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AN EXTENSION TO MULTIFUNCTIONS OF THE KEELER-MEIR'S FIXED POINT THEOREM

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Abstract. In this paper we prove the existence of fixed points for multifunctions verifying the property of weakly uniformly strict p-contraction. Then, as a corollary, we deduce a result on the existence of contractive fixed points for single-valued functions not necessarily with complete graph, which strictly contains a result due to Keeler and Meir.
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1. Introduction

The theory concerning the existence of contractive fixed points in complete metric spaces has been developed by several authors starting from the classical Banach's Theorem (see, e.g., [1], [3], [2], [4], [5]).
In particular, in [5] Keeler and Meir stated that the conclusion of the Banach's Theorem holds for functions acting on a complete metric space into itself and satisfying the property of *weakly uniformly strict contraction.*
More recently Leader ([6]), in the setting of metric spaces not necessarily complete, has improved the result in [5]. In order to obtain the existence of a (unique) contractive fixed point, he has considered functions *with complete*

graph and satisfying a more general property than the one required by Keeler and Meir (see Remark 2.3).

In the present paper we introduce a new extension to multifunctions of the property of weakly uniformly strict contraction, named *weakly uniformly strict p-contraction*. This extension, as we note in Remark 3.4, is different from the property involving the Hausdorff distance adopted in the existing literature (see e.g. [7], [8]).
In the framework of complete metric spaces we obtain a result on the existence of fixed points for multifunctions. Then, as a corollary, we deduce the existence of a (unique) contractive fixed point for single-valued functions not necessarily with complete graph.
We show that the corollary strictly contains the Keeler-Meir's fixed point theorem.
Moreover, we prove that there exist functions verifying the hypotheses of our corollary but not all the assumptions required by Leader in his theorem and vice versa.

2. Notations and preliminary results

Let X be a metric space and $f : X \to X$ be a given function.

Definition 2.1. For every $z \in X$ the sequence $(f^p(z))_{p\in \mathbb{N}_0}$, where $f^0(z) = z$ and $f^p(z) = f(f^{p-1}(z))$ for $p \geq 1$, is said to be an *orbit* of f (at z).

Definition 2.2. A point $x \in X$ is a *contractive fixed point* for f if it is a fixed point for f and if every orbit of f converges to x.

In [5] Keeler and Meir showed that the conclusion of the classic Banach's Theorem holds for *weakly uniformly strict contractions*, i.e. functions verifying the property

(KM) for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$d(f(x), f(y)) < \varepsilon \ , \quad (x, y) \in E(\varepsilon, \varepsilon + \delta(\varepsilon)) \ ,$$

where $E(\varepsilon, \varepsilon + \delta(\varepsilon)) = \{(x, y) \in X \times X \ : \ \varepsilon \leq d(x, y) < \varepsilon + \delta(\varepsilon)\}$.

Remark 2.1. Let us observe that (KM) is equivalent to the following property

(KM)' for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$d(f(x), f(y)) < \varepsilon \ , \quad (x, y) \in D(\varepsilon + \delta(\varepsilon)) \ ,$$

where $D(\varepsilon + \delta(\varepsilon)) = \{(x, y) \in X \times X \ : \ d(x, y) < \varepsilon + \delta(\varepsilon)\}$. □

We quote here the Keeler-Meir's fixed point theorem.

Theorem 2.1. *([5], Theorem) Let (X, d) be a complete metric space and let $f : X \to X$ be a function satisfying the property (KM). Then f has a (unique) contractive fixed point.*

Remark 2.2. Every function $f : X \to X$ satisfying (KM) is a contractive function, i.e. f has the property

$$d(f(x), f(y)) < d(x, y) \ , \quad x, y \in X \ , \quad x \neq y \ .$$

□

In 1983 Leader (see [6]) obtained, in the more general case of a not necessarily complete metric space (X, d), an interesting fixed point theorem for functions which satisfy the following property.

