INITIAL VALUE PROBLEM ON SEMI-LINE FOR DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. A slight extension of the continuation principle established in [1] is used to prove the existence of solution for the initial value problem for first-order differential equations with advanced argument on semi-line.

Key Words and Phrases: fixed point, continuation principle, initial value problem.

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1. Introduction

The aim of this paper is to present a natural application of the continuation principle established in [1] involving contractions in Gheorghiu’s sense, with respect to a family of pseudo-metrics.

The advanced argument in our differential equation makes necessary the use of two pseudo-metrics in the contraction condition.

In what follows we recall some notions and results from paper [1].

First recall the notion of contraction on a gauge space introduced by Gheorghiu [5]. Let \((X, \mathcal{P})\) be a gauge space with \(\mathcal{P} = \{p_\alpha\}_{\alpha \in A}\). A map \(F : D \subset X \to X\) is a contraction if there exists a function \(\varphi : A \to A\) and \(a \in \mathbb{R}^+_A, a = \{a_\alpha\}_{\alpha \in A}\) such that

\[
p_\alpha(F(x), F(y)) \leq a_\alpha p_{\varphi(\alpha)}(x, y) \quad \forall \alpha \in A, x, y \in D,
\]

\[
\sum_{n=1}^{\infty} a_\alpha a_{\varphi^2(\alpha)} \cdots a_{\varphi^{n-1}(\alpha)} p_{\varphi^n(\alpha)}(x, y) < \infty
\]

for every \(\alpha \in A\) and \(x, y \in D\). Here, \(\varphi^n\) is the \(n\)th iteration of \(\varphi\).
We present now a slow extension of some results from paper [1], whose proofs are similar.

**Theorem 1.1.** Let \( X \) be a set endowed with two separating gauge structures \( \mathcal{P} = \{ p_\alpha \}_{\alpha \in A} \) and \( \mathcal{Q} = \{ q_\beta \}_{\beta \in B} \), let \( D_0 \) and \( D \) be two subsets of \( X \) with \( D_0 \subset D \), and let \( F : D \to X \) be a map. Assume that \( F(D_0) \subset D_0 \) and \( D \) is \( \mathcal{P} \)-closed. In addition, assume that the following conditions are satisfied:

(i) there is a function \( \psi : A \to B \) and \( c \in (0, \infty)^A \), \( c = \{ c_\alpha \}_{\alpha \in A} \) such that

\[ p_\alpha(x,y) \leq c_\alpha q_{\psi(\alpha)}(x,y) \quad \forall \alpha \in A, x, y \in X; \]

(ii) \( (X, \mathcal{P}) \) is a sequentially complete gauge space;

(iii) if \( x_0 \in D \), \( x_n = F(x_{n-1}) \) for \( n = 1, 2, \ldots \), and \( \mathcal{P}\text{-}\lim_{n \to \infty} x_n = x \) for some \( x \in D \), then \( F(x) = x \);

(iv) \( F \) is a \( Q \) contraction on \( D_0 \).

Then \( F \) has at least one fixed point which can be obtained by successive approximation starting from any element of \( D_0 \).

For a map \( H : D \times [0,1] \to X \), where \( D \subset X \), we will use the following notations:

\[ \Sigma = \{ (x, \lambda) \in D \times [0,1] : H(x, \lambda) = x \}, \]

\[ S = \{ x \in D : H(x, \lambda) = x \text{ for some } \lambda \in [0,1] \}, \quad (1.1) \]

\[ \Lambda = \{ \lambda \in [0,1] : H(x, \lambda) = x \text{ for some } x \in D \}. \]

**Theorem 1.2.** Let \( X \) be a set endowed with the separating gauge structures \( \mathcal{P} = \{ p_\alpha \}_{\alpha \in A} \) and \( \mathcal{Q}^\lambda = \{ q_\beta^\lambda \}_{\beta \in B} \) for \( \lambda \in [0,1] \). Let \( D \subset X \) be \( \mathcal{P} \)-sequentially closed, \( H : D \times [0,1] \to X \) a map, and assume that the following conditions are satisfied:

