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MEASURES OF NONCOMPACTNESS, CONDENSING OPERATORS AND FIXED POINTS: AN APPLICATION-ORIENTED SURVEY

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Dedicated to the memory of Gabriele Darbo (1920-2004)

Abstract. In this survey we collect a large variety of applications of Darbo's celebrated fixed point principle to both linear and nonlinear problems, such as integral equations, boundary value problems, imbedding theorems, Schrödinger operators, essential spectra, integral transforms, substitution operators over complex domains, superposition operators in function spaces, differential equations in Banach spaces, functional-differential equations, Fredholm operators, Banach space geometry, and nonlinear spectral theory. The presentation is throughout elementary, being addressed to non-specialists, and does not require a profound background in nonlinear analysis.
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About 50 years ago, the Italian mathematician Gabriele Darbo published a fixed point theorem [25] which ensures the existence of a fixed point for so-called condensing operators and generalizes both the classical Schauder fixed point principle and (a special variant of) Banach's contraction mapping principle. Darbo's theorem is not only of theoretical interest, but has found a wealth of applications in both linear and nonlinear analysis. Typically, such applications are characterized by some "loss of compactness" which arises in many fields: imbedding theorems between Sobolev spaces with critical exponent, imbeddings over domains with irregular boundary, linear composition operators over the complex unit disc, integral equations with strongly singular kernels, differential equations over unbounded domains, functional-differential equations of neutral type or with deviating argument, linear differential operators with nonempty essential spectrum, nonlinear superposition operators between various function spaces, initial value problems in Banach spaces, and much more. In this connection, one often encounters some kind of "Golden Rule" which states that

Loss of compactness always occurs on the boundary.

Of course, we have to explain on the boundary *of what*; this will become clearer in the numerous examples which follow.

In spite of its importance, the class of condensing operators and its applications seems to be known only among some specialists in the field. We therefore think that it might be useful to collect, in a survey which mainly addresses to non-specialists, the basic facts on the theory, methods, and applications of this operator class. Here we restrict ourselves to the simplest, though application-oriented, parts of the theory, disregarding more sophisticated topics like topological degree, fixed point index, or bifurcation theory. Moreover, when writing this survey, we intentionally put the main emphasis on *examples and applications*, rather than abstract theorems, following the reliable and deserving didactical motto

Only wimps do the general case, real teachers tackle examples.

In fact, we do not aim for presenting results in their most general setting, but rather for providing a self-contained and comprehensible survey of some typical results and methods for the "working analyst", as well as a glimpse of the large variety of directions in which current research in this field is still moving.

This survey consists of a "linear part" and a "nonlinear part" and is organized as follows. Since condensing operators are intimately related to measures of noncompactness, we study first in Sections 1 and 2 general measures of noncompactness, including some special examples which have found some attention in applications. As the name suggests, such measures of noncompactness give an idea of the "lack of compactness" of a given set or operator, and this leads to interesting new results, or a new look at classical results, even in the linear case, as we shall show in Section 3. In Sections 4 and 6 this is illustrated by means of applications to (linear) differential equations, integral equations, multiplication operators, substitution operators, and imbedding theorems. There is also an important connection with classical Fredholm theory which we will discuss in Section 5.

In Section 7 we start with the nonlinear theory, with a particular emphasis on nonlinear superposition operators which offer a great variety of examples and (sometimes unexpected) counterexamples. Darbo's fixed point theorem for condensing operators, which is the *Leitmotiv* of this survey, is discussed in detail in Section 8. Combining this with the results for the linear operators obtained above, one gets numerous existence theorems for nonlinear problems of different type, as will be shown in Section 9. The subsequent Sections 10 and 11 are concerned with applications to initial value problems for differential equations in Banach spaces, as well as functional-differential equations and their periodic solutions. Fixed point free operators are discussed in Section 12; the study of such operators leads to several characteristics, defined in Section 13, which seem to have some importance in Banach space geometry. Finally, in the last Section 14 we briefly sketch how Darbo's fixed point principle may be related to the definition and study of spectra for nonlinear operators which is a rather new field of nonlinear analysis of growing importance.

1. Measures of noncompactness and ϕ-norms

Let X be a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In what follows, we write $B_r(X) = \{x \in X : ||x|| \leq r\}$ for the closed ball and $S_r(X) = \{x \in X : ||x|| = r\}$ for the sphere in X around 0 with radius $r > 0$. In particular, we use the shortcut $B_1(X) =: B(X)$ and $S_1(X) =: S(X)$ for the unit ball and sphere, respectively. Sometimes we have to consider the ball of radius r centered at $x_0 \in X$, which we denote by $B_r(X; x_0)$. Unless otherwise stated, *all operators considered in the sequel are assumed to be continuous.*

Throughout this survey, a nonnegative function ϕ defined on the bounded subsets of X will be called *Sadovskij functional* if it satisfies the following requirements ($M, N \subset X$ bounded, $\lambda \in \mathbb{K}$):

$$\phi(M \cup N) = \max\{\phi(M), \phi(N)\}, \tag{1.1}$$

$$\phi(M + N) \leq \phi(M) + \phi(N), \tag{1.2}$$

$$\phi(\lambda M) = |\lambda|\phi(M), \tag{1.3}$$

$$\phi(M) \leq \phi(N) \text{ for } M \subseteq N, \tag{1.4}$$

$$\phi([0,1] \cdot M) = \phi(M), \tag{1.5}$$

$$\phi(\overline{co}\, M) = \phi(M). \tag{1.6}$$

It is natural to call (1.1) the *set additivity*, (1.2) the *algebraic subadditivity*, (1.3) the *homogeneity*, (1.4) the *monotonicity*, (1.5) the *absorption invariance*, and (1.6) the *convex closure invariance* of ϕ. We remark that these axioms are not independent; for example, (1.4) follows from (1.1), and (1.5) follows from (1.6) if $\phi(\{0\}) = 0$.

A particularly important additional property of a Sadovskij functional is

$$\phi(M) = 0 \text{ if and only if } M \text{ is precompact}, \tag{1.7}$$

which we call the *regularity* of ϕ. A regular Sadovskij functional is called *measure of noncompactness*. This name is motivated by the fact that, loosely speaking, the smaller $\phi(M)$, the closer is M to being precompact (i.e., having compact closure). Apart from regularity, the most important property which plays a crucial role in both the theory and applications is the invariance property (1.6).

We used the name "Sadovskij functional" to emphasize the pioneering role of the survey article [73] in the axiomatic theory of measures of noncompactness. In the paper [15] the authors use another approach based on the Hausdorff distance $H(M, N) = \max\{D(M, N), D(N, M)\}$ of two bounded sets $M, N \subset X$, where, as usual,

$$D(M, N) = \inf\{r > 0 : M \subseteq N + B_r(X)\}.$$

A nonnegative function ϕ defined on the family of all bounded subsets of X is called *set quantity* in [15] if it satisfies (1.1), (1.2), (1.3) and (1.6). It is then shown that ϕ is a Sadovskij functional (in our terminology) which satisfies a Lipschitz condition

$$|\phi(M) - \phi(N)| \leq \phi(B(X))H(M, N)$$

with respect to the Hausdorff distance on X, and so ϕ is uniformly continuous. Now, suppose that $\mathcal{N}(X)$ is some family of bounded subsets of X with certain "good" additional properties (e.g., $\mathcal{N}(X)$ is stable under finite unions, algebraic sums, multiplication by scalars, and passing to the convex hull). Then it is shown in [15] that

$$\phi_{\mathcal{N}}(M) = \mathrm{dist}\,(M, \mathcal{N}(X)) = \inf\{H(M,N) :\ N \in \mathcal{N}(X)\}$$

is a set quantity, hence a Sadovskij functional. Clearly, $\phi_{\mathcal{N}}(M) = 0$ if and only if $\overline{M} \in \mathcal{N}(X)$, and so $\phi_{\mathcal{N}}$ is a measure of noncompactness (in our terminology) if $\mathcal{N}(X)$ is the family of all precompact subsets of X.

We give now a list of three important examples of measures of noncompactness which arise over and over in applications. The first example is the *Kuratowski measure of noncompactness* (or *set measure of noncompactness*) [55]

$$\begin{aligned} \alpha(M) = \inf\{\varepsilon > 0 : M &\text{ may be covered} \\ \text{by finitely many sets of diameter } &\leq \varepsilon\}, \end{aligned} \tag{1.8}$$

the second one is the *Istrăţescu measure of noncompactness* (or *lattice measure of noncompactness*) [47]

$$\begin{aligned} \beta(M) = \sup\{\varepsilon > 0 :\ &\text{there exists a sequence} \\ (x_n)_n \text{ in } M \text{ with } &\|x_m - x_n\| \geq \varepsilon \text{ for } m \neq n\}, \end{aligned} \tag{1.9}$$

and the third one is the *Hausdorff measure of noncompactness* (or *ball measure of noncompactness*) [45]

$$\gamma(M) = \inf\{\varepsilon > 0 :\ \text{there exists a finite } \varepsilon\text{-net for } M \text{ in } X\}, \tag{1.10}$$

where by a finite ε-net for M in X we mean, as usual, a set $\{z_1, z_2, \ldots, z_m\} \subset X$ such that the balls $B_\varepsilon(X; z_1), B_\varepsilon(X; z_2), \ldots, B_\varepsilon(X; z_m)$ cover M. These measures of noncompactness are mutually equivalent in the sense that

$$\gamma(M) \leq \beta(M) \leq \alpha(M) \leq 2\gamma(M) \tag{1.11}$$

for any bounded set $M \subset X$. Historically, (1.8) was the first measure of noncompactness introduced in nonlinear analysis in connection with metric spaces [55].

Clearly, the Hausdorff measure of noncompactness (1.10) may be viewed as a special case of the set quantity $\phi_{\mathcal{N}}$ mentioned above if we take for $\mathcal{N}(X)$ the family of all precompact subsets of X. Other choices of $\mathcal{N}(X)$ lead to other interesting set quantities which are also useful in applications. For instance, if $\mathcal{N}(X)$ is the family of all weakly precompact subsets of X, then $\phi_{\mathcal{N}}$ is the so-called *weak measure of noncompactness* introduced by De Blasi in [26]. Furthermore, if $\mathcal{N}(X)$ denotes the family of all weakly conditionally precompact subsets of X (i.e., every sequence in such a set admits a weak Cauchy subsequence), then $\phi_{\mathcal{N}}$ is nothing else but the set quantity introduced and studied by Falcón and Sadarangani in [34]. The paper [34] also contains some interesting connections with Banach space geometry, reflexivity criteria for normed spaces, and the "Cantor property" of nested families of subsets, which means that a decreasing sequence $(M_n)_n$ of bounded closed subsets of X satisfying $\phi_{\mathcal{N}}(M_n) \to 0$ as $n \to \infty$ has always a nonempty intersection. This property is important in Darbo's fixed point theorem which we will prove below (Theorem 8.1).

To show that the regularity axiom (1.7) is independent of the others, we bring a simple example of a Sadovskij functional which is not a measure of noncompactness.

Example 1.1. Let $X = L^p[0,1]$ $(1 \leq p < \infty)$ be the Lebesgue space of all (classes of) p-integrable real functions on $[0,1]$ with the usual norm, and denote by χ_D the characteristic function of a measurable subset $D \subseteq [0,1]$. It is not hard to see that then

$$\phi(M) := \limsup_{\mathrm{mes}\, D \to 0} \sup_{u \in M} ||\chi_D u|| \qquad (M \subset L^p \text{ bounded})$$

is a Sadovskij functional. However, ϕ is not a measure of noncompactness. In fact, any set M which is bounded in $L^q[0,1]$ for some $q > p$ satisfies $\phi(M) = 0$ in X, by the Hölder inequality. Clearly, such a set need not be precompact in X. $\square$

The Sadovskij functional in the preceding Example 1.1 has the property that the precompactness of $M \subset X$ implies $\phi(M) = 0$, but not vice versa. A certain "dual example" of a Sadovskij functional ϕ for which $\phi(M) = 0$ implies the precompactness of M, but not vice versa, is given in Example 2.4 below.

We point out that in the sequel we will retain the terms "Sadovskij functional" or "measure of noncompactness" also for set functions which do not have all the properties (1.1) – (1.7). It will become clear from time to time which of these properties we actually need. For instance, it is sometimes useful to require that the centers $z_1, z_2, \ldots, z_m$ of the covering balls in (1.10) to belong to the set M itself, not just to X. This leads to the set function

$$\gamma_0(M) = \inf\{\varepsilon > 0 : \text{ there exists a finite } \varepsilon\text{-net for } M \text{ in } M\}, \qquad (1.12)$$

which is usually called *inner Hausdorff measure of noncompactness* in the literature. This "measure of noncompactness" does not satisfy all the above axioms; for example, (1.1), (1.4), (1.5) and (1.6) are not true for (1.12), as the following example shows.

Example 1.2. Let X be any Banach space. If X is finite dimensional, then $\alpha(B(X)) = \alpha(S(X)) = \beta(B(X)) = \beta(S(X)) = \gamma(B(X)) = \gamma(S(X)) = 0$, by (1.7) and the classical Heine-Borel theorem. If X is infinite dimensional, however, we have

$$\alpha(B(X)) = \alpha(S(X)) = 2, \qquad \gamma(B(X)) = \gamma(S(X)) = 1. \qquad (1.13)$$

In fact, the equality $\gamma(B(X)) = 1$ follows from the trivial fact that $B(X)$ may be covered by itself, from a simple homogeneity argument using (1.3), and from (1.6). The proof of the equality $\alpha(B(X)) = 2$ is more complicated and requires a Ljusternik-Shnirel'man type argument. Now, the inner Hausdorff measure of noncompactness (1.12) still satisfies $\gamma_0(B(X)) = 1$; however, $\gamma_0(S(\ell^2)) = \sqrt{2}$, for example. So γ_0 is not invariant under passing to the convex hull.

From the well-known Riesz lemma it follows that $\beta(B(X)) = \beta(S(X)) \geq 1$ in every infinite dimensional Banach space X. Unfortunately, there is no "universal" formula for calculating the characteristic $\beta(B(X))$, but only formulas or estimates which require individual arguments in every space. For example (see [13, Theorem II.2.1 and Remark II.3.11]), one knows that $\beta(B(\ell^p)) = 2^{1/p}$ for $1 \leq p < \infty$, $\beta(B(L^p)) = 2^{1/p}$ for $1 \leq p \leq 2$, and $\beta(B(L^p)) = 2^{1-1/p}$ for $2 \leq p < \infty$. Loosely speaking, one could say that, in contrast to the measures of noncompactness α and γ, the measure of noncompactness β "feels the geometry" of the underlying space. $\square$

Comparing the usual and the inner Hausdorff measure of noncompactness one easily sees that

$$\gamma(M) \leq \gamma_0(M) \leq \alpha(M). \tag{1.14}$$

To get an idea of how to calculate these measures of noncompactness, we give some examples which are taken from [37].

