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INITIAL VALUE PROBLEM ON SEMI-LINE FOR DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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Abstract. A slight extension of the continuation principle established in [1] is used to prove the existence of solution for the initial value problem for first-order differential equations with advanced argument on semi-line.

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1. Introduction

The aim of this paper is to present a natural application of the continuation principle established in [1] involving contractions in Gheorghiu's sense, with respect to a family of pseudo-metrics.

The advanced argument in our differential equation makes necessary the use of two pseudo-metrics in the contraction condition.

In what follows we recall some notions and results from paper [1].

First recall the notion of contraction on a gauge space introduced by Gheorghiu [5]. Let $(X,\mathcal{P})$ be a gauge space with $\mathcal{P}=\{p_\alpha\}_{\alpha\in A}$. A map $F:D\subset X\to X$ is a *contraction* if there exists a function $\varphi:A\to A$ and $a\in\mathbb{R}_+^A, a=\{a_\alpha\}_{\alpha\in A}$ such that

$$p_\alpha(F(x),F(y))\le a_\alpha p_{\varphi(\alpha)}(x,y)\qquad \forall\alpha\in A, x,y\in D,$$

$$\textstyle\sum_{n=1}^{\infty}a_\alpha a_{\varphi(\alpha)}a_{\varphi^2(\alpha)}...a_{\varphi^{n-1}(\alpha)}p_{\varphi^n(\alpha)}(x,y)<\infty$$

for every $\alpha\in A$ and $x,y\in D$. Here, φ^n is the nth iteration of φ.

We present now a slow extension of some results from paper [1], whose proofs are similar.

Theorem 1.1. *Let X be a set endowed with two separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q} = \{q_\beta\}_{\beta\in B}$, let D_0 and D be two subsets of X with $D_0 \subset D$, and let $F : D \to X$ be a map. Assume that $F(D_0) \subset D_0$ and D is $\mathcal{P}$-closed. In addition, assume that the following conditions are satisfied:*

(i) there is a function $\psi : A \to B$ and $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ such that

$$p_\alpha(x, y) \le c_\alpha q_{\psi(\alpha)}(x, y) \qquad \forall\alpha \in A, x, y \in X;$$

(ii) $(X, \mathcal{P})$ is a sequentially complete gauge space;

(iii) if $x_0 \in D$, $x_n = F(x_{n-1})$ for $n = 1, 2, ...,$ and $\mathcal{P}\text{-}\lim_{n\to\infty} x_n = x$ for some $x \in D$, then $F(x) = x$;

(iv) F is a $\mathcal{Q}$ contraction on D_0.

Then F has at least one fixed point which can be obtained by successive approximation starting from any element of D_0.

For a map $H : D \times [0, 1] \to X$, where $D \subset X$, we will use the following notations:

$$\begin{aligned} &\Sigma = \{(x,\lambda) \in D \times [0,1] : H(x,\lambda) = x\},\\ &S = \{x \in D : H(x,\lambda) = x \text{ for some } \lambda \in [0,1]\},\\ &\Lambda = \{\lambda \in [0,1] : H(x,\lambda) = x \text{ for some } x \in D\}. \end{aligned} \tag{1.1}$$

Theorem 1.2. *Let X be a set endowed with the separating gauge structures $\mathcal{P} = \{p_\alpha\}_{\alpha\in A}$ and $\mathcal{Q}^\lambda = \{q^\lambda_\beta\}_{\beta\in B}$ for $\lambda \in [0, 1]$. Let $D \subset X$ be $\mathcal{P}$-sequentially closed, $H : D \times [0, 1] \to X$ a map, and assume that the following conditions are satisfies:*

