FIXED POINT CURVES GENERATED BY NONEXPANSIVE MAPPINGS

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Abstract. In this paper we consider a nonexpansive map \( T \) from a nonempty closed bounded and convex set \( K \) into \( K \) and investigate the properties of the fixed point curves generated from it.

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1. Introduction

Let \( X \) be a Banach space and \( T_1, T_2 \) two contraction maps from \( X \) into \( X \). In 1988 S. Nadler and K. Ushijima [8] considered the contractions \( T_t = (1-t)T_1 + tT_2 \) for each \( t \) in \([0,1]\) and investigated the properties of the curve generated by their fixed points. They proved a necessary and sufficient condition for a curve to be a fixed point curve for two contractions \( T_1, T_2 : \mathbb{R}^1 \to \mathbb{R}^1 \) and asked if this condition applies for every strictly convex Banach space.

In 1990 in [9] it was given an example which shows that this is not true, even in the case of contraction maps from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \).

In this paper we first give a necessary and sufficient condition for a curve in a Banach space \( X \) to be a fixed point curve for two contractions maps from \( X \) into \( X \). Then using the above technique, we consider a nonexpansive map \( T \) from a nonempty closed, convex and bounded set \( K \) of a Banach space \( X \) into \( K \) and investigate the properties of the fixed point curves of the contraction maps \( T_t = (1-t)y + tT \) for \( y \in K \) and \( t \in [0,1] \).
2. Preliminaries and notations

With a contraction map we mean a map $T$ from a Banach space $X$ into $X$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad 0 < L < 1,$$

for every $x, y \in X$, while a nonexpansive map is a map $T : X \to X$ which satisfies the condition

$$\|Tx - Ty\| \leq \|x - y\|,$$

for every $x, y \in X$.

Let $T_1, T_2 : X \to X$ be contraction maps. In [8] the authors constructed a curve

$$G : [0, 1] \to X$$

which maps every $t \in [0, 1]$ to the fixed point of the contraction map

$$T_t = (1 - t)T_1 + tT_2. \quad (1)$$

They proved that the condition

$$m|t_1 - t_2| \leq \|G(t_1) - G(t_2)\| \leq M|t_1 - t_2| \quad (2)$$

for $t_1, t_2 \in [0, 1]$ and $m > 0, M > 0$ is necessary for a curve to be the fixed point curve of the maps $T_t$ in (1) if $X$ is an arbitrary Banach space and necessary and sufficient if $X = \mathbb{R}^1$. They asked if condition (2) generalize to $\mathbb{R}^n$.

In [9] the author used the Archimedean spiral $G(t) = 2\pi t, t \in [0, 1]$, to show that this in tot true even if $X = \mathbb{R}^2$.

In Section 3 we give a necessary and sufficient condition on a curve to be a fixed point curve for the maps in (1) valid for every Banach space $X$ and with the aid of this condition we give another proof to Theorem 1.1 in [8].

In Section 4 we consider a nonexpansive map $T : K \to K$ where $K$ is a nonempty, bounded, closed and convex subset of a Banach space $X$. We consider the curves which define the fixed points of the contractions

$$T_t = (1 - t)y + tT, \quad y \in K \text{ fixed, } t \in [0, 1]$$

and give some results on the form of these curves and on the relations between such curves, for the various $y \in K$. 
3. The fixed point curve for two contractions

Let $X$ be a Banach space and $T_1, T_2 : X \rightarrow X$ two contraction maps,

$$
\|T_1 x - T_1 y\| \leq L_1 \|x - y\|, \quad \|T_2 x - T_2 y\| \leq L_2 \|x - y\|
$$

with $0 < L_1, L_2 < 1$. Then the maps

$$
T_t = (1 - t)T_1 + tT_2, \quad t \in [0, 1]
$$

are also contractions with $L = \max\{L_1, L_2\}$. So by the Banach contraction principle $T_t$ has a unique fixed point, say $x_t$. These fixed points $x_t, t \in [0, 1]$, form a curve which we name "the fixed point curve for $T_1, T_2$" and denote by $F(T_1, T_2)$.