Example 1.3. Let $X = C[0,1]$ be the Banach space of all continuous real functions on $[0,1]$, equipped with the usual maximum norm. By (1.13) in Example 1.2, for $M = B(X)$, say, we have then

$$\gamma(M) = \gamma_0(M) = 1, \quad \alpha(M) = 2.$$

On the other hand, the set $M := \{u \in B(X) :\ 0 = u(0) \leq u(t) \leq u(1) = 1\}$ satisfies

$$\gamma(M) = \frac{1}{2}, \quad \gamma_0(M) = \alpha(M) = 1.$$

Similarly, for the set $M := \{u \in B(X) :\ 0 \leq u(0) \leq \frac{1}{3},\ 0 \leq u(t) \leq 1,\ \frac{2}{3} \leq u(1) \leq 1\}$ we obtain

$$\gamma(M) = \frac{1}{2}, \quad \gamma_0(M) = \frac{2}{3}, \quad \alpha(M) = 1.$$

Finally, the (noncompact) set

$$M := \{u \in B(X) : 0 \leq u(t) \leq \frac{1}{2} \text{ for } 0 \leq t \leq \frac{1}{2}, \text{ and } \frac{1}{2} \leq u(t) \leq 1 \text{ for } \frac{1}{2} \leq t \leq 1\}$$

satisfies

$$\gamma(M) = \gamma_0(M) = \alpha(M) = \frac{1}{2}.$$

So all possible combinations of equality and strict inequality may occur in the estimates (1.14). □

Given two Banach spaces X and Y, a set $M \subseteq X$, a (linear or nonlinear) operator $A : M \to Y$, and a measure of noncompactness ϕ on X and Y, the characteristic

$$[A]_\phi = \inf \{k > 0 :\ \phi(A(N)) \leq k\phi(N) \text{ for bounded } N \subseteq M\} \tag{1.15}$$

is called the *ϕ-norm* of A. The ϕ-norm (1.15) has the property

$$[A]_\phi = 0 \text{ if and only if } A \text{ is compact}, \tag{1.16}$$

where an operator A is called *compact*, as usual, if $A(N) \subset Y$ is precompact for each bounded set $N \subseteq M$. Moreover, the ϕ-norm (1.15) is *subadditive* in the sense that

$$[A+B]_\phi \leq [A]_\phi + [B]_\phi \tag{1.17}$$

for two operators $A, B : M \to Y$. In infinite dimensional spaces X and Y, one may use the equivalent formula

$$[A]_\phi = \sup\left\{\frac{\phi(A(N))}{\phi(N)} : N \subseteq M \text{ bounded}, \ \phi(N) > 0\right\} \tag{1.18}$$

for calculating the ϕ-norm of A. For some measures of noncompactness (or Sadovskij functionals) ϕ, the ϕ-norm (1.15) may be connected to Lipschitz continuity. We say that a Sadovskij functional ϕ is *Lip-compatible*, if

$$[A]_\phi \leq Lip(A), \tag{1.19}$$

where

$$Lip(A) = \inf\{k > 0 : ||A(x) - A(y)|| \leq k||x-y|| \, (x, y \in M)\} \tag{1.20}$$

is the *minimal Lipschitz constant* of $A : M \to Y$. More generally, we say that ϕ is *weakly Lip-compatible*, if

$$[A]_\phi \leq cLip(A), \tag{1.21}$$

for some constant $c = c(\phi) > 0$ independent of A. For example, the ϕ-norms generated by the measures of noncompactness (1.8), (1.9) and (1.12) are Lip-compatible, but the ϕ-norm generated by (1.10) is only weakly Lip-compatible with $c(\gamma) = 2$.

Clearly, if $L : X \to Y$ is a bounded *linear* operator, then $Lip(L) = ||L||$, and so $[L]_\phi \leq ||L||$ if ϕ is Lip-compatible. We will discuss this inequality in more detail in Section 3 below.

The following example shows that there are somewhat "artificial" Sadovskij functionals which are not even weakly Lip-compatible.

Example 1.4. Let $X = c_0$ be the Banach space of all real sequences converging to zero, equipped with the supremum norm. For $x = (\xi_n)_n \in c_0$

we denote by $n(x)$ the number of coordinates $\xi_n \geq 1$ of the sequence x. It is not hard to see that the set function

$$\phi(M) := \frac{1}{1 + \min\limits_{x \in M} n(x)}$$

is then a Sadovskij functional in X which satisfies (1.1), (1.5) and (1.6), but not (1.2). Moreover, since (1.7) fails, ϕ is not a measure of noncompactness.

Now consider the linear operator $L = \frac{1}{2}I$, where I denotes the identity. Denoting by $x_k := e_1 + e_2 + \ldots + e_k$ the sum of the first k basis vectors in c_0, the set $M_k := \{x_k, x_{k+1}, x_{k+2}, \ldots\}$ satisfies $\phi(M_k) = \frac{1}{1+k}$. On the other hand, the trivial fact that $n(Lx_k) = 0$ for each k implies that $\phi(L(M_k)) = 1$, and so ϕ cannot be weakly Lip-compatible. □

We call an operator $A : M \to Y$ *ϕ-Lipschitz* if $[A]_\phi < \infty$. A particularly important case is $[A]_\phi < 1$; in this case A is called *ϕ-condensing*. If merely $[A]_\phi \leq 1$, it is natural to call A a *ϕ-nonexpansive* operator. We remark that α-condensing, β-condensing, and γ-condensing operators are also called set-contractions, lattice-contractions, and ball-contractions, respectively, in the literature.

Sometimes it is very easy to estimate (or even calculate) the ϕ-norm (1.15) of a given operator, simply by using the properties (1.1) – (1.7) of the underlying measure of noncompactness ϕ. We illustrate this by a simple, though important, example.

Example 1.5. Let X be a Banach space, and consider the *radial retraction* $\rho : X \to B_r(X)$ defined by

$$\rho(x) := \begin{cases} x & \text{if } \ ||x|| \leq r, \\ r\dfrac{x}{||x||} & \text{if } \ ||x|| > r. \end{cases} \tag{1.22}$$

Let ϕ be any measure of noncompactness on X which satisfies (1.1), (1.2), and (1.7) (for example, $\phi \in \{\alpha, \beta, \gamma\}$). Then the obvious geometric fact that $\rho(M) \subseteq \overline{co}\,(M \cup \{0\})$, for each bounded set $M \subset X$, implies that the retraction (1.22) is ϕ-nonexpansive. More precisely, the fact that $\rho(B_r(X)) = B_r(X)$ implies that $[\rho]_\phi = 1$ if X is infinite dimensional, and $[\rho]_\phi = 0$ if X is finite dimensional. □

We point out that the radial retraction (1.22) does *not* satisfy the estimate $Lip(\rho) \leq 1$, but only the weaker estimate $Lip(\rho) \leq 2$. So this retraction may serve as a first example for strict inequality in (1.19).

For some problems in nonlinear analysis it is useful to consider, together with the ϕ-norm (1.15), also the *lower ϕ-norm* of $A : M \to Y$ defined by

$$[A]_\phi^- = \sup\{k > 0 : \phi(F(N)) \geq k\phi(N) \text{ for bounded } N \subseteq M\}. \tag{1.23}$$

The lower ϕ-norm may be calculated between infinite dimensional spaces equivalently by the formula

$$[A]_\phi^- = \inf\left\{\frac{\phi(A(N))}{\phi(N)} : N \subseteq M \text{ bounded}, \ \phi(N) > 0\right\} \tag{1.24}$$

which is of course parallel to (1.18). The number (1.23) is closely related to the class of proper operators. Recall that an operator $A : M \to Y$ is called *proper* if the preimage $A^{-1}(K)$ of any compact subset $K \subset Y$ is a compact subset of M. Clearly, $[A]_\phi^- > 0$ implies that A is proper on closed bounded subsets of M, by (1.7). The converse, however, is not true:

Example 1.6. Let X be any infinite dimensional Banach space and $A : X \to X$ defined by $A(x) = ||x||x$. A straightforward calculation shows that A is a homeomorphism on X, hence proper. On the other hand, for the Hausdorff measure of noncompactness (1.10), say, one has $\gamma(S_r(X)) = r$, by (1.13), and $\gamma(A(S_r(X))) = \gamma(S_{r^2}(X)) = r^2$. Letting $r \to 0$ and using (1.24) we conclude that $[A]_\gamma^- = 0$. $\square$

We remark that a detailed account of the theory and applications of measures of noncompactness and corresponding classes of condensing operators, together with many more examples, may be found in the survey [1], the Lecture Notes [14], and the more recent monographs [3,13].

2. Measures of noncompactness in special spaces

If one wants to calculate the measure of noncompactness $\phi(M)$ of some set M in a specific Banach space X, it is sometimes easier to pass to a set function ϕ^* which is equivalent to ϕ in the sense that

$$a\phi(M) \leq \phi^*(M) \leq b\phi(M) \qquad (M \subset X \text{ bounded}) \tag{2.1}$$

for some constants $a, b > 0$ independent of M. The equivalence (2.1) plays a similar role for measures of noncompactness as the equivalence of norms. From (1.7) and (2.1) it follows, in particular, that $\phi^*(M) = 0$ precisely for all precompact subsets $M \subset X$.

In this section we construct such a set function γ^* which is equivalent to the Hausdorff measure of noncompactness (1.10) in the spaces $C = C[0,1]$, $L^p = L^p[0,1]$, ℓ^p, c and c_0. Recall that the *modulus of continuity* of a function $u \in C$ is defined by

$$\omega(u,\sigma) = \sup\{|u(s) - u(t)| : |s - t| \leq \sigma\}. \tag{2.2}$$

We have then $\omega(u,\sigma) \to 0$, as $\sigma \to 0$, since u is uniformly continuous on $[0,1]$. More generally, if this limit relation holds uniformly for u running over some bounded set $M \subset C$, then M is equicontinuous, and vice versa. Therefore the following result is not too surprising:

Theorem 2.1. *On the space $X = C$, the measure of noncompactness* (1.10) *is equivalent to*

$$\gamma^*(M) = \lim_{\sigma\to 0} \sup_{u\in M} \omega(u,\sigma), \tag{2.3}$$

and one even has $a = b = 2$ in (2.1), *i.e.,*

$$\gamma^*(M) = 2\gamma(M) \tag{2.4}$$

for all bounded sets $M \subset C$.

A parallel result in the Lebesgue space L^p reads as follows:

Theorem 2.2. *On the space $X = L^p$ $(1 \leq p < \infty)$, the measure of noncompactness* (1.10) *is equivalent to*

$$\gamma^*(M) = \lim_{h\to 0} \sup_{u\in M} \left\{ \int_0^1 |u(t+h) - u(t)|^p \, dt \right\}^{1/p}$$

in the sense that

$$\gamma(M) \leq \gamma^*(M) \leq 2\gamma(M)$$

for all bounded sets $M \subset L^p$.

Observe that Theorems 2.1 and 2.2 generalize (and imply) the well-known Arzelà-Ascoli and Kolmogorov compactness criteria in the spaces C and L^p, respectively.

We remark that, in contrast to (2.4), the equality $\gamma^*(M) = 2\gamma(M)$ does not hold in the space L^p, and so the claim 1.1.3 in [3] is false:

Example 2.1. Let $X = L^p$ $(1 \le p < \infty)$ and $M = \{u_\delta : 0 < \delta < \frac{1}{2}\}$, where $u_\delta(t) := \delta^{-1/p}$ for $\frac{1}{2} \le t \le \frac{1}{2} + \delta$, and $u_\delta(t) := 0$ otherwise. Then a straightforward calculation shows that $\gamma(M) = 1$, but $\gamma^*(M) = 2^{1/p}$. So (2.4) fails in L^p if $p > 1$. $\square$

The next theorem provide equivalent measures of noncompactness in some important sequence spaces, even with $a = b = 1$ in (2.1).

Theorem 2.3. *On the space $X = \ell^p$ $(1 \le p < \infty)$, the measure of noncompactness* (1.10) *is equal to*

$$\gamma^*(M) = \lim_{N\to\infty} \sup_{(\xi_n)_n \in M} \left\{ \sum_{n=N}^{\infty} |\xi_n|^p \right\}^{1/p},$$

while on the space $X = c$ and $X = c_0$, the measure of noncompactness (1.10) *is equal to*

$$\gamma^*(M) = \lim_{N\to\infty} \sup_{(\xi_n)_n \in M} |\xi_N - \hat{\xi}|,$$

where $\hat{\xi}$ denotes the limit of the sequence $(\xi_n)_n$.

Again, Theorem 2.3 contains a number of well-known compactness criteria in the sequence spaces ℓ^p, c, and c_0. Interestingly, such theorems may be obtained in the more general setting of Banach spaces with Schauder basis. Recall that a sequence $E := (e_n)_n$ of elements $e_n \in S(X)$ is called a *Schauder basis* in X if for every $x \in X$ there exists a unique scalar sequence $(c_n(x))_n$ such that $x = \sum_{n=1}^{\infty} c_n(x) e_n$. The numbers $c_n(x)$ are usually called the *coordinates* of x. To any Schauder basis E we may associate the sequence of canonical projections

$$P_n : X \to \operatorname{span}\{e_1, e_2, \ldots, e_n\}, \qquad P_n x := \sum_{k=1}^{n} c_k(x) e_k;$$

moreover, by $R_n x := x - P_n x$ we denote the corresponding "remainders". Clearly, P_n and R_n are bounded linear operators in X. From the closed graph theorem and the uniform boundedness theorem it follows that the sequences $(P_n)_n$ and $(R_n)_n$ are bounded in the operator norm, and so

$$C_E := \limsup_{n\to\infty} ||R_n|| < \infty. \tag{2.5}$$

The following theorem [13, Theorems II.4.2 and II.4.3] shows that one may define two measures of noncompactness on X in terms of the operators R_n, which are both equivalent to the three measures of noncompactness α, β and γ discussed in the preceding section.

Theorem 2.4. *The functions*

$$\mu(M) := \limsup_{n\to\infty} \sup_{x\in M} ||R_n x||, \qquad \nu(M) := \liminf_{n\to\infty} \sup_{x\in M} ||R_n x|| \tag{2.6}$$

are measures of noncompactness on X which satisfy

$$\nu(M) \leq \mu(M) \leq C_E\gamma(M) \leq C_E\nu(M) \qquad (M \subset X \text{ bounded}). \tag{2.7}$$

Theorem 2.4 shows that, whenever one knows the constant (2.5) for some Schauder basis $E = (e_n)_n$ in a space X, one may estimate the Hausdorff measure of noncompactness γ (and so also the measures of noncompactness α and β, by (1.11)) by one of the measures of noncompactness μ or ν which are sometimes easier to calculate.

Example 2.2. In the spaces $X = \ell^p$ $(1 \leq p < \infty)$ or $X = c_0$, the canonical sequences $e_n := (\delta_{k,n})_k$ form a Schauder basis E with $C_E = 1$. Thus, for these spaces all measures of noncompactness in (2.7) coincide, which is nothing else but a reformulation of Theorem 2.3. On the other hand, in the space $X = c$ the set $\hat{E} := E \cup \{(1, 1, 1, \ldots)\}$ is a Schauder basis with $||R_n|| = 2$ for each n, and so $C_{\hat{E}} = 2$. Indeed, the equalities

$$\gamma(B(c)) = 1, \qquad \mu(B(c)) = \nu(B(c)) = 2$$

hold in this space, and so the estimates (2.7) are sharp. It is also known [12] that, in the space $X = L^p[0, 1]$ for $1 \leq p < \infty$ and $p \neq 2$, equipped with the classical Haar system as Schauder basis, the two measures of noncompactness μ and ν from (2.6) do *not* coincide. □

We point out that the chain of estimates (1.11) may also be sharpened in special Banach spaces. For example, it has been shown in [33] that

$$\beta(M) = 2^{1/p}\gamma(M) \leq \alpha(M) \qquad (M \subset \ell^p \text{ bounded}), \tag{2.8}$$

and in [32] that

$$2^{1/p}\gamma(M) \leq \beta(M) \qquad (M \subset L^p \text{ bounded}), \tag{2.9}$$

where equality holds in (2.9) if the set M is compact in measure. Analogous formulas in ℓ^∞ and L^∞ are not true.