Definition 2.3. A map $f : X \to X$ is said to be *a function with complete graph* if for every Cauchy sequence $(x_n)_{n \in I\!N_0}$ such that $(f(x_n))_{n \in I\!N_0}$ is also a Cauchy sequence there exists a point $x \in X$ such that $x_n \longrightarrow x$ and $f(x_n) \longrightarrow f(x)$.

Moreover, he introduced the property

(L) for every $\varepsilon > 0$ there exist $\delta(\varepsilon) \in]0, +\infty]$ and a natural number $r = r(\varepsilon)$ such that

$$d(f^r(x), f^r(y)) < \varepsilon \ , \quad (x, y) \in D(\varepsilon + \delta(\varepsilon)) \ .$$

Remark 2.3. We note that in the particular case where (L) is true for $r(\varepsilon) = 1$, for every $\varepsilon > 0$, the property (L) reduces itself to (KM)'. □

We recall the following result due to Leader.

Theorem 2.2. *([6], Corollary 4) Let (X, d) be a metric space and $f : X \to X$ be a function with complete graph satisfying the property (L). Then f has a (unique) contractive fixed point.*

Remark 2.4. If a function $f : X \to X$ satisfies the assumption (KM), then it is continuous (see Remark 2.2). Therefore, being the metric space X complete, the hypotheses of Theorem 2.1 lead us to claim that f is a function with complete graph.
Hence, either in Theorem 2.1 and in Theorem 2.2 we are considering functions with complete graph.

□

Finally, we recall the following theorem which will be useful in the sequel.

Theorem 2.3. *([9], Lemma 3) Let (X, d) be a metric space and $f : X \to X$ be a given function. Then, for each p-iterate of f with $p > 1$ we have:*

(p1) if f^p has a unique fixed point, then f has a unique fixed point;

(p2) if there exists $z \in X$ such that every orbit of f^p converges to z, then every orbit of f converges to z;

(p3) if every orbit of f^p is a bounded sequence, then every orbit of f is a bounded sequence.

3. The existence theorems

In this section we provide a fixed point theorem for multifunctions. Further, we give a result on the existence of contractive fixed points for single-valued functions.

First of all, we introduce a new extension to multifunctions $F : X \to P(X)$ of the property of weakly uniformly strict contraction (KM), where $P(X) = \{S \subset X \ : \ S \neq \emptyset\}$.

Definition 3.1. We say that a multifunction $F : X \to P(X)$ is a *weakly uniformly strict p-contraction* if the following property holds:

($\mathcal{F}$) there exists $p \in I\!N$ such that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for each pair $(z, w) \in X \times X$ admitting the representation:

$$\exists z_0, \cdots, z_{p-1} \in X \text{ with } z \in F(z_{p-1}), z_{p-1} \in F(z_{p-2}), \cdots, z_1 \in F(z_0)$$

and

$$\exists w_0, \cdots, w_{p-1} \in X \text{ with } w \in F(w_{p-1}), w_{p-1} \in F(w_{p-2}), \cdots, w_1 \in F(w_0),$$

$(z_0, w_0) \in D(\varepsilon + \delta(\varepsilon))$ we have that

$$d(z, w) < \varepsilon \ .$$

(Here $D(\varepsilon + \delta(\varepsilon))$ is the same set introduced in (KM)'.)

Obviously our property reduces itself to (KM)' , and then to (KM), in the case where $p = 1$ and F is a single-valued function.

Remark 3.1. At first, we want to state precisely that if there exists $\varepsilon > 0$ such that for all $\delta > 0$ does not exist a dyad $(z, w) \in X \times X$ admitting the above representation, then the multifunction F has not the property $(\mathcal{F})$.