(i) for each $\lambda \in [0, 1]$, there exists a function $\varphi_\lambda : B \to B$ and $a^\lambda \in [0, 1)^B$, $a^\lambda = \{a^\lambda_\beta\}_{\beta\in B}$ such that

$$q^\lambda_\beta(H(x,\lambda), H(y,\lambda)) \le a^\lambda_\beta q^\lambda_{\varphi_\lambda(\beta)}(x, y), \tag{1.2}$$

$$\sum_{n=1}^{\infty} a^\lambda_\beta a^\lambda_{\varphi_\lambda(\beta)} a^\lambda_{\varphi^2_\lambda(\beta)} ... a^\lambda_{\varphi^{n-1}_\lambda(\beta)} < \infty \tag{1.3}$$

for every $\beta \in B$ and $x, y \in D$;

(ii) there exists $\rho > 0$ such that for each $(x, \lambda) \in \Sigma$, there is a $\beta \in B$ with

$$\inf\{q^\lambda_\beta(x, y) : y \in X\backslash D\} > \rho; \tag{1.4}$$

(iii) for each $\lambda \in [0,1]$*, there is a function* $\psi : A \to B$ *and* $c \in (0,\infty)^A$, $c = \{c_\alpha\}_{\alpha\in A}$ *such that*

$$p_\alpha(x,y) \le c_\alpha q^\lambda_{\psi(\alpha)}(x,y) \qquad \text{for all } \alpha \in A \text{ and } x,y \in X; \tag{1.5}$$

(iv) $(X,\mathcal{P})$ *is a sequentially complete gauge space;*

(v) if $\lambda \in [0,1]$, $x_0 \in D, x_n = H(x_{n-1},\lambda)$ *for* $n = 1,2,...,$ *and* $\mathcal{P}$-$\lim_{n\to\infty} x_n = x$*, then* $H(x,\lambda) = x$*;*

(vi) for every $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon) > 0$ *with*

$$q^\lambda_{\varphi^n_\lambda(\beta)}(x, H(x,\lambda)) \le (1 - a^\lambda_{\varphi^n_\lambda(\beta)})\varepsilon$$

for $(x,\mu) \in \Sigma$, $|\lambda - \mu| \le \delta$*, all* $\beta \in B$*, and* $n \in \mathbb{N}$*.*

In addition, assume that $H_0 := H(.,0)$ *has a fixed point. Then, for each* $\lambda \in [0,1]$*, the map* $H_\lambda := H(.,\lambda)$ *has at least a fixed point.*

2. The main result

We consider the problem Cauchy

$$\begin{cases} x'(t) = f(t, x(t+1)), t \in [0,\infty) \\ x(0) = 0. \end{cases} \tag{2.6}$$

This problem is equivalent to the Volterra integral equation

$$x(t) = \int_0^t f(s, x(s+1))ds, \qquad t \in [0,\infty) \tag{2.7}$$

Theorem 2.1. *Let* $(E, \|.\|)$ *be a Banach space, and let* $f : \mathbb{R}_+ \times E \to E$ *be a continuous function. Assume that the following conditions hold:*

(a) there exists $k \in L^1(\mathbb{R}_+, (0,\infty))$ *with* $|k|_{L^1(\mathbb{R}_+)} < 1$ *such that*

$$\|f(t,x) - f(t,y)\| \le k(t)\,\|x-y\|$$

for all $x,y \in E$*, and* $t \in \mathbb{R}_+$*;*

(b) for each $n \in \mathbb{N}\backslash\{0\}$ *there exists* $r_n > 0$ *such that, any continuous solution* x *of the equation*

$$x(t) = \lambda \left(\int_0^t f(s, x(s+1))ds \right), \qquad t \in \mathbb{R}_+$$

with $\lambda \in [0,1]$*, satisfies* $\|x(t)\| \le r_n$ *for any* $t \in [0,n]$*;*

(c) there exists $\alpha \in L^1(\mathbb{R}_+)$ *şi* $\beta : \mathbb{R}_+ \to (0,\infty)$ *nondecreasing such that*

$$\|f(t,x)\| \le \alpha(t)\beta(\|x\|)$$

for all $t \in \mathbb{R}_+$ *and* $x \in E$;

(d) there exists $C > 0$ *such that* $\dfrac{\beta(r_{k+1})}{1 - L_k} \leq C$ *for any* $k \in \mathbb{N}\backslash\{0\}$*, where* $L_k = \int_0^k k(s)ds$.