Interestingly, there is a certain "asymmetry" in the last two formulas, inasmuch the ℓ^p-estimate (2.8) holds for $1 \leq p < \infty$, but the L^p-estimate (2.9) only for $2 \leq p < \infty$. The following counterexample [29] illustrates this fact.

Example 2.3. Consider the Rademacher functions y_n in $L^p[0,1]$, and let $M := \{y_1, y_2, y_3, \ldots\}$. The proof of Theorem X.5.2 in [13] shows that

$$\gamma(M) = 1, \qquad \beta(M) = 2^{1-1/p} = 2^{1-1/p}\gamma(M).$$

But $2^{1-1/p} < 2^{1/p}$ for $p < 2$, and so (2.9) fails for these values of p. □

The reason for this counterexample is that (2.9) has, for general $p \in [1,\infty)$, to be replaced by the correct estimate [29]

$$2^{\min\{1/p,1-1/p\}}\gamma(M) \leq \beta(M) \leq 2^{\max\{1/p,1-1/p\}}\gamma(M). \tag{2.10}$$

To see this, choose $M_0 \subseteq M$ such that $\gamma(M_0) = \gamma(M)$ and M_0 is γ-minimal in the sense of [13, Definition III.2.1]. We can assume that M_0 is also β-minimal. From Corollary X.5.3 in [13] it follows that

$$\frac{\gamma(M)}{\beta(M)} \leq \frac{\gamma(M)}{\beta(M_0)} = \frac{\gamma(M_0)}{\beta(M_0)} \leq 2^{\max\{-1/p,-(1-1/p)\}} = 2^{-\min\{1/p,1-1/p\}}$$

which proves the first estimate in (2.10). On the other hand, by [13, Theorem III.2.10] for $\varepsilon > 0$ we may find $M_\varepsilon \subseteq M$ such that M_ε is β-minimal and $(1+\varepsilon)\beta(M_\varepsilon) \geq \beta(M)$. Again, we can assume that M_ε is also γ-minimal. Applying as above Corollary X.5.3 in [13] yields

$$\begin{aligned}\frac{\gamma(M)}{\beta(M)} &\geq \frac{\gamma(M)}{(1+\varepsilon)\beta(M_\varepsilon)} \geq \frac{\gamma(M_\varepsilon)}{(1+\varepsilon)\beta(M_\varepsilon)} \\ &\geq 2^{\min\{-1/p,-(1-1/p)\}} = 2^{-\max\{1/p,1-1/p\}}\end{aligned}$$

which gives the second estimate in (2.10).

Observe that Example 2.3 shows that the left inequality in (2.10) is sharp for $p < 2$, and the right inequality in (2.10) is sharp for $p > 2$ if the underlying measure is not purely atomic. Similarly, by imbedding the canonical ℓ^p basis into L^p one sees that (2.10) is sharp also for the remaining values of p.

Apart from the Chebyshev space $C[0,1]$ and the Lebesgue space $L^p[0,1]$, another function space which is important in applications is the *Hölder space* $C^\alpha[0,1]$ $(0 < \alpha < 1)$ of all continuous functions u for which the norm

$$||u|| := \max_{0\le t\le 1} |u(t)| + \sup_{0\le s<t\le 1} \frac{|u(s)-u(t)|}{|s-t|^\alpha} \tag{2.11}$$

is finite. Unfortunately, a simple compactness criterion in this space is not known, let alone a formula for some measure of noncompactness. (The compactness criterion given in [46, Ch. II, 4, Theorem 1] is false.) However, in the so-called *little Hölder space* $C_0^\alpha[0,1]$ of all $u \in C^\alpha[0,1]$ satisfying

$$\lim_{|s-t|\to 0} \frac{|u(s)-u(t)|}{|s-t|^\alpha} = 0, \tag{2.12}$$

the situation is different.

Theorem 2.5. *On the space* $X = C_0^\alpha$, *the measure of noncompactness* (1.10) *is equivalent to*

$$\gamma^*(M) = \lim_{|s-t|\to 0} \sup_{u\in M} \frac{|u(s)-u(t)|}{|s-t|^\alpha} = 0,$$

in the sense that

$$\gamma(M) \le \gamma^*(M) \le 2\gamma(M)$$

for all bounded sets $M \subset C_0^\alpha$.

Theorem 2.5 implies, in particular, that a bounded set $M \subset C_0^\alpha$ is precompact if and only (2.12) holds uniformly for $u \in M$. For example, any bounded subset $M \subset C^\beta$ for $\beta > \alpha$ is precompact in both C^α and C_0^α. This shows that the space C^β is compactly imbedded into the space C^α for $\beta > \alpha$; this is in contrast to the space L^q which is only continuously imbedded into L^p for $q > p$.

We point out that sometimes it is useful to introduce other measures of noncompactness in special spaces which take into account special properties of solutions of a certain problem. For example, if one is interested in monotonically increasing continuous solutions, one may use the following construction due to [16].

Example 2.4. In the space $X = C$, consider the function

$$\phi(M) := \gamma^*(M) + \delta(M) \qquad (M \subset X \text{ bounded}), \tag{2.13}$$

where $\gamma^*(M)$ is defined as in (2.3), and $\delta(M)$ by

$$\delta(M) := \sup_{u \in M} \sup \{|u(t) - u(s)| - u(t) + u(s) : 0 \leq s \leq t \leq 1\}.$$

It is not hard to see that (2.13) is a Sadovskij functional which has all properties (1.1) – (1.6) (with $\lambda > 0$ in (1.3)), and that $\delta(M) = 0$ if and only if M consists only of increasing functions. Of course, the equality $\phi(M) = 0$ implies $\gamma^*(M) = 0$, and so M is precompact. On the other hand, the converse is not true, since $\phi(\{u\}) > 0$ for any non-increasing continuous function. This shows that (2.13) is not a measure of noncompactness in the sense of (1.7).

To give an idea of how to calculate the Sadovskij functional (2.13), consider the sets $M := \{u_1, u_2, u_3, \ldots\}$ and $N := \{v_1, v_2, v_3, \ldots\}$, where $u_n(t) := t^n$ and $v_n(t) := (1-t)^n$. A simple calculation shows then that

$$\gamma(M) = \gamma(N) = \frac{1}{2}, \quad \gamma^*(M) = \gamma^*(N) = 1, \quad \delta(M) = 0, \quad \delta(N) = 2,$$

and so $\phi(M) = 1$ and $\phi(N) = 3$. □

3. Linear ϕ-Lipschitz operators

Let X and Y be two Banach spaces and $L : X \to Y$ a bounded linear operator. Apart from the ϕ-norm (1.15), there is another characteristic which measures how far the operator L is from being compact, namely its *essential norm*

$$|||L||| := \inf \{||L - K|| : \ K : X \to Y \text{ compact}\}. \tag{3.1}$$

Of course, (3.1) is nothing else but the distance of L from the closed two-sided ideal $\mathcal{K}(X, Y)$ of all compact operators in the Banach space $\mathcal{L}(X, Y)$ of all bounded linear operators; in particular, $|||L||| = 0$ if and only if L is compact. Equivalently, the number (3.1) may be viewed as norm of the class of L in the *Calkin algebra* $\mathcal{L}(X,Y)/\mathcal{K}(X,Y)$. From the algebraic subadditivity (1.17) of the ϕ-norm and the compactness of the zero operator it follows that

$$[L]_\phi \leq |||L||| \leq ||L|| \tag{3.2}$$

for any Lipschitz-compatible measure of noncompactness ϕ. Of course, every nonzero compact operator may serve as a trivial example for strict inequality in the second estimate in (3.2). The problem of finding an example of strict inequality in the first estimate is much harder; the following example for $\phi = \gamma$ is taken from [45].

Example 3.1. Consider the product space $X = \ell^2 \times c$ of all pairs (x, y) of square summable sequences $x = (\xi_1, \xi_2, \xi_3, \ldots)$ and convergent sequences $y = (\eta_1, \eta_2, \eta_3, \ldots)$, equipped with the Euclidean norm $||(x,y)||^2 = ||x||^2 + ||y||^2$, and let $L(x,y) := (0,x)$. It is evident that $L \in \mathcal{L}(X,X)$ with $||L|| = 1$. We claim that

$$|||L||| = 1, \qquad [L]_\gamma = \frac{1}{\sqrt{2}}. \tag{3.3}$$

In fact, since the space X has a Schauder basis $(e_n)_n$, it suffices to show that $||L - K|| \geq 1$ for any one-dimensional operator $K \in \mathcal{K}(X,X)$. But every such operator has the form

$$K(x,y) = \left(\sum_{n=1}^{\infty} \alpha_n \xi_n \right) x_0,$$

where $(\alpha_n)_n \in \ell^2$ and $x_0 \in X$ are fixed; for this K we have $K(e_n, 0) = |\alpha_n| x_0 \to 0$, as $n \to \infty$.

On the other hand, the second equality in (3.3) follows from the fact that $[L]_\gamma = \gamma(M)$, where $M = \{(0,y) : y \in B(c)\}$. It remains to note that for each $(0,y) \in M$ we can find a finite sequence $\tilde{y}$ such that $||y - \tilde{y}|| \leq 1/\sqrt{2}$, and the subset $\tilde{M} = \{(0,\tilde{y}) : y \in M\}$ is a compact $1/\sqrt{2}$-net for M in X. $\square$

It may also happen that $|||L||| = ||L||$ for some operator $L \in \mathcal{L}(X,Y)$. By (3.2) this means that L is "as noncompact as a linear operator may be". We give two examples of this type.

Example 3.2. Given a continuous function $a : [0,1] \to \mathbb{R}$, consider the *multiplication operator* A defined by $Au(t) = a(t)u(t)$. Obviously, this operator is continuous in the space C, equipped with the ususal maximum norm, and has norm $||A|| = ||a||$. Moreover, since $A^n u(t) = a(t)^n u(t)$ and $||a^n|| = ||a||^n$, the norm $||A||$ of A coincides with its spectral radius $r(A)$. Finally, we have the equalities

$$[A]_\gamma = |||A||| = ||A|| = ||a|| \tag{3.4}$$

for this operator. To see this, suppose that, without loss of generality, $a(t) \not\equiv 0$, fix $\varepsilon \in (0, ||a||)$, and choose an open interval $I \subset [0,1]$ such that $|a(t)| \geq ||a|| - \varepsilon$ for $t \in I$. Putting $b(t) := 1/a(t)$ on I and extending b constant outside I, we may consider the multiplication operator $Bv(t) := b(t)v(t)$ with norm $||B|| = ||b||$.

The set $M := \{u \in B(C) : u(t) \equiv 0 \text{ for } t \in [0,1] \setminus I\}$ is not precompact, and so $0 < \gamma(M) \leq 1$. For $u \in M$ we have $BAu = u$, by definition of B, hence

$$\gamma(M) = \gamma(B(A(M))) \leq ||B||\gamma(A(M)) = ||b||\gamma(A(M)) \leq \frac{\gamma(A(M))}{||a|| - \varepsilon}.$$

This implies that $[A]_\gamma \geq ||a|| - \varepsilon$, and so (3.4) is proved, since $\varepsilon > 0$ was arbitrary. □

The equalities (3.4) show that the multiplication operator generated by a continuous function is “extremely noncompact”; in particular, it may be compact only in the trivial case when $a(t) \equiv 0$.

Of course, an analogous result holds when $a \in L^\infty$ and the operator A is considered in the Lebesgue space L^p $(1 \leq p \leq \infty)$. In this case A is compact if and only if $a(t) = 0$ almost everywhere in $[0,1]$.

In the following example we discuss another “extremely noncompact” linear operator in this sense which plays a prominent role in complex analysis.

Example 3.3. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open complex unit disc, and $\mathbb{S}^1 = \partial\mathbb{D}$ its boundary. Recall that the *Hardy space* $H^2 = H^2(\mathbb{D})$ is defined as set of all complex functions $u \in L^2(\mathbb{S}^1)$ for which the norm

$$||u|| := \lim_{r \uparrow 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |\hat{u}(re^{i\tau})|^2 \, d\tau \right)^{1/2} = \sup_{r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |\hat{u}(re^{i\tau})|^2 \, d\tau \right)^{1/2}$$

makes sense and is finite; here $\hat{u}$ is the harmonic extension of u to $\overline{\mathbb{D}}$ defined through the classical Poisson kernel by

$$\hat{u}(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\tau) \mathrm{Re} \, \frac{e^{i\tau} + z}{e^{i\tau} - z} \, d\tau.$$

Equivalently, one may characterize $u \in H^2$ by the property that all Fourier coefficients of u on $\mathbb{S}^1$ with negative indices vanish (and thus $\hat{u}$ is analytic in $\mathbb{D}$).

Given a function $\varphi : \mathbb{D} \to \mathbb{D}$ (which we may consider as "change of variables"), the linear operator $C_\varphi := u \circ \varphi$, i.e., $C_\varphi u(z) = u(\varphi(z))$, is called the *composition operator* (or *substitution operator*) generated by φ. It is easy to see that its inverse (if it exists!) is given by $C_\varphi^{-1} = C_{\varphi^{-1}}$; so C_φ is invertible if and only if $\varphi \in Aut(\mathbb{D})$. For example, the substitution operator C_φ generated by the Möbius transform

$$\varphi(z) = \frac{a - z}{1 - \overline{a} z} \qquad (a \in \mathbb{D}) \tag{3.5}$$

coincides with its inverse, since $\varphi^{-1} = \varphi$.

It turns out that one may estimate (or even calculate) the norm and the essential norm (3.1) of the operator C_φ in the Hardy space $H^2(\mathbb{D})$ completely in terms of the corresponding function φ. Clearly, $||C_\varphi|| \geq 1$, since C_φ keeps every constant function fixed. Moreover, the norm satisfies the two-sided estimate [89]

$$\frac{1}{2} \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2} \leq ||C_\varphi|| \leq \left(\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \right)^{1/2};$$

in particular, $||C_\varphi|| = 1$ if $\varphi(0) = 0$. One may show that in the special case of the Möbius transform (3.5) the equality

$$||C_\varphi|| = \left(\frac{1 + |a|}{1 - |a|} \right)^{1/2} \tag{3.6}$$

holds; this means that the "size" of the norm is determined by the distance of a to the boundary of $\mathbb{D}$, and may actually attain any value in the interval $[1, \infty)$.