Thus for every $x_t \in F(T_1, T_2)$ we will have

$$
T_t x_t = (1 - t)T_1 x_t + tT_2 x_t = x_t.
$$

Relation (5) gives a necessary condition for the fixed point curve $F(T_1, T_2)$. According to it the vectors $T_1 x_t - x_t$ and $x_t - T_2 x_t$ are collinear and for every $t \in (0, 1)$

$$
\frac{\|T_1 x_t - x_t\|}{\|x_t - T_2 x_t\|} = \frac{t}{1 - t}.
$$

With the aid of (6) we prove the following:

**Theorem 3.1.** ([8], Theorem 1.1). Either $F(T_1, T_2)$ is constant or it is $1 - 1$.

**Proof.** Suppose $F(T_1, T_2)$ is not constant and for $t_1, t_2 \in [0, 1]$ $x_{t_1} = x_{t_2}$. From equation (6) it follows that

$$
\frac{t_1}{1 - t_1} = \frac{t_2}{1 - t_2}
$$

which implies that $t_1 = t_2$. \qed

The following theorem shows that condition (5) is also sufficient for the fixed point curve $F(T_1, T_2)$.

**Theorem 3.2.** Let $F : [0, 1] \rightarrow X$ be a map. The necessary and sufficient condition that $F$ is the fixed point curve for the contractions $T_1, T_2 : X \rightarrow X$, is that for every $t \in [0, 1]$ the point $F(t) := x_t$ satisfies the condition

$$
x_t = (1 - t)T_1 x_t + tT_2 x_t.
$$
Proof. The necessity follows immediately from (5). For the sufficiency suppose for a contradiction that for a $t_0 \in [0,1]$ there exists a $z_{t_0}$ which satisfies (5) i.e.

$$z_{t_0} = (1 - t_0)T_1 z_{t_0} + t_0 T_2 z_{t_0}$$

(7)

and such that $z_{t_0} \notin F(T_1, T_2)$. Now for this $t_0$, the point $x_{t_0} \in F(T_1, T_2)$, satisfies

$$x_{t_0} = (1 - t_0)T_1 x_{t_0} + t_0 T_2 x_{t_0}.$$  

(8)

¿From (7) and (8) it follows that

$$\|z_{t_0} - x_{t_0}\| = \|(1 - t_0)T_1 z_{t_0} + t_0 T_2 z_{t_0} - (1 - t_0)T_1 x_{t_0} - t_0 T_2 x_{t_0}\|$$

$$\leq (1 - t_0)\|T_1 z_{t_0} - T_1 x_{t_0}\| + t_0 \|T_2 z_{t_0} - T_2 x_{t_0}\|$$

$$\leq (1 - t_0)L_1\|z_{t_0} - x_{t_0}\| + t_0 L_2 \|z_{t_0} - x_{t_0}\|$$

$$\leq L\|z_{t_0} - x_{t_0}\|,$$

where $L_1, L_2$ are defined by (3) and $L = \max\{L_1, L_2\} < 1$, which is impossible. The proof of the theorem is complete. □

4. The fixed point curves for a nonexpansive map

Let $K$ be a nonempty, bounded, closed and convex subset in a Banach space $X$ and $T : K \rightarrow K$ a nonexpansive map. Fix $y \in K$. Then the maps

$$T_t := (1 - t)y + tT, \quad t \in [0,1]$$

(9)

are contraction maps and thus every $T_t$ has a fixed point, say $x_t$. The totallity of these fixed points form a curve, which we name "the fixed point curve of $T$ with respect to $y$" and denote by $F(y, T)$. Thus for every $y \in K$ we have a curve $F : [0,1] \rightarrow K$ with

$$F(t) = x_t \in F(y, T), \quad t \in [0,1].$$

(10)

For $t = \frac{1}{n}, n \in \mathbb{N}$, every $F(y, T)$ define a sequence $\{x_n\}$. Such sequences are very important in the fixed point theory and widely investigated, see [1, 2, 3, 4, 5, 6, 7].

In this section we characterize and give some properties of the curves $F(y, T)$.
Lemma 4.1. For every $y \in K$ the fixed point curve $F(y, T)$ is continuous in $(0, 1)$.