(i) for each \( \lambda \in [0,1] \), there exists a function \( \varphi_\lambda : B \to B \) and \( a^\lambda \in (0,1)^B \), \( a^\lambda = \{ a_\beta^\lambda \}_{\beta \in B} \) such that

\[ q_\beta^\lambda(H(x,\lambda),H(y,\lambda)) \leq a_\beta^\lambda q_{\varphi_\lambda(\beta)}(x,y), \quad (1.2) \]

\[ \sum_{n=1}^{\infty} a_\beta^\lambda a_{\varphi_\lambda(\beta)}^\lambda \ldots a_{\varphi_\lambda^{n-1}(\beta)}^\lambda < \infty \quad (1.3) \]

for every \( \beta \in B \) and \( x,y \in D \);

(ii) there exists \( \rho > 0 \) such that for each \( (x,\lambda) \in \Sigma \), there is a \( \beta \in B \) with

\[ \inf \{ q_\beta^\lambda(x,y) : y \in X \setminus D \} > \rho; \quad (1.4) \]
(iii) for each \( \lambda \in [0, 1] \), there is a function \( \psi : A \to B \) and \( c \in (0, \infty)^A \), \( c = \{c_\alpha\}_{\alpha \in A} \) such that
\[
p_\alpha(x, y) \leq c_\alpha q_\psi(\alpha)(x, y) \quad \text{for all } \alpha \in A \text{ and } x, y \in X; \quad (1.5)
\]
(iv) \((X, \mathcal{P})\) is a sequentially complete gauge space;
(v) if \( \lambda \in [0, 1] \), \( x_0 \in D, x_n = H(x_{n-1}, \lambda) \) for \( n = 1, 2, \ldots \), and \( \mathcal{P}/ \lim_{n \to \infty} x_n = x \), then \( H(x, \lambda) = x \);
(vi) for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) with
\[
q_\psi^\lambda(\beta)(x, H(x, \lambda)) \leq (1 - a_\gamma^\lambda(\beta))\varepsilon
\]
for \((x, \mu) \in \Sigma, \ |\lambda - \mu| \leq \delta, \ \text{all } \beta \in B, \ \text{and } n \in \mathbb{N}.

In addition, assume that \( H_0 := H(., 0) \) has a fixed point. Then, for each \( \lambda \in [0, 1] \), the map \( H_\lambda := H(., \lambda) \) has at least a fixed point.

2. The main result

We consider the problem Cauchy
\[
\begin{align*}
x'(t) &= f(t, x(t + 1)), \quad t \in [0, \infty) \\
x(0) &= 0.
\end{align*}
\]
This problem is equivalent to the Volterra integral equation
\[
x(t) = \int_0^t f(s, x(s + 1))ds, \quad t \in [0, \infty) \quad (2.7)
\]

**Theorem 2.1.** Let \((E, \|\|)\) be a Banach space, and let \( f : \mathbb{R}_+ \times E \to E \) be a continuous function. Assume that the following conditions hold:

(a) there exists \( k \in L^1(\mathbb{R}_+, (0, \infty)) \) with \( |k|_{L^1(\mathbb{R}_+)} < 1 \) such that
\[
\|f(t, x) - f(t, y)\| \leq k(t)\|x - y\|
\]
for all \( x, y \in E \), and \( t \in \mathbb{R}_+ \);

(b) for each \( n \in \mathbb{N}\setminus\{0\} \) there exists \( r_n > 0 \) such that, any continuous solution \( x \) of the equation
\[
x(t) = \lambda \left(\int_0^t f(s, x(s + 1))ds\right), \quad t \in \mathbb{R}_+
\]
with \( \lambda \in [0, 1] \), satisfies \( \|x(t)\| \leq r_n \) for any \( t \in [0, n] \);

(c) there exists \( \alpha \in L^1(\mathbb{R}_+) \) and \( \beta : \mathbb{R}_+ \to (0, \infty) \) nondecreasing such that
\[
\|f(t, x)\| \leq \alpha(t)\beta(\|x\|)
\]
for all $t \in \mathbb{R}^+$ and $x \in E$;

(d) there exists $C > 0$ such that \( \frac{\beta(r_{k+1})}{1 - L_k} \leq C \) for any $k \in \mathbb{N}\setminus\{0\}$, where

\[
L_k = \int_0^k k(s)ds.
\]

Then there exists at least one solution $x \in C(\mathbb{R}^+, E)$ of the integral equation (2.7).