Then, let us consider the complete metric space $X = \{0\} \cup \{\frac{1}{n} : n \in I\!N\}$ endowed with the induced Euclidean metric d and the multifunction $F : X \to P(X)$ defined by

$$F(x) = \begin{cases} \{0\} \cup \{\frac{1}{2n}\} & , \quad x = \frac{1}{2n+1},\ n \in I\!N \\ \{0\} & , \quad \text{otherwise} . \end{cases}$$

It is easy to verify that F satisfies the property $(\mathcal{F})$ as $p = 2$.
Therefore, F is an example of multifunction for which all the assumptions of the following Theorem 3.1 are verified. □

Now we can state and prove our main result, that is an extension to multifunctions of the Keeler-Meir's fixed point theorem.

Theorem 3.1. *Let (X, d) be a complete metric space and $F : X \to P(X)$ be a multifunction satisfying the property $(\mathcal{F})$. Then F has a fixed point.*

Proof. By means of the Zermelo's axiom of choice, there exists a function $f : X \to X$ such that

$$f(x) \in F(x) \ , \quad \forall x \in X \ . \tag{1}$$

First of all, we prove that the function f^p satisfies the property (KM), where p is the number given by $(\mathcal{F})$.
Fixed $\varepsilon > 0$, let $\delta(\varepsilon)$ be the positive number provided by $(\mathcal{F})$.
Let us fix $z_0, w_0 \in X$ such that $(z_0, w_0) \in D(\varepsilon + \delta(\varepsilon))$ and define

$$z_1 = f(z_0), \cdots, z_{p-1} = f(z_{p-2}) \ ; \qquad w_1 = f(w_0), \cdots, w_{p-1} = f(w_{p-2}) \ .$$

Of course, by using (1), we get

$$z_1 \in F(z_0), \cdots, z_{p-1} \in F(z_{p-2}) \ ; \qquad w_1 \in F(w_0), \cdots, w_{p-1} \in F(w_{p-2}) \ .$$

Moreover, it is easy to see that

$$z_k = f^k(z_0) \ , \quad w_k = f^k(w_0) \ , \quad k = 1, \cdots, p-1 \ .$$

By construction and by using $(\mathcal{F})$, the points

$$z := f^p(z_0) \ , \quad w := f^p(w_0)$$

are such that

$$d(f^p(z_0), f^p(w_0)) = d(z,w) < \varepsilon \ .$$

Then, we can deduce that the p-iterate of f satisfies the property (KM)'.

Now, let us show that f has a unique fixed point.
It is easy to see that if $p = 1$ then f has a unique (contractive) fixed point (see Remark 2.1 and Theorem 2.1).
In the case $p > 1$, we can apply Remark 2.1 and Theorem 2.1 to the p-iterate of f so that the function f^p has a unique contractive fixed point.
Therefore, by using *(p1)* of Theorem 2.3, we obtain that there exists a unique fixed point for f.

Obviously, if $y \in X$ is the unique fixed point for f, from (1) we get $y = f(y) \in F(y)$ and this concludes the proof. $\square$

The proof of the previous theorem leads us to state the following result for single-valued functions.

Corollary 3.1. *Let (X,d) be a complete metric space and $f : X \to X$ be a given function. If there exists a positive natural number p such that function f^p satisfies the property (KM), then f has a (unique) contractive fixed point.*

Proof. If $p = 1$ the theorem is Theorem 2.1.

Let us consider $p > 1$. In this case, by proceeding as in the second part of the proof of the previous theorem, we have that f^p has a unique contractive fixed point, say $\bar{x}$, and that f has a unique fixed point $y \in X$.
Now, by means of *(p2)* of Theorem 2.3, we get that every orbit of f converges to $\bar{x}$. In particular, this implies that the constant orbit $(f^n(y))_{n \in \mathbb{N}_0}$ converges to $\bar{x}$.
Then $\bar{x} = y$ and so we can conclude that $\bar{x}$ is a contractive fixed point also for f. $\square$

Remark 3.2. Our Corollary 3.1 strictly contains the Keeler-Meir's fixed point theorem, here Theorem 2.1.
First of all we note that in the particular case $p = 1$ our assumptions coincide with the hypotheses of Theorem 2.1.
Moreover, the hypothesis (KM) may be violated by f while the hypotheses of our Corollary 3.1 are fulfilled, also in the class of contractive functions, as the following example shows.

