimate $||\partial_\tau w^\varepsilon||_p \leq C$. Taking into account that the initial values for w^ε do not depend on ε, we obtain that the solutions $w_\varepsilon(z)$ are uniformly bounded in W_p^1 (if they exist). Since W_p^1 is compactly embedded in $C^\alpha(S^1)$ for $0 < \alpha < 1 - \frac{1}{p}$, we obtain also a uniform a priori estimate for $w_\varepsilon(z)$ in $C^\alpha(S^1)$. Using regularity results (see [3, 8, 9] and exploiting for the odd operator

$$(A_{\varepsilon,0}u)(e^{i\tau}) \quad = \quad (Hu)(e^{i\tau}) - f_\varepsilon(\tau, u(e^{i\tau}))$$

the Borsuk-Ulam property, we find the existence of a Hölder continuous solution $w_\varepsilon(z) = u_\varepsilon(z) + iv_\varepsilon(z)$ of the nonlinear (RHP) (4.1). As in case 1, we can select a sequence $\varepsilon_j \to 0$ so that the corresponding solutions $w_{\varepsilon_j}(z)$ converge to a function $w_0 \in C^\alpha(S^1)$ which satisfies the original boundary condition. This proves Theorem 4.2. □

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References

[1] C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere,* Acad. Polon. Sci. Ser. Math., **14** (1966), 27-31.

[2] M.A. Efendiev, *Nonlinear Hilbert problem with smoothly immersion curves in* $\mathbb{R}^2$, (Russian), Izv. Nauk. Azerb SSR, Ser. Fiz-Tekhn. and Mat. Nauk, **4 3** (1982), 7-8.

[3] M.A. Efendiev, W.L. Wendland, *Nonlinear Riemann-Hilbert problems for multiply connected domains*, *J. Nonlinear Analysis*, **27** (1996), 37-58.

[4] M.A. Efendiev, W.L. Wendland, *Nonlinear Riemann-Hilbert problems without transversality*, Math. Nachrichten, **183** (1997), 73-89.

[5] M.A. Efendiev, W.L. Wendland, *Nonlinear Riemann-Hilbert problems and the similarity principle for multiply connected domains*, In: Analysis, Numerics and Applications of Differential and Integral Equations, (eds. Bach et al.) Pitman Research Notes in Math. Ser. 379, Longman, London, 1998, 81-86.

[6] P. M. Fitzpatrick and J. Pejsachowicz, *An extension of the Leray-Schauder degree for fully nonlinear elliptic problems.* In: Proceedings of Symposia in Pure Mathematics, **45**, Amer. Math. Soc. Providence, R. I. (1986), 425-439.

[7] M. Reissig, E. Wegert, *Nonlinear boundary value problems for elliptic systems in the plane*, Complex Variables, **27** (1995), 193-210.

[8] A.I. Šnirel'man, *The degree of quasi-ruled mapping and a nonlinear Hilbert problem*, Math. USSR-Sbornik, **18** (1972), 373-396.

[9] E. Wegert, *Nonlinear Boundary Value Problems for Holomorphic Functions and Singular Integral Equations*, Akademie-Verlag, Berlin, 1992.

[10] E. Zeidler, *Nonlinear Functional Analysis and its Applications; Vol.I - Fixed Point Theorems.* Springer-Verlag, Berlin, 1986.

[11] E. Zeidler, *Nonlinear Functional Analysis and its Applications; Vol.IV - Applications to Mathematical Physics*. Springer-Verlag, Berlin 1988.

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