To calculate the essential norm (3.1) and α-norm (1.15) of C_φ, we have to recall a notion from geometric function theory. The *Nevanlinna counting* *function* N_φ of $\varphi : \mathbb{D} \to \mathbb{D}$ is defined by

$$N_\varphi(w) := \sum_{\substack{0<|z|<1 \\ \varphi(z)=w}} \log \frac{1}{|z|} \qquad (w \in \mathbb{D}).$$

Now, the "size" of the essential norm (3.1) of C_φ essentially depends on the asymptotic behaviour of $N_\varphi(w)$ as w approaches the boundary; more precisely, the equality (see [62,77,89])

$$[C_\varphi]_\alpha = |||C_\varphi||| = \lim_{|w|\uparrow 1} \left(\frac{N_\varphi(w)}{\log \frac{1}{|w|}} \right)^{1/2} \tag{3.7}$$

holds. In particular, for the Möbius transform (3.5) one has $N_\varphi(w) = \log \varphi(w)$, and so

$$[C_\varphi]_\alpha = |||C_\varphi||| = \left(\frac{1+|a|}{1-|a|} \right)^{1/2}.$$

A comparison with (3.6) shows that the essential norm and the usual norm of C_φ coincide for φ as in (3.5), and so the composition operator generated by a Möbius transform is as "noncompact" as a linear operator may be. On the other hand, for the function $\varphi(z) := \frac{1}{2}(z+1)$, say, we have

$$||C_\varphi|| = \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right)^{1/2} = \sqrt{3}, \qquad [C_\varphi]_\alpha = |||C_\varphi||| = \lim_{r\uparrow 1} \frac{\log(2r-1)}{\log r} = 2,$$

while for the function $\varphi(z) := \frac{1}{3}(z+1)$ we have

$$||C_\varphi|| = \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|} \right)^{1/2} = 2, \qquad [C_\varphi]_\alpha = |||C_\varphi||| = 0.$$

In general, one may show that C_φ is compact if the image $\varphi(\mathbb{D})$ of $\mathbb{D}$ under φ is bounded away from $\mathbb{S}^1$. □

Formula (3.7) shows that the composition operator C_φ is compact in the Hardy space H^2 if and only if $N_\varphi(w) = o(\log 1/|w|)$ as $|w| \uparrow 1$. There is a generalization to Hardy spaces H^p which reads as follows [78]: *In case $p \le q$, the operator $C_\varphi : H^p \to H^q$ is bounded if and only if $N_\varphi(w) = O([\log 1/|w|]^{2q/p})$ as $|w| \uparrow 1$, and $C_\varphi : H^p \to H^q$ is compact if and only if $N_\varphi(w) = o([\log 1/|w|]^{2q/p})$ as $|w| \uparrow 1$.*

Many linear compact operators may be represented as matrix operators in sequence spaces or integral operators in function spaces. While we will treat integral operators in the next section, we will briefly discuss matrix operators in ℓ^p the following theorem and example. Suppose that a linear operator $A : \ell^p \to \ell^q$ $(1 \leq p, q \leq \infty)$ is given by some infinite matrix $(\alpha_{ij})_{ij}$, i.e., $A(\xi_1, \xi_2, \xi_3, \ldots) = (\eta_1, \eta_2, \eta_3, \ldots)$ with

$$\eta_i = \sum_{j=1}^{\infty} \alpha_{ij} \xi_j \qquad (i = 1, 2, 3, \ldots).$$

In this case one may consider the *mixed norms*

$$[\alpha]_{p,q} := ||(||(\alpha_{ij})_j||_{p'})_i||_q, \qquad [\alpha]^*_{p,q} := ||(||(\alpha_{ij})_i||_q)_j||_{p'} \tag{3.8}$$

of this matrix, where $p' = p/(p-1)$ as usual. It is well-known that $A \in \mathcal{L}(\ell^p, \ell^q)$ if one of the numbers in (3.8) is finite; moreover, the smaller of these numbers may serve as an upper bound for the norm of A. In some cases one of the numbers in (3.8) are even equal to the norm of A; for example, $||A||_{\ell^1 \to \ell^q} = [\alpha]^*_{1,q}$ for $1 \leq q \leq \infty$, and $||A||_{\ell^p \to \ell^\infty} = [\alpha]_{p,\infty}$ for $1 \leq p \leq \infty$.

Concerning the compactness of the operator A which corresponds to the matrix $(\alpha_{ij})_{ij}$, we use the abbreviation

$$\gamma_{p,q}(\alpha) := \begin{cases} \inf\limits_{k,m} \sup\limits_{j \geq m} \sum\limits_{i=k}^{\infty} |\alpha_{ij}|^q & \text{if} \quad p = 1 \text{ and } 1 \leq q < \infty, \\ \inf\limits_{k,m} \sup\limits_{j \geq m} \sup\limits_{i \geq k} |\alpha_{ij}| & \text{if} \quad p = 1 \text{ and } q = \infty, \\ \inf\limits_{k,m} \sup\limits_{i \geq k} \sum\limits_{j=m}^{\infty} |\alpha_{ij}|^{p'} & \text{if} \quad 1 \leq p < \infty \text{ and } q = \infty. \end{cases} \tag{3.9}$$

We summarize with the following

Theorem 3.1. *Suppose that one of the numbers in* (3.8) *is finite. Then the operator A given by the matrix $(\alpha_{ij})_{ij}$ is always compact between ℓ^p and ℓ^q in case $1 < p \leq \infty$ and $1 \leq q < \infty$. In all the other cases the estimate*

$$[A]_\gamma \leq \gamma_{p,q}(\alpha)$$

holds, with $\gamma_{p,q}(\alpha)$ given by (3.9). *In particular, A is compact if $\gamma_{p,q}(\alpha) = 0$.*

Example 3.4. We illustrate Theorem 3.1 by means of the matrix operator

$$A_\tau := \begin{pmatrix} 1 & \tau & \tau^2 & \tau^3 & \tau^4 & \dots \\ 1-\tau & 2(1-\tau)\tau & 3(1-\tau)\tau^2 & 4(1-\tau)\tau^3 & . & \dots \\ (1-\tau)^2 & 3(1-\tau)^2\tau & 6(1-\tau)^2\tau^2 & . & . & \dots \\ (1-\tau)^3 & 4(1-\tau)^3\tau & . & . & . & \dots \\ (1-\tau)^4 & . & . & . & . & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix}$$

generated by a fixed real number $\tau \in (0,1)$. For simplicity, we restrict ourselves to the cases $p = q = 1$ and $p = q = \infty$. Since the increasing diagonals of this matrix contain the binomial expansion of $((1-\tau)+\tau)^n = 1$, the sum of each column is $1/\tau$, and the sum of each row is $1/(1-\tau)$. So from our calculation above we see that

$$||A||_{\ell^1\to\ell^1} = [\alpha]^*_{1,1} = \sup_j \sum_i |\alpha_{ij}| = \frac{1}{1-\tau},$$

$$||A||_{\ell^\infty\to\ell^\infty} = [\alpha]_{\infty,\infty} = \sup_i \sum_j |\alpha_{ij}| = \frac{1}{\tau}.$$

Moreover, by Theorem 3.1 we get for the γ-norm of A in ℓ^1 the estimate

$$[A]_\gamma \le \gamma_{1,1}(\alpha) = \inf_{k,m} \sup_{j\ge m} \sum_{i=k}^{\infty} \alpha_{ij} = \inf_{k,m} \sup_{j\ge m} \frac{\tau^{j-1}}{(j-1)!} \sum_{i=k}^{\infty} \frac{(i+j-2)!}{(i-1)!}(1-\tau)^{i-1},$$

and in ℓ^∞ the estimate

$$[A]_\gamma \le \gamma_{\infty,\infty}(\alpha) = \inf_{k,m} \sup_{i\ge k} \sum_{j=m}^{\infty} \alpha_{ij} = \inf_{k,m} \sup_{i\ge k} \frac{(1-\tau)^{i-1}}{(i-1)!} \sum_{j=m}^{\infty} \frac{(i+j-2)!}{(j-1)!}\tau^{j-1}.$$

The remaining cases for p and q may be treated taking into account the monotonicity behaviour of the expression

$$\alpha_{i+1j+1} = \begin{pmatrix} i+j \\ j \end{pmatrix} (1-\tau)^i\tau^j = \frac{(i+j)!}{i!j!}(1-\tau)^i\tau^j$$

with respect to i and j, and using the corresponding mixed norms (3.8). $\square$

Similarly, one may show that a matrix operator A is bounded from c_0 into itself if and only if $||A|| = [\alpha]_{\infty,\infty} < \infty$ and

$$\lim_{i\to\infty} \alpha_{ij} = 0 \qquad (j = 1, 2, 3, \ldots).$$

Moreover, building on Theorem 2.3 one may give upper estimates for the γ-norm $[A]_\gamma$ of A, and hence sufficient compactness criteria, in terms of asymptotic properties of the matrix elements α_{ij}.

To conclude this section, we briefly compare the ϕ-norm of a linear operator $L \in \mathcal{L}(X, Y)$ and its adjoint operator $L^* \in \mathcal{L}(Y^*, X^*)$ defined, as usual, by the duality $\langle Lx, y^*\rangle = \langle x, L^*y^*\rangle$ for $x \in X$ and $y^* \in Y^*$.

Theorem 3.2. *For $L \in \mathcal{L}(X, X)$, the estimates*

$$\frac{1}{2}[L]_\phi \le [L^*]_\phi \le 2[L]_\phi \qquad (\phi \in \{\alpha, \beta, \gamma\}) \tag{3.10}$$

and

$$[L^*]_\alpha \le [L]_\gamma, \quad [L]_\alpha \le [L^*]_\gamma$$

hold in every Banach space X. If X is a Hilbert space, one has

$$[L^*]_\phi = [L]_\phi, \quad [L]_\phi = [L^*L]_\phi^{1/2} \qquad (\phi \in \{\alpha, \beta, \gamma\}), \tag{3.11}$$

and so $[L]_\alpha \le [L]_\gamma$. Finally, if X is a Hilbert space and L is normal, one even has

$$[L]_\phi = \lim_{n\to\infty} [L^n]_\phi^{1/n} \qquad (\phi \in \{\alpha, \beta, \gamma\}).$$

We do not present the proof of this theorem (see [31, Chapter 1]) which in part relies on the fact that the ϕ-norm of a linear operator may be simply calculated by $[L]_\phi = \phi(L(S(X)))$. Instead, we use the operator from Example 3.1 to show that the numbers $[L]_\phi$ and $[L^*]_\phi$, in contrast to the norms $||L||$ and $||L^*||$, in general do *not* coincide.

Example 3.5. Let $L : \ell^2 \times c \to \ell^2 \times c$ be defined as in Example 3.1. Then the adjoint operator $L^* : \ell^1 \times \ell^2 \to \ell^1 \times \ell^2$ is given by $L^*(x, y) = (0, Ux)$, where $U \in \mathcal{L}(\ell^2, \ell^2)$ is the left-shift operator $U(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_2, \xi_3, \xi_4, \ldots)$. It is readily seen that

$$[L^*]_\gamma = \gamma(B(\ell^1)) = 1 > \frac{1}{\sqrt{2}} = [L]_\gamma,$$

by (3.3), and so (3.11) may fail in Banach spaces which are not Hilbert. □

Considering the operator L from Example 3.1 in $X = \ell^p \times c$, for p arbitrarily large, and its adjoint L^* in $X^* = \ell^1 \times \ell^{p/(p-1)}$, one may show by a similar reasoning that $[L^*]_\gamma \geq 2^{1/p}[L]_\gamma$, and so the second estimate in (3.10) is sharp. Likewise, replacing L by L^* we may conclude that $[L]_\gamma \geq [L^{**}]_\gamma \geq 2^{1/p}[L^*]_\gamma$, and so the first estimate in (3.10) is sharp as well.

We remark that, apart from the ϕ-norm (1.15) and the essential norm (3.1), there are other possibilities to measure the "noncompactness" of a bounded linear operator. For instance, in Section 2.4 of [3] the authors consider the so-called λ-*norm*

$$[L]_\lambda := \inf_{\operatorname{codim} U < \infty} ||L|_U||, \tag{3.12}$$

where the infimum runs over all subspaces U of X of finite codimension. However, it is also shown in [3, Theorem 2.5.2] that

$$[L]_\lambda = [L^*]_\gamma,$$

so the definition (3.12) does not give any new information on L. From Theorem 3.2 it follows, in particular, that $[L]_\lambda = [L^*]_\lambda$ in any Hilbert space X.

4. Applications to imbeddings and integral operators

In this section we show that even for linear operators one may get interesting information based on the ϕ-norm (1.15) which characterizes the "noncompactness" of these operators. We start with imbeddings between Sobolev spaces.

Given some bounded domain $\Omega \subset \mathbb{R}^N$, by $W_2^k = W_2^k(\Omega)$ we denote the Sobolev space of all measurable functions $u : \Omega \to \mathbb{R}$ whose (distributional) derivatives $D^\alpha u$ belong to $L^2 = L^2(\Omega)$ for $0 \leq |\alpha| \leq k$. For $0 \leq m \leq k$, the *imbedding* $J_\Omega : W_2^k(\Omega) \hookrightarrow W_2^m(\Omega)$ is a bounded linear operator with norm

$$||J_\Omega|| = \sup \left\{ \sum_{|\alpha| \leq m} \left(\int_\Omega |D^\alpha u(x)|^2 \, dx \right)^{1/2} : \, ||u||_{W_2^k(\Omega)} \leq 1 \right\}. \tag{4.1}$$

Now, if the boundary $\partial\Omega$ of Ω is sufficiently smooth, the imbedding J_Ω is even *compact* in case $m < k$. Conversely, there are examples of "irregular" domains Ω for which J_Ω is not compact, hence $\alpha(J_\Omega) > 0$. Of course, since the imbedding cannot have norm greater than 1, one has the simple upper estimate $\alpha(J_\Omega) \leq 1$. So the question arises how to get more precise estimates for $\alpha(J_\Omega)$ building on the "geometry" of $\partial\Omega$.

This problem was completely solved by Amick [6] in the following way. Given $\varepsilon > 0$, denote by Ω_ε the "boundary layer" consisting of all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \varepsilon$, and define $\Gamma_\Omega(\varepsilon) > 0$ by

$$\Gamma_\Omega(\varepsilon) := \sup \left\{ \sum_{|\alpha| \leq m} \left(\int_{\Omega_\varepsilon} |D^\alpha u(x)|^2)\, dx \right)^{1/2} : ||u||_{W_2^k(\Omega)} \leq 1 \right\}. \tag{4.2}$$

So the only difference to (4.1) is that the integral is taken only over the boundary layer Ω_ε instead of Ω, and so $\Gamma_\Omega(\varepsilon) \leq ||J_\Omega||$ for sufficiently small $\varepsilon > 0$. Clearly, the function (4.2) is monotonically increasing in ε, and so the limit

$$\Gamma_\Omega := \lim_{\varepsilon \downarrow 0} \Gamma_\Omega(\varepsilon) \tag{4.3}$$

exists. It turns out that the number (4.3) is zero if and only if the imbedding J_Ω is compact. More generally, in [6] the following remarkable formula has been proved.

Theorem 4.1. *With* Γ_Ω *as in* (4.3), *the equality*

$$[J_\Omega]_\alpha = \Gamma_\Omega$$

holds, where α *denotes the Kuratowski measure of noncompactness* (1.8).

In the paper [6] one may also find an example of an extremely "badly behaved" domain Ω for which the worst case $\Gamma_\Omega = 1$ is true. There is also an interesting connection with the well-known *Poincaré inequality*: one has $\Gamma_\Omega < 1$ (i.e., the imbedding J_Ω is α-condensing) if and only if the Poincaré inequality holds over Ω. By the way, the fact that only the boundary layer Ω_ε has an influence on the behaviour of $\Gamma_\Omega(\varepsilon)$ from (4.2) illustrates what we meant with our "Golden Rule" in the Introduction.