Example 3.1. Let $X = [0, 1]$ be the complete metric space endowed with the metric $d : X \times X \to [0, +\infty[$ defined as

$$d(x, y) = \begin{cases} \max\{x, y\} & , \ x \neq y \\ 0 & , \ x = y \end{cases}$$

and let $f : X \to X$ be the function defined by

$$f(x) = \begin{cases} \dfrac{1}{n+1} & , \ x \in \left]\dfrac{1}{n+1}, \dfrac{1}{n}\right] \ , \ n \in I\!N \\ 0 & , \ x = 0 \ . \end{cases}$$

First of all, we note that f is a contractive function. In fact, fixed $x, y \in X$ with $x \neq y$, we have:
if $f(x) = f(y)$ then

$$d(f(x), f(y)) = 0 < d(x, y) \ ;$$

if $f(x) \neq f(y)$, since f is a non decreasing function such that $f(x) < x$ for every $x \in X \setminus \{0\}$, we can write

$$d(f(x), f(y)) = f(\max\{x, y\}) = f(d(x, y)) < d(x, y) \ .$$

Let us prove that f does not satisfy the property (KM)' (see Remark 2.1). Suppose, on the contrary, that f verifies (KM)'. Thus, in particular for $\varepsilon = \frac{1}{2}$, there exists a number $\tilde{\delta} = \delta(\frac{1}{2}) > 0$ such that

$$d(f(x), f(y)) < \frac{1}{2} \ , \quad (x, y) \in D\left(\frac{1}{2} + \tilde{\delta}\right) \ . \tag{2}$$

Of course, (2) is true for $\tilde{x} = \frac{1}{2}$ and $\tilde{y} = \frac{1}{2} + \min\{\frac{1}{2}, \frac{\tilde{\delta}}{2}\}$, being $(\tilde{x}, \tilde{y}) \in D(\frac{1}{2} + \tilde{\delta})$. By the choice of $\tilde{y} \in \left]\frac{1}{2}, 1\right]$ and (2), we have the following contradiction

$$f(\tilde{y}) = \frac{1}{2} = d(f(\tilde{x}), f(\tilde{y})) < \frac{1}{2} \ .$$

On the other hand, f satisfies the hypotheses of our Corollary 3.1. To this aim, we first observe that

$$f(x) \leq \frac{1}{2}\ , \quad x \in X\ .$$

Now, we go to prove that the iterate function given by $p = 2$ has the property (KM)'.

For every fixed $\varepsilon > 0$ we consider the positive number $\delta(\varepsilon)$ defined as

$$\delta(\varepsilon) = \begin{cases} \dfrac{1}{2n(n-1)} & ,\ \varepsilon = \dfrac{1}{n},\ n \in I\!N,\ n \geq 2 \\ \\ \dfrac{1}{2}\left(\dfrac{1}{n} - \varepsilon\right) & ,\ \dfrac{1}{n+1} < \varepsilon < \dfrac{1}{n},\ n \in I\!N \\ \\ 1 & ,\ \varepsilon \geq 1\ . \end{cases} \tag{3}$$

Moreover, let us fix $(x, y) \in D(\varepsilon + \delta(\varepsilon))$.

If $f^2(x) = f^2(y)$, obviously it is

$$d(f^2(x), f^2(y)) < \varepsilon\ . \tag{4}$$

If $f^2(x) \neq f^2(y)$, then (4) is verified in the case $\varepsilon \geq 1$, whereas, in the case $\varepsilon < 1$, we have two different situations:

if there exists a natural number $m \in I\!N$ such that $\varepsilon = \frac{1}{m}$, by using (3) we get

$$d(x, y) = \max\{x, y\} < \frac{1}{m} + \frac{1}{2m(m-1)} < \frac{1}{m-1}$$

and, since f^2 is nondecreasing, we obtain

$$d(f^2(x), f^2(y)) = f^2(\max\{x, y\}) \leq f^2\left(\frac{1}{m-1}\right) = \frac{1}{m+1} < \varepsilon\ ;$$