Then there exists at least one solution $x \in C(\mathbb{R}_+, E)$ *of the integral equation* (2.7).

Proof. For the proof we use Theorem 1.2.

Let $X = \{x \in C(\mathbb{R}_+, E) : x(0) = 0\}$. For each $n \in \mathbb{N}\backslash\{0\}$ we define the map $|.|_n : X \to \mathbb{R}_+$, by $|x|_n = \max_{t\in[0,n]} \|x(t)\|$. This map is a semi-norm on X, and let $d_n : X \times X \to \mathbb{R}_+$ be given by

$$d_n(x, y) = |x - y|_n = \max_{t\in[0,n]} \|x(t) - y(t)\|.$$

It is easy to show that d_n is a pseudo-metric on X and the family $\{d_n\}_{n\in\mathbb{N}\backslash\{0\}}$ defines on X a gauge structure, separated and complete by sequences.

Here $\mathcal{P} = \mathcal{Q}^\lambda = \{d_n\}_{n\in\mathbb{N}\backslash\{0\}}$ for each $\lambda \in [0, 1]$. Let D be the closure in X of the set

$$\{x \in X : \text{ there exists } n \in \mathbb{N}\backslash\{0\} \text{ such that } d_n(x, 0) \leq r_n + \delta\ \}$$

where $\delta > 0$ is a fixed number. We define $H : D \times [0, 1] \to X$, by $H(x, \lambda) = \lambda A(x)$, where

$$A(x)(t) = \int_0^t f(s, x(s + 1))ds.$$

First we verify *condition (i)* from Theorem 1.2.

Let $t \in [0, n]$. We estimate

$$\begin{aligned}
\|H(x, \lambda)(t) - H(y, \lambda)(t)\| &\leq \lambda \int_0^t \|f(s, x(s + 1)) - f(s, y(s + 1))\| \, ds \\
&\leq \int_0^t k(s) \|x(s + 1) - y(s + 1)\| \, ds \\
&\leq \max_{s\in[0,n]} \|x(s + 1) - y(s + 1)\| \int_0^t k(s)ds \\
&\leq \max_{\tau\in[0,n+1]} \|x(\tau) - y(\tau)\| \int_0^n k(s)ds \\
&= L_n d_{n+1}(x, y).
\end{aligned}$$

If we take the maximum with respect to t, we obtain

$$d_n(H(x,\lambda),H(y,\lambda)) \le L_n d_{n+1}(x,y)$$

for all $x, y \in D$ and all $n \in \mathbb{N}\backslash\{0\}$. Hence, condition (i) in Theorem 1.2 holds with $\varphi_\lambda = \varphi$ where $\varphi : \mathbb{N}\backslash\{0\} \to \mathbb{N}\backslash\{0\}$ is defined by $\varphi(n) = n+1$. In addition the series $\sum_{n=1}^{\infty} L_n L_{n+1}...L_{2n}$ is finite since from assumption (a) we know that $|k|_{L^1(\mathbb{R}_+)} < 1$ and so $L_n \le |k|_{L^1(\mathbb{R}_+)} < 1$.

Condition (ii) in our case becomes: there exists $\rho > 0$ such that for any solution $(x,\lambda) \in D \times [0,1]$, to $x = H(x,\lambda)$, there exists $n \in \mathbb{N}\backslash\{0\}$ with

$$\inf\{d_n(x,y) : y \in X\backslash D\} > \rho. \tag{2.8}$$

If $y \in X\backslash D$, we have that $d_n(y,0) > r_n + \delta$ for each $n \in \mathbb{N}\backslash\{0\}$. So there exists at least one $t \in [0,n]$ with

$$\|x(t) - y(t)\| \ge \|y(t)\| - \|x(t)\| > r_n + \delta - r_n = \delta.$$

Hence $d_n(x,y) > \delta$ and (2.8) holds for any $\rho \in (0,\delta)$.