We pass now to the second group of applications, viz. *linear integral operators*. It is well-known that Volterra-type integral operators of the type

$$Ku(s) = \int_0^s k(s,t)u(t)\, dt \qquad (0 \leq s \leq 1) \tag{4.4}$$

are often compact, provided that the generating kernel function k is sufficiently "well-behaved". However, if k exhibits some singular behaviour near the end-points 0 or 1 of the underlying interval, the operator K may be noncompact, and so the problem arises to estimate or even calculate its measure of noncompactness. As a sample result, we consider an operator with degenerate kernel function.

Example 4.1. Consider the Volterra operator (4.4) with kernel function $k(s,t) = \sigma(s)\tau(t)$, i.e.

$$Ku(s) = \sigma(s) \int_0^s \tau(t)u(t)\,dt \qquad (0 \le s \le 1). \tag{4.5}$$

For $1 < p < \infty$ and $0 \le a \le b \le 1$, put

$$\omega_{a,b}(s) := \left(\int_s^b |\sigma(t)|^p\,dt \right)^{1/p} \left(\int_a^s |\tau(t)|^{p'}\,dt \right)^{1/p'},$$

where $p' = p/(p-1)$, and

$$\omega := \sup_{0<s<1} \omega_{0,1}(s).$$

In [48] it was shown that the norm of the operator (4.5) in the space L^p satisfies the two-sided estimate

$$\omega \le ||K|| \le p^{1/p}(p')^{1/p'}\omega,$$

and these estimates are sharp. Moreover, let

$$\omega_0 := \lim_{\varepsilon \downarrow 0} \left[\sup_{0<s<\varepsilon} \omega_{0,\varepsilon}(s) + \sup_{1-\varepsilon<s<1} \omega_{1-\varepsilon,1}(s) \right]. \tag{4.6}$$

In [79] (for $p = 2$) and [48] (for general p) it was shown that the γ-norm of the operator (4.5) in L^p can be estimated by

$$2^{-(1+1/p)}\omega_0 \le [K]_\gamma \le p^{1/p}(p')^{1/p'}\omega_0; \tag{4.7}$$

in particular, K is compact in L^p if and only if $\omega_0 = 0$. In other words, only the "boundary behaviour" of the function $\omega_{0,1}$ decides on the compactness or noncompactness of the operator (4.5), again in accordance with our "Golden Rule" stated in the Introduction. □

In the next example we consider another noncompact integral operator which is of fundamental importance in Fourier analysis and distribution theory.

Example 4.2. Consider the (truncated) singular *Hilbert transform*

$$Hu(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{s-t}\,dt \qquad (-1 \leq s \leq 1). \tag{4.8}$$

The operator H in (4.8) is bounded in any Lebesgue space $L^p[-1,1]$ for $1 < p < \infty$, but neither in L^1 nor in L^∞. To the best of our knowledge, the most precise upper and lower estimates for its norm $||H||_p$ in the space L^p which are presently known are [60]

$$\begin{cases} -\cot\dfrac{\pi}{p} \leq ||H||_p \leq \tan\dfrac{\pi}{2p} & \text{if } 1 < p \leq \dfrac{4}{3}, \\ 1 \leq ||H||_p \leq \tan\dfrac{\pi}{2p} & \text{if } \dfrac{4}{3} \leq p \leq 2, \\ 1 \leq ||H||_p \leq \cot\dfrac{\pi}{2p} & \text{if } 2 \leq p \leq 4, \\ \cot\dfrac{\pi}{p} \leq ||H||_p \leq \cot\dfrac{\pi}{2p} & \text{if } 4 \leq p < \infty. \end{cases} \tag{4.9}$$

Observe, in particular, that

$$||H||_2 = \min\{||H||_p : 1 < p < \infty\} = 1$$

and

$$\lim_{p\to 1} ||H||_p = \lim_{p\to\infty} ||H||_p = \infty. \tag{4.10}$$

On the other hand,

$$[H]_\alpha = [H]_\gamma \equiv 1, \tag{4.11}$$

no matter what p is. This discrepancy between the α-norm (4.11) of H and the fact that its norm "blows up" if we want to consider this operator either in a particularly "large" Lebesgue space (i.e., for p close to 1), or in a particularly "narrow" Lebesgue space (i.e., for p very large), may become important in applications (see Example 9.3 below).

5. Fredholm operators and essential spectra

Recall that an operator $L \in \mathcal{L}(X,Y)$ is called *right semi-Fredholm* ($L \in \Phi_+(X,Y)$) if its nullspace $N(L) = \{x \in X : Lx = 0\}$ is finite dimensional and its range $R(L) = \{Lx : x \in X\}$ is closed. Similarly, L is called *left semi-Fredholm* ($L \in \Phi_-(X,Y)$) if its range $R(L)$ is closed and has finite codimension. Every operator $L \in \Phi(X,Y) := \Phi_+(X,y) \cap \Phi_-(X,Y)$ is called *Fredholm operator*, the number $\operatorname{ind} L := \dim N(L) - \operatorname{codim} R(L) \in \mathbb{Z}$ its *index*.

The following theorem establishes a connection between Fredholmness and the lower ϕ-norm (1.23). We restrict ourselves to the Kuratowski measure of noncompactness (1.8). As before, L^* denotes the adjoint operator $L^* \in \mathcal{L}(Y^*, X^*)$ of $L \in \mathcal{L}(X,Y)$.

Theorem 5.1. *One has $L \in \Phi_+(X,Y)$ if and only if $[L]_\alpha^- > 0$, and $L \in \Phi_-(X,Y)$ if and only if $[L^*]_\alpha^- > 0$. Consequently, L is a Fredholm operator if and only if both $[L]_\alpha^- > 0$ and $[L^*]_\alpha^- > 0$.*

The proof of Theorem 5.1 uses the fact that the finite dimensional subspace $N(L)$ has a closed complemented subspace X_0, and that the canonical isomorphism $L|_{X_0} : X_0 \to R(L)$ has a positive lower α-norm $[L|_{X_0}]_\alpha^-$.

Fredholm operators may also be characterized by another construction which may be found in [73] or [3]. Given a Banach space X with norm $||\cdot||$, let us denote by BX the Banach space of all bounded sequences $(x_n)_n$ in X, equipped with the norm

$$||(x_n)_n|| := \sup_n ||x_n||.$$

Clearly, the subset KX of all sequences $(x_n)_n \in BX$ whose range $\{x_1, x_2, x_3, \ldots\}$ is precompact in X is a closed subspace of BX, and so we may consider the quotient space $X^\# := BX/KX$ with the natural norm

$$||(x_n)_n + KX||^\# := \inf\{||(\tilde{x}_n)_n|| : (x_n - \tilde{x}_n)_n \in KX\} = \operatorname{dist}((x_n)_n, KX).$$

It is not hard to see that then $||(x_n)_n + KX||^\# = \gamma(\{x_1, x_2, x_3, \ldots\})$, i.e., the quotient norm of a sequence $(x_n)_n \in BX$ in $X^\#$ is nothing else but the measure of noncompactness of its range in X. More generally, every measure of noncompactness ϕ on X induces in a natural way a norm $||\cdot||_\phi^\#$ on $X^\#$ by means of the formula

$$||(x_n)_n + KX||_\phi^\# := \phi(\{x_1, x_2, x_3, \ldots\}). \tag{5.1}$$

It turns out that the functorial map $X \mapsto X^\#$ may be used to give an easy characterization of Fredholm operators. Given two Banach spaces X and Y with corresponding quotient spaces $X^\#$ and $Y^\#$, respectively, we associate to each operator $L \in \mathcal{L}(X, Y)$ an operator $L^\# : X^\# \to Y^\#$ by putting

$$L^\#((x_n)_n + KX) := (Lx_n)_n + KY.$$

Since L maps BX into BY and KX into KY, the operator $L^\#$ is well-defined. It is not hard to see that the map $L \mapsto L^\#$ is a covariant additive functor, i.e., $(L_1 + L_2)^\# = L_1^\# + L_2^\#$ and $(L_2L_1)^\# = L_2^\# L_1^\#$. The following theorem may be regarded as a refomulation of Theorem 5.1 in terms of the map $L \mapsto L^\#$, by (5.1); the proof of this theorem may be found in Section 2.3 of [3].

Theorem 5.2. *One has $L \in \Phi_+(X, Y)$ if and only if $L^\# : X^\# \to Y^\#$ is injective, and $L \in \Phi_-(X, Y)$ if and only if $(L^*)^\# : (Y^*)^\# \to (X^*)^\#$ is injective. Consequently, L is a Fredholm operator if and only if $L^\# : X^\# \to Y^\#$ is bijective.*

Now we recall a well-known connection between Fredholm operators and essential spectra. Let X be a Banach space and $L \in \mathcal{L}(X, X)$. The *essential spectrum* $\sigma_e(L)$ is the spectrum of L induced in the Calkin algebra $\mathcal{L}(X, X)/\mathcal{K}(X, X)$ over X, i.e.,

$$\sigma_e(L) = \bigcap_{K \text{compact}} \sigma(L - K). \tag{5.2}$$

Equivalently, the scalars $\lambda \in \mathbb{K} \setminus \sigma_e(L)$ are characterized by the property that $\lambda I - L$ is a Fredholm operator of index zero; therefore, the set (5.2) is sometimes called the *Fredholm spectrum* of L. Likewise, it is reasonable to call the number

$$r_e(L) := \sup\{|\lambda| : \lambda \in \sigma_e(L)\}$$

the *essential spectral radius* or *Fredholm spectral radius* of L. Interestingly, this radius satisfies a Gel'fand-type formula, as the usual spectral radius, with the norm $||L||$ replaced by the α-norm (1.15). In fact, if X is a complex Banach space and $L \in \mathcal{L}(X,X)$, then the equalities

$$r_e(L) = \inf_n [L^n]_\alpha^{1/n} = \lim_{n\to\infty} [L^n]_\alpha^{1/n} \tag{5.3}$$

is true; for a short proof see [27]. If X is a real Banach space one has to pass to the so-called *complexification* $X_{\mathbb{C}}$ of X which consists, by definition, of all ordered pairs $(x,y) \in X \times X$, written usually $x+iy$. The set $X_{\mathbb{C}}$ is equipped with the algebraic vector space operations $(x+iy)+(u+iv) := (x+u)+i(y+v)$ and $(\mu+i\nu)(x+iy) := (\mu x - \nu y) + i(\nu x + \mu y)$ $(\mu, \nu \in \mathbb{R})$. A natural norm on $X_{\mathbb{C}}$ is the so-called *projective tensor norm* defined by

$$||x+iy|| := \inf \sum_{k=1}^{n} |\lambda_k|\,||z_k||,$$

where the infimum is taken over all possible representations of the form $x+iy = \lambda_1 z_1 + \ldots + \lambda_n z_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in X$. Equivalently [54], one may define a norm on $X_{\mathbb{C}}$ by

$$||x+iy|| := \max_{0\le t\le 2\pi} ||(\cos t)x + (\sin t)y||.$$

Given a real Banach space X and an operator $L \in \mathcal{L}(X,X)$, one may extend L to an operator $L_{\mathbb{C}} \in \mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}})$ putting $L(x+iy) := Lx + iLy$. It is readily seen that then

$$||L_{\mathbb{C}}|| = ||L||, \qquad [L_{\mathbb{C}}]_\alpha = [L]_\alpha, \qquad [L_{\mathbb{C}}]_\alpha^- = [L]_\alpha^-,$$

and so L and $L_{\mathbb{C}}$ have the same essential spectral radius, by (5.3).

It is well-known that, although the spectral radius $r(L)$ of a bounded linear operator L may be strictly less than its norm, for any $\varepsilon > 0$ one may always find a norm $||\cdot||^*$ which is equivalent to the original norm $||\cdot||$ on X and such that $r(L) \le ||L||^* \le r(L) + \varepsilon$, where $||L||^*$ denotes the operator norm of L in $(X, ||\cdot||^*)$. Roughly speaking, this means that one may interpret the spectral radius of an operator $L \in \mathcal{L}(X,X)$ in the form

$$r(L) = \inf\{||L||^* : \, ||\cdot||^* \sim ||\cdot||\},$$

where the infimum is taken over all norms which are equivalent to the original norm on X. There is a parallel result for the essential spectral radius (5.3) called *Leggett's theorem* [56] which states that for any $\varepsilon > 0$ one may always find a norm $||\cdot||^*$ which is equivalent to the projective tensor norm $||\cdot||$ on

$X_{\mathbb{C}}$ and such that $r_e(L) \leq [L_{\mathbb{C}}]^*_\alpha \leq r_e(L) + \varepsilon$, where $[L_{\mathbb{C}}]^*_\alpha$ denotes the α-norm (1.15) of $L_{\mathbb{C}}$ in $(X_{\mathbb{C}}, ||\cdot||^*)$, and $r_e(L)$ is given by (5.3).

There is another characteristic for bounded linear operators which is of some interest in spectral theory. Given $L \in \mathcal{L}(X, X)$, the number

$$\wr L \wr := \begin{cases} ||L^{-1}||^{-1} & \text{if } L \text{ is bijective,} \\ 0 & \text{otherwise} \end{cases}$$

is called *inner norm* of L (see [3]). This number is closely related to the *inner spectral radius*

$$r_i(L) := \inf\{|\lambda| : \lambda \in \sigma(L)\},$$

inasmuch as the formula

$$r_i(L) = \lim_{n\to\infty} \wr L^n \wr^{1/n} \tag{5.4}$$

holds true, which is of course the analogue to the classical Gel'fand formula for the spectral radius. In fact, for any bijection $L \in \mathcal{L}(X, X)$ we have

$$r_i(L) = \inf\{|\lambda| : \lambda \in \sigma(L)\}$$

$$= \frac{1}{\sup\{|\lambda|^{-1} : \lambda \in \sigma(L)\}} = \frac{1}{\sup\{|\mu| : \mu \in \sigma(L^{-1})\}}$$

$$= \frac{1}{r(L^{-1})} = \frac{1}{\lim\limits_{n\to\infty} ||L^{-n}||^{1/n}} = \lim_{n\to\infty} ||L^{-n}||^{-1/n} = \lim_{n\to\infty} \wr L^n \wr^{1/n} .$$

If L is not bijective, then L^n is not bijective either for any n, and both sides in (5.4) are zero.

We illustrate our discussion by means of a classical integral equation which involves a nice "interaction" between complex analysis, Fredholm theory, and essential spectra.