**Proof.** Fix $t_0 \in (0, 1)$ and let $x_t$ denote the fixed point of the contraction defined by (9). Then

$$\|x_t - x_{t_0}\| = \|(1 - t)y + tx_t - (1 - t_0)y - t_0Tx_{t_0}\|$$

$$\leq |t - t_0|\|y\| + |t - t_0|\|Tx_t\| + t_0\|Tx_t - Tx_{t_0}\|$$

or

$$(1 - t_0)\|x_t - x_{t_0}\| \leq |t - t_0| (\|y\| + \|Tx_t\|).$$

Since $K$ is bounded we have

$$\lim_{t \to t_0} \|x_t - x_{t_0}\| = 0.$$

□

Lemma 4.2. The function $G : [0, 1] \to \mathbb{R}$ with

$$G(t) = \|x_t - Tx_t\|, \quad x_t \in F(y, T), \quad y \in K \quad (11)$$

is continuous and $\lim_{t \to 1} G(t) = 0$.

**Proof.** For $t \in [0, 1]$ we have

$$\|x_t - Tx_t\| = \|(1 - t)y + tx_t - Tx_t\| = (1 - t)\|y - Tx_t\|$$

and since $K$ is bounded $\lim_{t \to 1} G(t) = 0$.

□

The next theorem gives a characterization of the fixed point curve $F(y, T)$.

**Theorem 4.3.** Let $K$ be a nonempty, bounded, closed and convex subset of a Banach space $X$, $y \in K$ and $T : K \to K$ a nonexpansive map. Then a point $x \in K$ lies in the fixed point curve $F(y, T)$ if and only if there exists a $t \in [0, 1]$ such that

$$(1 - t)(y - x) = t(x - Tx). \quad (12)$$
Proof. ($\Rightarrow$) It is clear since every point $x$ of the fixed point curve $F(y, T)$ is a fixed point $x_t$ of a contraction $T_t$ defined by (9), i.e. if satisfies an equation of the form
\[ x = x_t = T_t x_t = (1 - t)y + tTx, \quad t \in [0, 1]. \]  

($\Leftarrow$) Suppose for a contradiction that there exists an $x_0 \in K - F(y, T)$ such that
\[ (1 - t_0)y + t_0Tx_0 = x_0, \quad t_0 \in (0, 1). \]  
Then there exists also a point $x_{t_0} \in F(y, T)$ with
\[ (1 - t_0)y + t_0Tx_{t_0} = x_{t_0}, \quad t_0 \in (0, 1). \]  
Equations (14) and (15) imply that
\[ x_0 - t_0Tx_0 = x_{t_0} - t_0Tx_{t_0} \]
and further that
\[ \|x_0 - x_{t_0}\| = t_0\|Tx_0 - Tx_{t_0}\| \]
which is impossible, since $T$ is nonexpansive and $t_0 \in (0, 1)$.

Corollary 4.4. With the assumptions of Theorem (4.3), the fixed point curve $F(y, T)$ is $1 - 1$.

Proof. For $x_{t_1}, x_{t_2} \in F(y, T)$ equation (12) gives
\[ (1 - t_1)(y - x_{t_1}) = t_1(x_{t_1} - Tx_{t_1}) \]
and
\[ (1 - t_2)(y - x_{t_2}) = t_2(x_{t_2} - Tx_{t_2}) \]
which for $x_{t_1} = x_{t_2}$ implies that $t_1 = t_2$.

Corollary 4.5. The function $G : y \rightarrow F(y, T)$ is $1 - 1$.

Corollary 4.6. If the fixed point curves $F(y_1, T)$ and $F(y_2, T)$ intersect at a point $x_0$, then the points $y_1, y_2, x_0$ and $Tx_0$ are collinear.

Corollary 4.7. The function $G(t) = \|x_t - Tx_t\|, \; t \in [0, 1]$ is strictly decreasing.
**Proof.** From Lemma 4.2 we have that $G(t)$ is continuous, $\lim_{t \to 1} G(t) = 0$ and $G(0) > 0$, and from Theorem 4.3 that $G(t)$ is $1 - 1$. \qed

Our last corollary gives an information about maximal values of $\|x_t - Tx_t\|$.

**Corollary 4.8.** Let $x_0 \in K$. Then

$$\sup \{ \|x_t - Tx_t\| : x_t = (1 - \lambda)x_0 + \lambda Tx_0, \lambda \in \mathbb{R} \} = \|x_s - Tx_s\|,$$

with $x_s = (1 - \lambda_0)x_0 + \lambda_0 Tx_0$, where

$$\lambda_0 = \inf \{ \lambda \in \mathbb{R} : (1 - \lambda)x_0 + \lambda Tx_0 \in K \}.$$

**References**


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