Proof. For the proof we use Theorem 1.2.

Let $X = \{x \in C(\mathbb{R}^+, E) : x(0) = 0\}$. For each $n \in \mathbb{N}\setminus\{0\}$ we define the map $|.|_n : X \to \mathbb{R}^+$, by $|x|_n = \max_{t \in [0,n]} \|x(t)\|$. This map is a semi-norm on $X$, and let $d_n : X \times X \to \mathbb{R}^+$ be given by

\[
d_n(x, y) = |x - y|_n = \max_{t \in [0,n]} \|x(t) - y(t)\|.
\]

It is easy to show that $d_n$ is a pseudo-metric on $X$ and the family $\{d_n\}_{n \in \mathbb{N}\setminus\{0\}}$ defines on $X$ a gauge structure, separated and complete by sequences.

Here $P = \mathcal{Q}_\lambda = \{d_n\}_{n \in \mathbb{N}\setminus\{0\}}$ for each $\lambda \in [0, 1]$. Let $D$ be the closure in $X$ of the set

\[
\{x \in X : \text{there exists } n \in \mathbb{N}\setminus\{0\} \text{ such that } d_n(x, 0) \leq r_n + \delta \}
\]

where $\delta > 0$ is a fixed number. We define $H : D \times [0, 1] \to X$, by $H(x, \lambda) = \lambda A(x)$, where

\[
A(x)(t) = \int_0^t f(s, x(s + 1))ds.
\]

First we verify condition (i) from Theorem 1.2.

Let $t \in [0, n]$. We estimate

\[
\|H(x, \lambda)(t) - H(y, \lambda)(t)\| \leq \lambda \int_0^t \|f(s, x(s + 1)) - f(s, y(s + 1))\| \, ds
\]

\[
\leq \int_0^t k(s) \|x(s + 1) - y(s + 1)\| \, ds
\]

\[
\leq \max_{s \in [0, n]} \|x(s + 1) - y(s + 1)\| \int_0^t k(s) \, ds
\]

\[
\leq \max_{\tau \in [0, n+1]} \|x(\tau) - y(\tau)\| \int_0^n k(s) \, ds
\]

\[
= L_n d_{n+1}(x, y).
\]
If we take the maximum with respect to \( t \), we obtain

\[
d_n(H(x, \lambda), H(y, \lambda)) \leq L_n d_{n+1}(x, y)
\]

for all \( x, y \in D \) and all \( n \in \mathbb{N} \setminus \{0\} \). Hence, condition (i) in Theorem 1.2 holds with \( \varphi_\lambda = \varphi \) where \( \varphi : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\} \) is defined by \( \varphi(n) = n + 1 \). In addition the series \( \sum_{n=1}^{\infty} L_n x_{n+1} \cdots x_n \) is finite since from assumption (a) we know that \( |k| L^1(\mathbb{R}_+) < 1 \) and so \( L_n \leq |k| L^1(\mathbb{R}_+) < 1 \).

Condition (ii) in our case becomes: there exists \( \rho > 0 \) such that for any solution \( (x, \lambda) \in D \times [0, 1] \), to \( x = H(x, \lambda) \), there exists \( n \in \mathbb{N} \setminus \{0\} \) with

\[
\inf\{d_n(x, y) : y \in X\setminus D\} > \rho.
\]  

(2.8)

If \( y \in X\setminus D \), we have that \( d_n(y, 0) > r_n + \delta \) for each \( n \in \mathbb{N} \setminus \{0\} \). So there exists at least one \( t \in [0, n] \) with

\[
\|x(t) - y(t)\| \geq \|y(t)\| - \|x(t)\| > r_n + \delta - r_n = \delta.
\]

Hence \( d_n(x, y) > \delta \) and (2.8) holds for any \( \rho \in (0, \delta) \).