if there exists a natural number m such that $\varepsilon \in \left]\frac{1}{m+1}, \frac{1}{m}\right[$, by taking into account of (3) we can write

$$d(x, y) < \varepsilon + \delta(\varepsilon) = \varepsilon + \frac{1}{2}\left(\frac{1}{m} - \varepsilon\right) < \frac{1}{m}$$

and this implies, by using again that f^2 is nondecreasing, the following inequality

$$d(f^2(x), f^2(y)) = f^2(\max\{x, y\}) \leq f^2\left(\frac{1}{m}\right) = \frac{1}{m+2} < \varepsilon\ .$$

Hence, f^2 satisfies the desired property (KM)'. □

Remark 3.3. We note that there exist functions verifying the hypotheses of our Corollary 3.1 but not all the assumptions required in Theorem 2.2 and vice versa. It is demonstrated by the following examples.

Example 3.2. Let $X = [0, 2]$ be the complete metric space endowed with the Euclidean metric, say d. Let $f : X \to X$ be the function defined as

$$f(x) = \begin{cases} 0 & , \ x \in [0, 1] \\ \\ 1 & , \ x \in]1, 2] \ . \end{cases}$$

Being $f^2(x) = 0, \ x \in X$, then f obviously satisfies the assumptions required in Corollary 3.1.

In order to see that f is not with complete graph, we fix the Cauchy sequence $(x_n)_{n\in I\!N}$, where $x_n = 1 + \frac{1}{n}$. Of course $(f(x_n))_{n\in I\!N}$ is also a Cauchy sequence; moreover, $(x_n)_{n\in I\!N}$ converges to $x = 1$, but $(f(x_n))_{n\in I\!N}$ converges to $1 \neq f(1) = 0$. Therefore, it is not possible to deduce the existence of a contractive fixed point by applying Theorem 2.2.

Let us prove now that there exist functions satisfying the conditions required in Theorem 2.2, but for which does not exist any natural number p such that f^p verifies the property (KM). This is true also in complete metric spaces.

Example 3.3. Let $X = \{0\} \cup B \cup I\!N$ be the complete metric space endowed with the euclidean metric d, where $B = \{\frac{1}{n} \ : \ n \in I\!N\}$. Let $f : X \to X$ be the function defined by

$$f(x) = \begin{cases} 0 & , \ x = 0 \\ \\ \dfrac{x}{1+2x} & , \ x \in C \\ \\ \dfrac{1}{x} & , \ x \in D \cup E \\ \\ \dfrac{1}{2} & , \ x \in F \end{cases}$$

where

$$C = \left\{ \frac{1}{2n} \ : \ n \in I\!N \right\} \ , \quad D = \left\{ \frac{1}{2n+1} \ : \ n \in I\!N \right\} \ ,$$

$$E = \{2n \ : \ n \in I\!N\} \ , \quad F = \{2n-1 \ : \ n \in I\!N\} \ .$$

First of all, let us prove that f is a function with complete graph. To this aim, let us fix a Cauchy sequence $(x_n)_{n\in I\!N}$ such that $(f(x_n))_{n\in I\!N}$ is also a Cauchy sequence.
If there exists $x \in X$ such that definitively $x_n = x$, then $(x_n)_{n\in I\!N}$ trivially converges to x and $(f(x_n))_{n\in I\!N}$ converges to $f(x)$.
Otherwise, the sequence $(x_n)_{n\in I\!N}$ satisfies the following properties.

(P1) The set $\{x_n \ : \ x_n \in I\!N, \ n \in I\!N\}$ is finite.
In fact, if by contradiction we suppose not, then $(x_n)_{n\in I\!N}$ is not a Cauchy sequence.