Example 5.1. We denote by $L^1(\mathbb{R}, \mathbb{C})$ the usual Banach space of complex integrable functions on the real line. Given $k, v \in L^1(\mathbb{R}, \mathbb{C})$, consider the *Wiener-Hopf equation*

$$\lambda u(s) - \int_{-\infty}^{+\infty} k(s-t)u(t)\,dt = v(s) \qquad (-\infty < s < \infty), \tag{5.5}$$

where $\lambda \in \mathbb{C}$ is fixed. We may rewrite (5.5) as operator equation $\lambda u - Lu = v$, where L is the linear operator defined by the integral term in (5.5). Applying the Fourier transforms to both sides of (5.5) yields

$$\lambda \hat{u}(\sigma) - \hat{k}(\sigma)\hat{u}(\sigma) = \hat{v}(\sigma) \qquad (-\infty < \sigma < \infty), \tag{5.6}$$

since the Fourier transform maps convolutions into pointwise products. Now, from the continuity of $\hat{k}$ and the fact that

$$\hat{k}(\pm\infty) := \lim_{\sigma \to \pm\infty} \hat{k}(\sigma) = 0$$

it follows that $\Gamma_\lambda := \{\lambda - \hat{k}(\sigma) : -\infty \leq \sigma \leq +\infty\}$ is a closed contour in the complex plane. Clearly, in case $\hat{k}(\sigma) \neq \lambda$ equation (5.6) may be solved to get $\hat{u}$ and, subsequently, u, provided we know the antitransform of the function $\sigma \mapsto \hat{v}(\sigma)/(\lambda - \hat{k}(\sigma))$. In fact, it may be shown that $\lambda I - L$ is a Fredholm operator on $L^1(\mathbb{R}, \mathbb{C})$ if and only if $\hat{k}(\sigma) \neq \lambda$ for all $\sigma \in [-\infty, +\infty]$. For the index of this operator we get then the nice formula

$$\operatorname{ind}(\lambda I - L) = -w(\Gamma_\lambda, 0)$$

involving the *winding number* $w(\Gamma_\lambda, 0)$ of Γ_λ around the origin. Moreover, one may show that

$$\sigma(L) = \sigma_e(L) = \{\hat{k}(\sigma) : -\infty \leq \sigma \leq +\infty\}$$

and so

$$r_e(L) = \max\{|\hat{k}(\sigma)| : -\infty \leq \sigma \leq +\infty\}.$$

For the special example $k(s) = e^{-|s|}$, say, we have $\hat{k}(\sigma) = 2/(\sigma^2 + 1)$ and thus $\sigma(L) = \sigma_e(L) = [0, 2]$, since Γ_λ does not wind around the origin for $\lambda \in \mathbb{C} \setminus [0, 2]$.

From Theorem 5.1 it follows that $[\lambda I - L]_\alpha^- > 0$ for $\lambda \notin [0, 2]$, while from Theorem 5.2 it follows that the operator $(\lambda I - L)^\# = \lambda I - L^\#$ is a bijection for $\lambda \notin [0, 2]$. Unfortunately, these two conditions are hard to verify directly even in this simple example. □

Our discussion suggests that there might be a relation between the essential spectrum of an invertible operator, on the one hand, and the ϕ-norm of its inverse, on the other. This is in fact true; we illustrate this in the simplest case of a Hilbert space X. Let $L \in \mathcal{L}(X, X)$ be a self-adjoint (i.e., $\langle Lu, u \rangle =$

$\langle u, Lu\rangle$), densely defined (i.e., $\overline{D(L)} = X$) and positive (i.e., $\langle Lu, u\rangle \geq 0$) operator, and suppose that $0 \notin \sigma(L)$, so the inverse operator $R = L^{-1}$ exists on the range $R(L)$ of L. Under these hypotheses, the essential spectrum of L has the form $\sigma_e(L) = [\sigma_0, \infty)$, with a suitable $\sigma_0 > 0$. Now, since $0 \notin \sigma(L)$, we may consider the inverse operator $R = L^{-1}$ in X. Let $\{E_\lambda\}_\lambda$ be the spectral decomposition of L, and fix $0 < \varepsilon < \sigma_0$. We have then

$$R = \int_{-\infty}^{+\infty} \lambda^{-1}\, dE_\lambda = \int_{|\lambda|>\varepsilon} \lambda^{-1}\, dE_\lambda + \int_{|\lambda|\leq\varepsilon} \lambda^{-1}\, dE_\lambda =: R_1 + R_2. \tag{5.7}$$

But $||R_1|| \leq 1/\varepsilon$, by construction, and R_2 is finite dimensional, hence compact. This implies that

$$[R]_\gamma \leq [R_1]_\gamma + [R_2]_\gamma \leq ||R_1|| + 0 \leq \frac{1}{\varepsilon}.$$

Since $\varepsilon < \sigma_0$ was arbitrary, we conclude that

$$[L^{-1}]_\gamma \leq \frac{1}{\inf\ \sigma_e(L)}. \tag{5.8}$$

This result is very suggestive: *the smaller the essential spectrum of* L*, the "more compact" the inverse operator* L^{-1}. This "quantifies" the classical result that, loosely speaking, the operators with purely discrete spectrum (i.e., $\sigma_e(L) = \emptyset$) are those which have compact resolvents. Some applications of this result to differential operators will be given in the next section.

6. Applications to differential operators

As we have seen, estimating or calculating the measure of noncompactness of the inverse of a differential operator amounts to estimating or calculating the infimum of the essential spectrum of this operator. We consider now some special examples where this may be made more precise.

Example 6.1. Consider the second order *Sturm - Liouville operator*

$$Lu(t) = \frac{d}{dt}p(t)\frac{d}{dt}u(t) + q(t)u(t) \qquad (0 \leq t < 1)$$

in the Hilbert space $L^2 = L^2[0,1]$. Here we assume that p is twice differentiable and singular near $t = 1$ in the sense that

$$\int_0^1 \frac{dt}{\sqrt{p(t)}} = \infty;$$

thus, the coefficient function p may not be removed by means of the classical Liouville transform. As a consequence, the essential spectrum of L has the form $\sigma_e(L) = [\sigma_0, \infty)$, where

$$\sigma_0 = \lim_{t \to 1} \left\{ \frac{1}{4} p''(t) - \frac{1}{16} \frac{p'(t)^2}{p(t)} + q(t) \right\}$$

(see, e.g., [30, p. 1500]). Now, if $\sigma_0 > 0$ and $0 \notin \sigma(L)$, the operator $R = L^{-1}$ exists and satisfies $[R]_\alpha \leq 1/\sigma_0$, by (5.8). □

Example 6.2. Consider the symmetric differential operator of order $2m$

$$Lu(t) = \sum_{k=0}^{m} (-1)^k a_k \frac{d^{2k}}{dt^{2k}} u(t) = a_0 u(t) - a_1 u''(t) + - \ldots + (-1)^m a_m u^{(2m)}(t)$$

$(-\infty < t < \infty)$, with constant real coefficients $a_0, a_1, \ldots, a_m$. Here the essential spectrum of L in $L^2 = L^2(\mathbb{R})$ has the form $\sigma_e(L) = [\sigma_0, \infty)$, where

$$\sigma_0 = \inf_{\tau > 0} \sum_{k=0}^{m} a_k \tau^{2k},$$

(see, e.g., [61]). In particular, if all coefficients $a_0, a_1, \ldots, a_m$ are nonnegative, we simply have $\sigma_e(L) = [a_0, \infty)$. In other words, all terms containing derivatives of u do not affect the compactness behaviour of L^{-1} (if it exists). □

Example 6.3. The preceding scalar examples also extend to some ordinary differential operators with variable coefficients, and even to elliptic partial differential operators. As a model example, consider the *Schrödinger type operator*

$$Lu(x) = -\Delta u(x) + q(x) u(x) \qquad (x \in \mathbb{R}^N), \tag{6.1}$$

where the potential q is assumed to be bounded away both from 0 and ∞, i.e., $0 < c \leq q(x) \leq C < \infty$. By adoperating the classical technique of Weyl sequences, one may calculate the number $\sigma_0 = \inf \sigma_e(L)$ also in this case, and then apply the estimate (5.8) for the γ-norm of L^{-1} in case $0 \notin \sigma(L)$.

In a physical interpretation, the eigenvalues of L (= "ground states" or "bound states") form the discrete part, and the elements in $\sigma_e(L)$ (= "scattering states") the essential part of the whole spectrum. So the resolvent operator $R = L^{-1}$ is "more" compact (i.e., $[R]_\gamma$ is small) if there are only "high energy scattering states" (i.e., $\inf \sigma_e(L)$ is large). If there are no scattering states at all (i.e., $\sigma_e(L) = \emptyset$), then the resolvent of L is compact, and one only has a sequence of discrete energy levels of finite multiplicity. □

Many other more sophisticated examples of type (6.1) in which it is possible to give lower estimates for the essential spectrum of L, may be found in Chapter 10 of the book [31].

7. Nonlinear ϕ-Lipschitz operators

In this section we start with the "nonlinear part" of this survey. Many nonlinear problems in function spaces may be written as operator equation

$$Lu = F(u), \tag{7.1}$$

where L is some linear operator (e.g., a differential operator), and F denotes the *Nemytskij operator* (or *superposition operator*)

$$F(u)(x) := f(x, u(x)) \tag{7.2}$$

generated by some function of two variables $f : \Omega \times \mathbb{R} \to \mathbb{R}$, with Ω being some suitable domain in $\mathbb{R}^N$. Important examples have been given in Section 6, where L is an (ordinary or partial) differential operator on some function space, and so (7.1) may be regarded as an (ordinary or partial) differential equation. We will discuss such problems in Section 9 below.

Now, suppose that the operators L and F are continuous between two Banach spaces X and Y, and the linear part L is *invertible* on its range. Then equation (7.1) may be equivalently transformed into the equation

$$u = L^{-1}F(u), \tag{7.3}$$

which is a *fixed point problem* for the operator $A = L^{-1}F$ in the space X. Later (see Theorems 8.1 and 8.2 below) we will see that, loosely speaking, this fixed point problem has a solution whenever the ϕ-norms $[L^{-1}]_\phi$ and $[F]_\phi$ are not "too big", where ϕ is a suitable measure of noncompactness on X. So we are led to the problem of estimating (or even calculating) the ϕ-norm $[L^{-1}]_\phi$ in terms of the coefficients of the differential operator L, as well as the ϕ-norm $[F]_\phi$ in terms of the generating function f. The first problem has been solved in part in Section 6 for some differential operators. In this section we will focus on the problem of calculating $[F]_\gamma$ for the Nemytskij operator (7.2).

To this end, we consider, apart from the Lipschitz constant

$$Lip(F) = \inf \{k > 0 : \, ||F(u) - F(v)|| \leq k||u - v|| \; (u, v \in X)\}$$

of the operator $F : X \to X$, the Lipschitz constant

$$Lip(f(x, \cdot)) := \inf \{k > 0 : \, |f(x, u) - f(x, v)| \leq k|u - v| \; (x \in \Omega; \, u, v \in \mathbb{R})\}$$

of the scalar function $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$. It is not hard to see that, in case of the function spaces $X = C[0, 1]$ or $X = L^p[0, 1]$, the estimate $Lip(F) \leq Lip(f(x, \cdot))$ is true. However, the following result [7,8] is not completely trivial:

Theorem 7.1. *In the function spaces $X = C[0, 1]$ and $X = L^p[0, 1]$ for $1 \leq p \leq \infty$, the equality*

$$[F]_\gamma = Lip(F) = Lip(f(x, \cdot)) \tag{7.4}$$

is true.

Of course, one has to prove here only the estimate $[F]_\gamma \geq Lip(f(x, \cdot))$, and this may be done by constructing special sets $M \subset X$ for which $\gamma(M) = 1$ and $\gamma(F(M)) = Lip(f(x, \cdot))$ (see [7]).

One might ask whether or not the equality (7.4) holds in *all* function spaces. This is far from being true. In the following two theorems we give results which are in sharp contrast to Theorem 7.1; for the proof see Chapters 7 and 8 of the book [11]. Here we consider the space $C^1 = C^1[0, 1]$ of all continuously differentiable real functions u, equipped with the norm

$$||u|| := \max_{0 \leq t \leq 1} |u(t)| + \max_{0 \leq t \leq 1} |u'(t)|,$$

and the Hölder space $C^\alpha = C^\alpha[0, 1]$ $(0 < \alpha < 1)$ equipped with the norm (2.11).

Theorem 7.2. *In the function space $X = C^1$, one has $[F]_\gamma < \infty$ whenever f is continuously differentiable with bounded derivatives. More precisely, in this case the estimate*

$$[F]_\gamma \leq \sup_{-\infty < u < \infty} \max_{0 \leq t \leq 1} \left| \frac{\partial f(t, u)}{\partial u} \right|$$

holds. On the other hand, one has $Lip(F) < \infty$ if and only if the function f has the form

$$f(t,u) = a(t) + b(t)u \tag{7.5}$$

with two functions $a, b \in C^1$.

Theorem 7.3. *In the function space $X = C^\alpha$, one has $[F]_\gamma < \infty$ whenever the function f satisfies a Lipschitz condition in both variables. On the other hand, one has $Lip(F) < \infty$ if and only if the function f has the form* (7.5) *with two functions $a, b \in C^\alpha$.*

Theorems 7.2 and 7.3 exhibit a striking "degeneracy" phenomenon of the Nemytskij operator in the spaces C^1 and C^α (which, by the way, holds in many other function spaces): *the Nemytskij operator* (7.2) *satisfies a Lipschitz condition in the norm of these spaces only if the function $f(t, \cdot)$ is affine.* So, if one wants to apply the Banach-Caccioppoli contraction mapping theorem in these spaces to some problem involving Nemytskij operators, one can do so only if this problem is actually *linear*! On the other hand, Theorems 7.2 and 7.3 also show that, in contrast, a more general fixed point theorem which we will discuss in the next section, applies to a fairly large class of nonlinear problems in these spaces.

8. The Darbo fixed point theorem

Probably the most important fixed point theorems in nonlinear analysis are the *Schauder fixed point theorem* and the *Banach-Caccioppoli fixed point theorem.* The first one states that a compact operator A which maps a nonempty, closed, convex, bounded subset M of a Banach space into itself, has a fixed point in M. The second states (in a special form) that the same is true if the compactness requirement on A is replaced by the condition $Lip(A) < 1$, see (1.20), which means that A is a *contraction.* In this case, one even has uniqueness of the fixed point in M, and the fixed point may be constructed explicitly by means of the usual successive approximations.

These two fixed point theorems are apparently completely independent of each other, since the crucial condition on A is quite different: in Banach's theorem the operator A has to decrease distances, but may map balls into (smaller) "massive" balls, while in Schauder's theorem the operator A may enlarge distances, but must map balls into strongly "rarefied" subsets, i.e. without interior points. It turns out, however, that both theorems may be regarded as special cases of another fixed point theorem which "bridges" their apparently quite different character: this is Darbo's celebrated fixed point theorem which he formulated in 1955 for the case of the Kuratowski measure of noncompactness [25]:

Theorem 8.1. *Let X be a Banach space, $M \subset X$ a nonempty, closed, convex, bounded subset, and $A : M \to M$ an α-condensing operator, i.e., $[A]_\alpha < 1$. Then A has a fixed point in M.*

Since Theorem 8.1 is at the very heart of this survey, we briefly sketch the idea of the proof. Define a sequence $(M_n)_n$ of subsets of M by putting

$$M_1 := \overline{co}\, A(M),\ M_2 := \overline{co}\, A(M_1), \ldots, M_{n+1} := \overline{co}\, A(M_n),$$

and let M_∞ denote the intersection of all sets M_n. It is easy to see that M_∞ is compact and $A(M_\infty) \subseteq M_\infty$. The only nontrivial part is to show that $M_\infty \neq \emptyset$; this follows from a Cantor type intersection property of the measure of noncompactness α which was proved in [55]. So Schauder's fixed point theorem implies that A has a fixed point in $M_\infty \subseteq M$ as claimed.