Condition (iii) in Theorem 1.2 is trivial since $\mathcal{P} = \mathcal{Q}^\lambda$.

Condition (iv) in Theorem 1.2 becomes: $(X, \{d_n\}_{n\in\mathbb{N}\backslash\{0\}})$ is a gauge space sequentially complete since E is a Banach space.

Condition (v): Let $\lambda \in [0,1]$, $x_0 \in D$, $x_n = H(x_{n-1},\lambda)$ for $n = 1,2,...$ and assume $\mathcal{P}\text{-}\lim_{n\to\infty} x_n = x$. We shall prove that $H(x,\lambda) = x$.

For $m \in \mathbb{N}\backslash\{0\}$ and $t \in [0,m]$ we have

$$\begin{aligned}
\|H(x,\lambda)(t) - x(t)\| &= \|H(x,\lambda)(t) - x_n(t) + x_n(t) - x(t)\| \\
&\le \|H(x,\lambda)(t) - x_n(t)\| + \|x_n(t) - x(t)\| \\
&= \|H(x,\lambda)(t) - H(x_{n-1},\lambda)(t)\| + \|x_n(t) - x(t)\| \\
&\le \int_0^t k(s)\, \|x(s+1) - x_{n-1}(s+1)\|\, ds + \max_{t\in[0,m]} \|x_n(t) - x(t)\| \\
&\le L_m \max_{s\in[0,m]} \|x(s+1) - x_{n-1}(s+1)\| + d_m(x_n,x) \\
&= L_m \max_{\tau\in[0,m+1]} \|x(\tau) - x_{n-1}(\tau)\| + d_m(x_n,x) \\
&= L_m d_{m+1}(x,x_{n-1}) + d_m(x_n,x).
\end{aligned}$$

Consequently

$$d_m(H(x,\lambda),x) \le L_m d_{m+1}(x,x_{n-1}) + d_m(x_n,x)$$

for all $m \in \mathbb{N}\backslash\{0\}$. Letting $n \to \infty$, we deduce that $d_m(H(x,\lambda),x) = 0$ for each $m \in \mathbb{N}\backslash\{0\}$ and since the family $\{d_m\}_{m\in\mathbb{N}\backslash\{0\}}$ is separated we have $H(x,\lambda) = x$.

Condition (vi) becomes: for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$d_{\varphi^n(m)}(x, H(x,\lambda)) \leq (1 - L_{\varphi^n(m)})\varepsilon$$

for each $(x,\mu) \in D \times [0,1]$, $H(x,\mu) = x$, $|\lambda - \mu| \leq \delta$, and $n, m \in \mathbb{N}\backslash\{0\}$.

We have $f^n(m) = n + m$. Let $t \in [0, n+m]$ and use conditions (c) and (d) to obtain

$$\begin{aligned}\|x(t) - H(x,\lambda)(t)\| &= \|H(x,\mu)(t) - H(x,\lambda)(t)\| \\ &= |\mu - \lambda| \left\| \int_0^t f(s, x(s+1))ds \right\| \\ &\leq |\mu - \lambda| \int_0^t \alpha(s)\beta(\|x(s+1)\|)ds \\ &\leq |\mu - \lambda|\, \beta(r_{m+n+1}) \int_0^\infty \alpha(s)ds \\ &\leq |\mu - \lambda|\, |\alpha|_{L^1(\mathbb{R}_+)}\, C(1 - L_{m+n}).\end{aligned}$$

So condition (vi) is true with $\delta(\varepsilon) = \dfrac{\varepsilon}{C\,|\alpha|_{L^1(\mathbb{R}_+)}}$.

In addition $H(.,0) = 0$. So $H(.,0)$ has a fixed point.

Therefore, all the assumptions of Theorem 1.2 are satisfied. Now the conclusion follows from Theorem 1.2 □

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