(P2) The set $\{x_n \ : \ n \in I\!N\}$ is infinite.
In fact, again by contradiction, suppose that the set $\{x_n \ : \ n \in I\!N\}$ is finite. Thus, being the sequence $(x_n)_{n\in I\!N}$ not definitively constant, there exist at least two elements of the set, x_{h^1}, x_{h^2}, with $x_{h^1} \neq x_{h^2}$, and there exist two subsequences $(x_{n^1_k})_{k\in I\!N}$, $(x_{n^2_k})_{k\in I\!N}$ of $(x_n)_{n\in I\!N}$ such that

$$x_{n^1_k} = x_{h^1} \quad \text{ and } \quad x_{n^2_k} = x_{h^2} \ , \quad n \in I\!N \ .$$

Then it contradicts that $(x_n)_{n\in I\!N}$ is a Cauchy sequence.

(P3) The set $\{x_n \ : \ n \in I\!N\} \cap D$ is finite and the set $\{x_n \ : \ n \in I\!N\} \cap C$ is infinite.
In fact, taking into account of the properties (P1) and (P2), it is sufficient to show that the set $\{x_n \ : \ n \in I\!N\} \cap D$ is finite.
On the contrary, we suppose that the set $\{x_n \ : \ n \in I\!N\} \cap D$ is infinite. Thus, we can deduce that
for every $k \in I\!N$ there exist two natural numbers j_k, m_k with $j_k, m_k \geq k$ such that

$$x_{j_k}, \, x_{m_k} \in D \ , \quad x_{j_k} \neq x_{m_k} \ .$$

By using the definition of f, we have that for every $k \in I\!N$ there exist two odd natural numbers p_k, q_k with $p_k \neq q_k$ such that

$$f(x_{j_k}) = p_k \ , \quad f(x_{m_k}) = q_k \ .$$

Therefore, the sequence $(f(x_n))_{n\in I\!N}$ cannot be a Cauchy sequence and then we have the contradiction.

Now, by using the properties (P1) - (P3), we go to conclude that also in this case, where $(x_n)_{n\in I\!N}$ is not definitively constant, there exists $x \in X$ such that $x_n \longrightarrow x$ and $f(x_n) \longrightarrow f(x)$.
Of course, the sequence $(x_n)_{n\in I\!N}$ converges to an $x \in X$.
The value of x cannot be greater than 0. In fact, by means of the property (P3), we can say that there exists a subsequence $(x_{n_k})_{k\in I\!N}$ of $(x_n)_{n\in I\!N}$ with $x_{n_k} \in C$, $k \in I\!N$, and such that $\{x_{n_k} \ : \ k \in I\!N\}$ is infinite.
Then, there exists $\bar{k} \in I\!N$ such that for every $k \in I\!N$ with $k \geq \bar{k}$ it is $x_{n_k} > \frac{x}{2}$.
Fixed $m \in I\!N$ such that $\frac{1}{2m} < \frac{x}{2}$ and taking into account that $x_{n_k} \in C$, $k \in I\!N$, the following inclusion holds

$$\{x_{n_k} \ : \ k \geq \bar{k}\} \subset \left\{ \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2(m-1)} \right\} .$$

This inclusion contradicts that $\{x_{n_k} \ : \ k \in I\!N\}$ is infinite. Therefore, x must be equal to 0.
Moreover, from the reasonings above, we have estabilished that there exists an infinite set $\{x_{n_k} \ : \ k \in I\!N\} \subset C$.
By applying the definition of f we can write the estimate

$$0 < f(x_{n_k}) < x_{n_k} \ , \quad k \in I\!N$$

so we can deduce that $f(x_{n_k}) \longrightarrow 0 = f(0)$. By means of the uniqueness of the limit algorithm, also the whole sequence $(f(x_n))_{n\in I\!N}$ converges to 0. Therefore we can conclude that f is a function with complete graph.