Let us make some remarks on Theorem 8.1. First of all, the difficulty in proving $M_\infty \neq \emptyset$ may be overcome by fixing in advance a point $x_0 \in M$ and defining the sequence $(M_n)_n$ alternatively by

$$M_1 := \overline{co}\,[A(M) \cup \{x_0\}],\ M_2 := \overline{co}\,[A(M_1) \cup \{x_0\}], \ldots,$$

$$M_{n+1} := \overline{co}\,[A(M_n) \cup \{x_0\}].$$

We also point out that, in case $M = B(X)$, one may assume without loss of generality that A vanishes on the boundary $S(X)$. In fact, from $A : B(X) \to B(X)$ one may pass to the operator $\tilde{A} : B(X) \to B(X)$ defined by

$$\tilde{A}(x) := \begin{cases} \frac{1}{2}A(2x) & \text{if} \quad ||x|| \leq \frac{1}{2}, \\ (1 - ||x||)A\left(\frac{x}{||x||}\right) & \text{if} \quad \frac{1}{2} < ||x|| \leq 1. \end{cases}$$

Clearly, this operator satisfies $[\tilde{A}]_\alpha = [A]_\alpha$ and, in addition, $\tilde{A}(x) \equiv 0$ on $S(X)$. It is easy to see that $\tilde{A}$ cannot have fixed points x^* of norm $||x^*|| > \frac{1}{2}$, since for any such x^* we would have

$$\frac{1}{2} < ||x^*|| = (1 - ||x^*||) \left|\left| A\left(\frac{x^*}{||x^*||}\right)\right|\right| \leq \frac{1}{2}\left|\left| A\left(\frac{x^*}{||x^*||}\right)\right|\right| \leq \frac{1}{2}.$$

So we see that the homeomorphism $x \mapsto 2x$ is a 1-1 correspondence between the fixed points of $\tilde{A}$ in $B_{1/2}(X)$ and the fixed points of A in $B(X)$. This fact will be used in Section 14 below in connection with nonlinear spectral theory.

A scrutiny of the proof of Theorem 8.1 shows that we did not use the special definition of the Kuratowski measure of noncompactness α, but only its regularity (1.7), its homogeneity (1.3), and its convex closure invariance (1.6). So Theorem 8.1 holds true for any measure of noncompactness ϕ satisfying these conditions, in particular, for the measures of noncompactness (1.9) and (1.10). Moreover, we even did not use the full regularity of the underlying measure of noncompactness ϕ, but merely the "only if" part of (1.7), i.e., the fact that $\phi(M) = 0$ implies the precompactness of M. An application of this type will be given in Example 9.5 below.

Since both contracting and compact operators are condensing, Theorem 8.1 unifies the Banach-Caccioppoli fixed point theorem and Schauder's fixed point principle. The common aspect of these two fixed point principles becomes transparent only after introducing the notions of measures of noncompactness and condensing operators. By the way, one may realize the methodological combination of Banach's and Schauder's theorems also in the proof of Theorem 8.1: the crucial role of compactness (borrowed from Schauder's theorem) comes with the definition of $\gamma(M)$, while the method of considering successive iterations (borrowed from Banach's theorem) comes with the definition of the sequence $(M_n)_n$.

Darbo's fixed point theorem contains two ingredients: first, one has to check the estimate $[A]_\alpha < 1$ (which is a topological condition), and then one has to find an invariant closed convex bounded set $M \subset X$ (which is a geometrical condition). Following an idea of Ioan A. Rus, one may also combine these two conditions, replacing (1.15) by the characteristic

$$[A]_\phi^{\circlearrowleft} := \inf\{k > 0 : \phi(A(N)) \leq k\phi(N) \text{ for bounded } N \subseteq M \text{ with } A(N) \subseteq N\}.$$

The proof of Theorem 8.1 shows that all subsets arising in the construction of M_∞ are invariant under A, and so one actually needs the condition $[A]_\alpha^\circlearrowleft < 1$ which is weaker, at least formally, than $[A]_\alpha < 1$. In other areas of nonlinear analysis (like nonlinear spectral theory, see below), however, one needs the stronger condition $[A]_\phi < 1$.

There is a certain generalization of Theorem 8.1 to a larger class of operators which was proved in case of the Hausdorff measure of noncompactness (1.10) in [69]. Let us call an operator $A : M \to Y$ *weakly ϕ-condensing* if

$$\phi(A(N)) < \phi(N) \qquad (N \subseteq M \text{ bounded}, \ \phi(N) > 0). \tag{8.1}$$

There are in fact operators which are weakly condensing, but not condensing (see Example 12.1 below). So the following fixed point theorem which is due to Sadovskij [69,72] is a proper extension of Theorem 8.1.

Theorem 8.2. *Let X be a Banach space, $M \subset X$ a nonempty, closed, convex, bounded subset, and $A : M \to M$ a weakly γ-condensing operator in the sense of* (8.1). *Then A has a fixed point in M.*

Interestingly, although the class of weakly condensing operators is strictly larger than that of condensing operators, one may prove Theorem 8.2 by means of Theorem 8.1 (see the last remark in Section 12).

We may summarize our discussion as follows: Schauder's theorem extends Brouwer's theorem (from $\mathbb{R}^N$ to infinite dimensional Banach spaces), Darbo's theorem extends Schauder's theorem (from compact to condensing operators), and Sadovskij's theorem extends Darbo's theorem (from condensing to weakly condensing operators). Nevertheless, since each of these theorems may be proved by means of the preceding one, they are actually *all equivalent*.

In the following example we discuss a special class of condensing operators which frequently arises in applications. In fact, a linear variant of this has already been considered in problem (5.7).

Example 8.1. Suppose that A is a continuous operator in a Banach space X which admits a representation as a sum $A = A_1 + A_2$, where A_1 is a contraction and A_2 is compact. If ϕ is a Lip-compatible measure of noncompactness on X, see (1.19), we get for every bounded subset $M \subset X$

$$\phi(A(M)) \leq \phi(A_1(M)) + \phi(A_2(M)) \leq [A_1]_\phi \phi(M) \leq Lip(A_1)\phi(M),$$

which shows that A is condensing. More generally, one may show that an operator of the form $A(x) = \Psi(x, x)$ is condensing, if $\Psi : X \times X \to X$ is continuous, $\Psi(x, \cdot)$ is compact for each $x \in X$, and $\Psi(\cdot, y)$ is a contraction (with contraction constant $Lip(\Psi(\cdot, y)) < 1$ independent of y) for each $y \in X$. This will be applied later (see Example 10.2 below) to initial value problems in infinite dimensional Banach spaces. □

Interestingly, there exists a certain converse of Example 8.1 for *linear* operators ([76], see also Section 2.6 of [3]) which states that, roughly speaking, the linear condensing operators are all of the form "contraction + compact":

Theorem 8.3. *Let $L \in \mathcal{L}(X, X)$ with $[L]_\gamma < 1$. Then L admits a representation as a sum $L = L_1 + L_2$, where $r(L_1) < 1$ and L_2 is compact.*

We remark that the proof of Theorem 8.3 is even constructive. Indeed, it builds on the fact that the set $\{\lambda \in \sigma(L) : |\lambda| > [L]_\gamma\}$ consists only of finitely many eigenvalues of L of finite multiplicity [5]. So if we fix $\rho > 0$ with $r_e(L) < \rho < 1$, the operator

$$P = \frac{1}{2\pi i} \int_{|\lambda|=\rho} (\lambda I - L)^{-1} \, d\lambda$$

has a finite dimensional range. Consequently, the operator $L_1 := LP$ satisfies $r(L_1) = r(LP) < r_e(L) < 1$, while the operator $L_2 := L(I - P)$ is compact, having finite rank. In the special case of a self-adjoint positive operator in a Hilbert space, we have employed a similar reasoning in (5.7).

The condition $r(L_1) < 1$ in Theorem 8.3 may appear weaker than the corresponding condition $||R_1|| < 1$ in (5.7). However, as we already mentioned, one may always pass to an equivalent norm $|| \cdot ||^*$ on X such that the norm of L_1 in the space $(X, || \cdot ||^*)$ is strictly less than 1, and the compactness of the other operator L_2 is not affected by this change of norms. In this way we arrive at an exact converse of Example 8.1.

We return now to the above fixed point theorems for condensing and weakly condensing operators. In view of Theorem 8.2, we point out that an analogous generalization of the Banach-Caccioppoli fixed point theorem to *weak contractions*, i.e., operators A satisfying

$$||A(x) - A(y)|| < ||x - y|| \qquad (x, y \in M,\, x \neq y) \tag{8.2}$$

is *false*, as the following Example 8.2 shows. This is simply due to the somewhat surprising fact that a weak contraction is not necessarily weakly α-condensing, although a contraction is of course α-condensing, see (1.19).

Example 8.2. Let $X = c_0$ be the Banach space of all sequences $x = (\xi_n)_n$ converging to zero with the supremum norm. The operator A defined by

$$A(x) = A(\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) = \\ (\tfrac{1}{2}(1 + ||x||), \tfrac{3}{4}\xi_1, \tfrac{7}{8}\xi_2, \ldots, (1 - 2^{-n})\xi_{n-1}, \ldots) \tag{8.3}$$

maps the ball $B(c_0)$ into itself and is a weak contraction in the sense of (8.2). However, A has no fixed point. Indeed, if $\hat{x} = (\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3, \ldots)$ were a fixed point of A in $B(c_0)$, then

$$\hat{\xi}_1 = \tfrac{1}{2}(1 + ||\hat{x}||),\ \hat{\xi}_2 = \tfrac{1}{2}\tfrac{3}{4}(1 + ||\hat{x}||),\ \hat{\xi}_3 = \tfrac{1}{2}\tfrac{3}{4}\tfrac{7}{8}(1 + ||\hat{x}||),\ \ldots$$

and, in particular, $\hat{\xi}_n \geq \frac{1}{2}$ for all n. So the point $\hat{x}$ cannot belong to the space X.

From Theorem 8.2 it follows that the operator (8.3) cannot be weakly γ-condensing. This may also be verified directly by observing that $A(e_n) = e_1 + (1 - 2^{-(n+1)})e_{n+1}$, with $(e_n)_n$ being the canonical basis in X. □

We give another example of this type which is even more surprising, since it is *linear*.

Example 8.3. In the Banach space $X = C[0, 1]$, consider again the second set from Example 1.3, i.e.,

$$M = \{u \in B(X) :\ 0 = u(0) \leq u(t) \leq u(1) = 1\},$$

which is clearly closed, bounded, and convex. The linear operator $A : M \to M$ defined by $Au(t) = tu(t)$ maps M into itself and satisfies $[A]_\gamma = ||A|| = 1$, by (3.4). Actually, as was observed in [88], A is even a weak contraction. On the other hand, the zero function $u(t) \equiv 0$ is the only fixed point of A, but does not belong to M. So from Theorem 8.2 it follows that the operator A cannot be weakly γ-condensing. $\square$

9. Applications to nonlinear problems

Building on our calculations of the ϕ-norm of various (linear or nonlinear) operators in the preceding sections, we give now a series of examples to illustrate these calculations. We start with a nonlinear integral equation of Volterra-Hammerstein type.

Example 9.1. Consider the nonlinear integral equation

$$u(s) = \sigma(s) \int_0^s \tau(t) f(t, u(t))\, dt \qquad (0 \leq s \leq 1) \tag{9.1}$$

in the space $L^p[0,1]$ for $1 < p < \infty$. The operator A defined by the right hand side of (9.1) may be written as composition $A = KF$ of the nonlinear Nemytskij operator (7.2) and the linear Volterra operator (4.5). Consequently, a sufficient condition for this operator to be γ-condensing, say, is $[K]_\gamma [F]_\gamma < 1$. But we have already estimated these two numbers in the space L^p, see (4.7) and (7.4). As a result, we see that the γ-norm of A satisfies the two-sided estimate

$$2^{-(1+1/p)} \omega_0 k \leq [A]_\gamma \leq p^{1/p} (p')^{1/p'} \omega_0 k,$$

where ω_0 is defined by (4.6), and

$$k := \sup_{0 \leq t \leq 1} \sup_{u \neq v} \frac{|f(t,u) - f(t,v)|}{|u - v|}$$

is the smallest Lipschitz constant of the scalar function $f(t, \cdot)$. $\square$

In the next problem we combine Theorem 7.1 with Example 6.3 and get an existence result for a nonlinear Schrödinger equation.

Example 9.2. Consider the nonlinear stationary Schrödinger type equation

$$-\Delta u(x) + q(x) u(x) = f(x, u(x)) \qquad (x \in \mathbb{R}^N) \tag{9.2}$$

in the space $L^2(\mathbb{R}^N)$, where the potential q is supposed to fulfill the same assumptions as in Example 6.3. Writing equation (9.2) in the form (7.3), with L given by (6.1) and F given by (7.2), we already know that $[L^{-1}]_\gamma \leq$

$(\inf\, \sigma_e(L))^{-1}$ and $[F]_\gamma = Lip(f(x,\cdot))$. So the crucial estimate $[L^{-1}]_\gamma[F]_\gamma < 1$ means precisely that

$$\sup_{x\in\mathbb{R}^N} \sup_{u\neq v} \frac{|f(x,u)-f(x,v)|}{|u-v|} < \inf\, \sigma_e(L). \tag{9.3}$$

This condition exhibits an interesting "interaction" between the linear left hand side and the nonlinear right hand side of (9.2): *the asymptotic slope of the nonlinearity has to remain strictly below the essential spectrum of the linear part.* We point out that a similar phenomenon is known for the "discrete part" of the spectrum under the name *non-resonance condition*: whenever the asymptotic slope of the nonlinearity does not "hit" an eigenvalue of the linear part, one has existence (and sometimes even uniqueness) of solutions; conversely, hitting an eigenvalue leads to such unpleasant phenomena like bifurcation or non-existence. Condition (9.3) (which could be called, by analogy, *essential non-resonance*) shows that it is not really important to avoid eigenvalues: what really matters is to keep away from the essential spectrum! In a physical interpretation this means that the "oscillations" of the nonlinearity in (9.2) have to stay strictly below the scattering states of the system. □

The next example involves the Hilbert transform (4.8) whose α-norm and γ-norm is given by (4.11).

Example 9.3. Consider the nonlinear strongly singular integral equation

$$u(s) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(t,u(t))}{s-t}\,dt \qquad (-1 \le s \le 1) \tag{9.4}$$

in the space $L^p[-1,1]$ for $1 < p < \infty$. Writing the operator A defined by the right hand side of (9.4) again as composition $A = HF$ of the nonlinear Nemytskij operator (7.2) and the linear Hilbert transform (4.8), we see that the condition $[H]_\gamma[F]_\gamma < 1$ is equivalent to the condition $Lip(f(t,\cdot)) < 1$, by (4.11) and Theorem 7.1. So if this condition is fulfilled, and we find an invariant closed ball for the operator A, we get existence of solutions, by Darbo's fixed point theorem, no matter what p is. On the other hand, if we look for solutions of (9.4) either in a particularly "large" Lebesgue space (i.e., for p close to 1), or in a particularly "narrow" Lebesgue space (i.e., for p very large), it becomes more and more difficult to apply Banach's contraction mapping principle, by (4.10). This problem is not as artificial as it may seem at first glance. In fact, one usually tries to prove existence of solutions in a possibily narrow space, and uniqueness in a possibly large space; if one is interested in both existence and uniqueness, one has to find some kind of "compromise", and the unpleasant "blow up relation" (4.10) may become important. □

We may summarize our discussion in the preceding example as follows: the operator A defined by the right hand side of (9.4) is a contraction in L^p if $Lip(f(t,\cdot)) < ||H||_p^{-1}$, where $||H||_p$ may be estimated by (4.9), and it is condensing if just $Lip(f(t,\cdot)) < 1$. Of course, the first condition severely restricts the class of admissible nonlinearities f, but there are of course plenty of nonlinear functions f satisfying this condition. In the next example this is not the case.