Condition (iii) in Theorem 1.2 is trivial since \( \mathcal{P} = \mathcal{Q}_\lambda \).

Condition (iv) in Theorem 1.2 becomes: \( (X, \{d_n\}_{n \in \mathbb{N} \setminus \{0\}}) \) is a gauge space sequentially complete since \( E \) is a Banach space.

Condition (v): Let \( \lambda \in [0, 1] \), \( x_0 \in D \), \( x_n = H(x_{n-1}, \lambda) \) for \( n = 1, 2, \ldots \) and assume \( n \rightarrow \infty \) \( \mathcal{P} \)-lim \( x_n = x \). We shall prove that \( H(x, \lambda) = x \).

For \( m \in \mathbb{N} \setminus \{0\} \) and \( t \in [0, m] \) we have

\[
\|H(x, \lambda)(t) - x(t)\| = \|H(x, \lambda)(t) - x_n(t) + x_n(t) - x(t)\|
\]

\[
\leq \|H(x, \lambda)(t) - x_n(t)\| + \|x_n(t) - x(t)\|
\]

\[
= \|H(x, \lambda)(t) - H(x_{n-1}, \lambda)(t)\| + \|x_n(t) - x(t)\|
\]

\[
\leq \int_0^t k(s) \|x(s+1) - x_{n-1}(s+1)\| ds + \max_{t \in [0,m]} \|x_n(t) - x(t)\|
\]

\[
\leq L_m \max_{s \in [0,m]} \|x(s+1) - x_{n-1}(s+1)\| + d_m(x, x)
\]

\[
= L_m \max_{\tau \in [0,m+1]} \|x(\tau) - x_{n-1}(\tau)\| + d_m(x, x)
\]

\[
= L_m d_{m+1}(x, x_{n-1}) + d_m(x, x).
\]

Consequently

\[
d_m(H(x, \lambda), x) \leq L_m d_{m+1}(x, x_{n-1}) + d_m(x, x)
\]
for all $m \in \mathbb{N}\{0\}$. Letting $n \to \infty$, we deduce that $d_m(H(x, \lambda), x) = 0$ for each $m \in \mathbb{N}\{0\}$ and since the family $\{d_m\}_{m \in \mathbb{N}\{0\}}$ is separated we have $H(x, \lambda) = x$.

Condition (vi) becomes: for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_{\varphi^n(m)}(x, H(x, \lambda)) \leq (1 - L_{\varphi^n(m)})\varepsilon$$

for each $(x, \mu) \in D \times [0, 1]$, $H(x, \mu) = x$, $|\lambda - \mu| \leq \delta$, and $n, m \in \mathbb{N}\{0\}$.

We have $f^n(m) = n + m$. Let $t \in [0, n + m]$ and use conditions (c) and (d) to obtain

$$\|x(t) - H(x, \lambda)(t)\| = \|H(x, \mu)(t) - H(x, \lambda)(t)\|$$

$$= |\mu - \lambda| \left\|\int_0^t f(s, x(s + 1))ds\right\|$$

$$\leq |\mu - \lambda| \int_0^t \alpha(s)\beta(\|x(s + 1)\|)ds$$

$$\leq |\mu - \lambda| \beta(r_{m+n+1}) \int_0^\infty \alpha(s)ds$$

$$\leq |\mu - \lambda| |\alpha|_{L^1(\mathbb{R}_+)} C(1 - L_{m+n}).$$

So condition (vi) is true with $\delta(\varepsilon) = \frac{\varepsilon}{C |\alpha|_{L^1(\mathbb{R}_+)}}$.

In addition $H(., 0) = 0$. So $H(., 0)$ has a fixed point.

Therefore, all the assumptions of Theorem 1.2 are satisfied. Now the conclusion follows from Theorem 1.2

\[ \square \]

References


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