Next, let us show that f satisfies the assumption (L) of Theorem 2.2. Let us fix $\varepsilon > 0$ and choose $r = r(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 2$ (where $[a]$ denotes the integer part of a real number a). Then we are able to prove that

$$d(f^r(x), f^r(y)) < \varepsilon \tag{5}$$

even for $(x, y) \in X \times X$.
The definition of f leads us to consider the following cases:

• $x = 0$: of course it is

$$f^r(x) = 0 < \varepsilon \ ;$$

• $x \in C$: in this case, the definition of C implies that there exists $m \in I\!N$ such that $x = \frac{1}{2m}$ and so

$$f(x) = f\left(\frac{1}{2m}\right) = \frac{1}{2m+2} = \frac{x}{1+2x} \in C \ .$$

It is easy to see that all the iterates of f have the following expression

$$f^p(x) = \frac{x}{1+2px} \in C \ , \quad p \in I\!N \ . \tag{6}$$

Therefore $f^r(x) = \dfrac{x}{1+2rx}$ and, moreover, the following estimate holds

$$f^r(x) < \frac{1}{r} < \varepsilon \ ;$$

- $x \in D$: in this case we have

$$f(x) = \frac{1}{x} \in F \ , \quad f^2(x) = \frac{1}{2} \in C \ . \tag{7}$$

So the previous case implies

$$f^r(x) = \frac{1}{2(r-1)} < \varepsilon \ ;$$

- $x \in E$: here we have

$$f(x) = \frac{1}{x} \in C$$

and then, as above, we can apply the case for $x \in C$ and obtain

$$f^r(x) = \frac{1}{2(r-1)+x} < \varepsilon \ ;$$

- $x \in F$: in this setting we have

$$f(x) = \frac{1}{2} \in C \ .$$

Hence, again as above, we get

$$f^r(x) = \frac{1}{2r} < \varepsilon \ .$$

Therefore, we can conclude that

$$f^r(x),\, f^r(y) \in [0, \varepsilon[\ , \quad x, y \in X$$

and (5) is proved.

Finally, let us show that the hypothesis of our Corollary 3.1 is not fulfilled. The proof is by contradiction; indeed we suppose the existence of a natural number p such that f^p verifies the property (KM)'.
The number p cannot be equal to 1. In fact, in correspondence to $\varepsilon = 1$ we should have that

$$\text{there exists } \delta(1) > 0 \text{ such that } d(f^p(x), f^p(y)) < 1 \ , \quad (x, y) \in D(1 + \delta(1)) \ .$$

But it is not possible since, if $\bar{x} = \frac{1}{2}$ and $\bar{y} = \frac{1}{3}$, then $(\bar{x}, \bar{y}) \in D(1 + \delta(1))$ and $d(f^p(\bar{x}), f^p(\bar{y})) = \frac{11}{4}$.

Let us prove that neither $p \geq 2$ is acceptable. Indeed, with the same reasoning as above, in correspondence to $\varepsilon = \frac{1}{2p}$ there exists $\bar{\delta} = \delta\left(\frac{1}{2p}\right)$ such that, in particular for $\tilde{x} = \frac{1}{n}$, n even, and $\tilde{y} = \frac{1}{m}$, m odd, with $(\tilde{x}, \tilde{y}) \in D\left(\frac{1}{2p} + \bar{\delta}\right)$ we have

$$d(f^p(\tilde{x}), f^p(\tilde{y})) < \frac{1}{2p} \ . \tag{8}$$

Since $\tilde{x} \in C$ and $\tilde{y} \in D$, from (6) and (7) the estimate (8) can be written as

$$\frac{n+2}{(2p+n)2(p-1)} < \frac{1}{2p} \ .$$

On the other hand, the following inequality holds

$$\lim_{n \to +\infty} \frac{n+2}{(2p+n)2(p-1)} = \frac{1}{2(p-1)} > \frac{1}{2p}$$

which leads again to a contradiction. □

Remark 3.4. We wish to conclude by observing that our property $(\mathcal{F})$ is different from the property involving the Hausdorff distance adopted in the existing literature (see e.g. [7], [8]). It is shown from our Example 3.2 just reading that function f as a multifunction. □

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