Example 9.4. Consider again the nonlinear strongly singular integral equation (9.4), but now in the Hölder space $C^\alpha[-1,1]$. One knows various sufficient conditions under which the Hilbert transform (4.8) maps this space into itself and is bounded (see, e.g., Chapter 5 of the monograph [46]). Moreover, upper estimates of the γ-norm $[F]_\gamma$ of the Nemytskij operator (7.2) in the Hölder space C^α are also known (see [11, Chapter 7]). So one may prove existence of Hölder continuous solutions of (9.4), by Darbo's fixed point theorem, for a fairly large class of nonlinear functions f.

On the other hand, from Theorem 7.3 we see that only *affine* functions f generate Lipschitz continuous Nemytskij operators in the space C^α. This means that one can *never* prove existence of Hölder continuous solutions of (9.4), adopting Banach's contraction mapping principle, unless equation (9.4) is actually linear. □

We conclude this section by considering an example of a nonlinear integral equation, where one may use the Sadovskij functional ϕ from Example 2.4 in order to derive the existence of increasing continuous solutions. In a more general setting, this and many related examples may be found in the thesis [21].

Example 9.5. Consider the nonlinear integral equation of Uryson-Volterra type

$$u(s) = a(s) + \int_0^s k(s,t,u(t))\,dt \qquad (0 \le s \le 1). \tag{9.5}$$

Here $a : [0,1] \to \mathbb{R}$ is continuous and increasing, and $k : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and such that $k(\cdot, t, u)$ is increasing for all $(t,u) \in [0,1]\times\mathbb{R}$. The crucial hypothesis is the growth condition

$$|k(s,t,u)| \le \kappa(|u|) \qquad (0 \le s,t \le 1,\ -\infty < u < \infty)$$

for the nonlinear kernel function in (9.5), where $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing. The author of [21] shows then that the integral operator defined by the right-hand side of (9.5) is condensing with respect to the Sadovskij functional (2.13) and leaves some closed ball $B_r(X)$ in the space $X = C[0,1]$ invariant, provided that $||a||+\kappa(r) \le r$. So Darbo's fixed point theorem (Theorem 8.1 for ϕ instead of α) implies that equation (9.5) has a continuous increasing solution u. □

10. Initial value problems in Banach spaces

Let X be a real Banach space, $x_0 \in X$ fixed, and $f : [-a,a]\times B_r(X;x_0) \to X$ a continuous function. It is well-known that then the initial value problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0 \tag{10.1}$$

need not have a solution if X is infinite dimensional, and so the classical Cauchy-Peano theorem does not carry over to infinite dimensions. We recall one of the first examples of this type which is due to Dieudonné [28]:

Example 10.1. In the Banach space $X = c_0$, consider the initial value problem

$$\dot{\xi}_k = \sqrt{|\xi_k|} + \frac{1}{k}, \quad \xi_k(0) \equiv 0 \qquad (k = 1, 2, 3, \ldots). \tag{10.2}$$

One can show that this initial value problem has no solution in the space c_0. This is not a peculiarity of the space c_0; indeed, Godunov [40] has proved that a similar example can be constructed in *any* infinite dimensional Banach space. The best result in this direction has been proved by Saint-Raymond [74]: given an arbitrary infinite dimensional Banach space X, there exists a continuous function $f : \mathbb{R} \times X \to X$ such that the initial value problem (10.1) has no solution in a neighbourhood of zero for *any* initial value $x_0 \in X$.

It turns out that, in order to get a Cauchy-Peano type existence theorem in arbitrary Banach spaces, one has to impose some compactness condition on the right-hand side f of (10.1). For instance, if the function $f(t,\cdot)$ maps every ball into a precompact set, then (10.1) is locally solvable, precisely as in the finite dimensional case. More generally, the following is true:

Theorem 10.1. *Let $f:[-a,a]\times B_r(X;x_0)\to X$ be a continuous function which satisfies*

$$\alpha(f(t,M))\le k\alpha(M)\qquad (|t|\le a;\ M\subseteq B_r(X;x_0)).\tag{10.3}$$

Then the initial value problem (10.1) *has a solution $x:[-h,h]\to X$, where $0<h\le\min\{a,r/C,1/k\}$ and $C:=\sup\{||f(t,x)||:|t|\le a,\ ||x-x_0||\le r\}$.*

The idea of the proof of Theorem 10.1 is almost the same as in the finite dimensional case. The problem of looking for solutions of the initial value problem (10.1) is replaced by the equivalent problem of finding fixed points of the Picard operator

$$A(x)(t)=x_0+\int_0^t f(s,x(s))\,ds\qquad (-h\le t\le h)$$

in the space $\hat{X}:=C([-h,h],X)$. In order to apply Darbo's fixed point principle (Theorem 8.1), one uses the fact that the Kuratowski measure of noncompactness $\hat{\alpha}(M)$ of a bounded equicontinuous subset $M\subset\hat{X}$ may be calculated by

$$\hat{\alpha}(M)=\sup_{-h\le t\le h}\alpha(M(t)),$$

where $M(t):=\{x(t):x\in M\}$, and α denotes the Kuratowski measure of noncompactness in X (see [4]). We illustrate Theorem 10.1 by means of another initial value problem, again in the sequence space c_0:

Example 10.2. In the Banach space $X=c_0$, consider the initial value problem

$$\dot{\xi}_k=\xi_{k+1}\sqrt{|\xi_1|}+\frac{1}{k},\quad \xi_k(0)\equiv 0\qquad (k=1,2,3,\ldots).\tag{10.4}$$

For $r>0$, define $\psi:c_0\times B_r(c_0)\to c_0$ by

$$\psi((\xi_1,\xi_2,\xi_3,\ldots),(\eta_1,\eta_2,\eta_3,\ldots)):=(\xi_2\sqrt{|\eta_1|}+1,\xi_3\sqrt{|\eta_1|}+\tfrac{1}{2},\xi_4\sqrt{|\eta_1|}+\tfrac{1}{3},\ldots).$$

It is not hard to see that ψ is continuous, $\psi(x,\cdot)$ is compact for any $x \in B_r(c_0)$, and $\psi(\cdot,y)$ is Lipschitz continuous with $Lip(\psi(\cdot,y)) \leq \sqrt{r}$ for each $y \in c_0$. Thus, from the last part of Example 8.1 we conclude that the function $f : B_r(c_0) \to c_0$ defined by $f(x) := \psi(x,x)$ satisfies $[f]_\gamma \leq \sqrt{r}$. Consequently, the initial value problem (10.4) has a solution $x : [-h,h] \to c_0$ for $0 < h \leq \min\{a, r/C, 1/\sqrt{r}\}$ and $C := \sup\{||\psi(x,x)|| : ||x|| \leq r\}$, by Theorem 10.1. □

The following existence result is slightly more general than Theorem 10.1 and was proved in [71].

Theorem 10.2. *Let $f : [-a,a] \times B_r(X;x_0) \to X$ be a continuous function which satisfies*

$$\alpha(f(t,M)) \leq L(t,\alpha(M)) \qquad (|t| \leq a;\ M \subseteq B_r(X;x_0)), \tag{10.5}$$

where $L : [-a,a] \times \mathbb{R} \to \mathbb{R}$ is such that the only solution of the scalar initial value problem

$$\dot{u}(t) = L(t,u(t)), \quad u(0) = 0 \tag{10.6}$$

is $u(t) \equiv 0$. Then the assertion of Theorem 10.1 holds.

Obviously, Theorem 10.2 contains Theorem 10.1 as a special case, since the initial value problem $\dot{u} = ku$, $u(0) = 0$ has only the trivial solution $u(t) \equiv 0$. It would be interesting to have an example of an initial value problem to which Theorem 10.2 applies, but Theorem 10.1 does not. As far as we know, such an example has not yet been given in the literature.

11. Functional-differential equations

In this section we discuss applications of Darbo's fixed point theorem to equations of the form

$$\dot{x}(t) = f(t, x_t, \dot{x}_t), \tag{11.1}$$

where the subscript t denotes the shift by $t \in \mathbb{R}$, i.e., $x_t(s) = x(t+s)$ and $\dot{x}_t(s) = \dot{x}(t+s)$. The real function f on the right-hand side of (11.1) is supposed to be defined on $\mathbb{R} \times C[-h,0] \times C[-h,0]$ for some $h > 0$. Since the equation in (11.1) contains derivatives of the unknown solution x as well as of the shifted solution x_t, it is called *functional-differential equation* (more precisely, *functional-differential equation of neutral type*). This equation is supposed to hold in the "future", i.e., on some interval $[0, T]$; on the other hand, instead of the usual "pointwise" initial condition in the Cauchy problem, here the suitable initial condition for (11.1) involves the values of the solution in the "past", i.e., has the form

$$x(t) = \varphi(t) \qquad (-h \leq t \leq 0). \tag{11.2}$$

Here $\varphi \in C[-h, 0]$ is a given initial function. If one is interested in classical (i.e., C^1-) solutions of (11.1)/(11.2), the initial function φ has to satisfy in the "present", i.e. for $t = 0$, the "glueing condition"

$$\lim_{t \uparrow 0} \dot{\varphi}(t) = f(0, \varphi, \dot{\varphi}). \tag{11.3}$$

We will assume throughout this section that the glueing condition (11.3) is fulfilled; this may always be achieved by passing, if necessary, from $f(t, x_t, \dot{x}_t)$ to the function

$$g(t, x_t, \dot{x}_t) := f(t, x_t, \dot{x}_t) + \nu_\tau(t) \left[\lim_{t \uparrow 0} \dot{x}(t) - f(0, x, \dot{x}) \right],$$

where $\nu_\tau(t) = 1 - t/\tau$ for $0 \leq t \leq \tau$ and $\nu_\tau(t) \equiv 0$ for $t \geq \tau$, with $\tau > 0$ being sufficiently small.

The fact that the equation (11.1) contains the shift functions x_t and $\dot{x}_t$ means, physically speaking, that the "reaction" of the system described by (11.1) occurs with a certain time delay. There are many problems in mechanics, physics, engineering, chemistry, and biology which lead to such type of equations.

To solve the problem (11.1)/(11.2), one tries as usual to associate to this problem a nonlinear (e.g., integral) operator in such a way that every solution of (11.1)/(11.2) is a fixed point of this integral operator, and vice versa. However, there is an essential difference between ordinary differential equations like (10.1) and functional-differential equations like (11.1). In fact, ordinary differential equations lead to *compact* integral operators, at least in finite dimensional spaces. (This is the reason why we considered equation (10.1) in infinite dimensional spaces.) On the other hand, the fact that the right-hand side of equation (11.1) depends on the "prehistory" of the hypothetic solution leads to *noncompact* integral operators, even in the scalar case. This explains the fact that such problems are good examples for illustrating the applicability of Darbo's fixed point principle, where Schauder's fixed point principle fails. A detailed description of the theory and applications of problem (11.1)/(11.2) may be found in [2] or Section 4.3 of [3]. Here we restrict ourselves to some typical methods, results and examples.

For every $y \in C[-h,0]$ and fixed $\varphi \in C^1[-h,0]$ we denote by $\tilde{y}$ the C^1-function defined by

$$\tilde{y}(t) := \varphi(0) + \int_0^t y(s)\,ds.$$

By means of the map $y \mapsto \tilde{y}$, we define a nonlinear operator A by

$$A(y)(t) := \begin{cases} \dot{\varphi}(t) + f(0,\tilde{y},y) - \dot{\varphi}(0) & \text{for } -h \le t \le 0, \\ f(t,\tilde{y}_t,y_t) & \text{for } 0 \le t \le T \end{cases} \tag{11.4}$$

on the Banach space $X := \{y \in C[-h,T] : y(t) \equiv \dot{\varphi}(t) \text{ for } -h \le t \le 0\}$. With this terminology, the following equivalence holds.

Theorem 11.1. *If x is a solution of problem* (11.1)/(11.2), *then $y := \dot{x}$ is a fixed point of the operator* (11.4) *in X. Conversely, if $y \in X$ is a fixed point of the operator* (11.4), *then $x := \tilde{y}$ is a solution of problem* (11.1)/(11.2).

Now, by using the explicit formulas for the Hausdorff measure of noncompactness in the space $C[-h,T]$ which we derived in Section 2 (see Theorem 2.1), one may show that the operator A given by (11.4) is in fact condensing on every bounded subset of X. So, if one finds an invariant closed convex bounded subset for this operator, Darbo's fixed point principle implies that problem (11.1)/(11.2) has a solution. For details we refer to the papers [65,73], see also [2].

From the viewpoint of applications it is more interesting to study existence of *periodic solutions* of the functional-differential equation (11.1). There are several ways to connect the existence problem for periodic solutions to fixed points of condensing operators; we describe three of them.

Let us suppose first that problem (11.1)/(11.2) has, for any initial function $\varphi \in C^1[-h,0]$, a *unique solution* $x \in C^1[-h,T]$ for some fixed $T > 0$; we denote this solution by $x(\varphi;\cdot)$ to emphasize the dependence on φ. Then one may define the *shift operator* (along the trajectories of (11.1)) $U_T : C^1[-h,0] \to C^1[-h,0]$ by

$$U_T(\varphi)(s) := x(\varphi; s+T), \tag{11.5}$$

i.e., U_T associates to each initial function φ the value of the corresponding solution at time T. Then the existence problem for T-periodic solutions of (11.1) is equivalent to the problem of finding fixed points of U_T. In fact, if $x(\varphi;\cdot)$ is T-periodic, then clearly $U_T(\varphi)(s) = x(\varphi; s+T) \equiv x(\varphi; s)$ for all $s \in [-h,0]$. Conversely, from $x(\varphi;\cdot) = U_T(\varphi) = x_T(\varphi;\cdot)$ and our uniqueness assumption it follows that $x(\varphi;\cdot)$ is T-periodic on $[-h,0]$.

We point out that the shift operator of an ordinary differential equation in $\mathbb{R}^n$, say (which associates to each initial *point* $x_0 \in \mathbb{R}^n$ the value of the corresponding solution at time T), is compact, because it acts in a finite dimensional space. In case of a functional-differential equation, however, we deal with initial *functions*, and so the shift operator acts in an infinite dimensional function space. It is therefore not surprising that the operator (11.5) is usually not compact; so the question arises whether or not it is condensing with respect to some measure of noncompactness.

This is in fact true, as was shown in [52] by means of the following special construction. Let α denote the usual Kuratowski measure of noncompactness (1.8) in the space $C[-h,0]$. For fixed $k > 1$ we define a special measure of noncompactness $\tilde{\alpha}_k$ on $C^1[-h,0]$ by

$$\tilde{\alpha}_k(M) := \alpha(\{y : y(t) = \frac{1}{h}\,((k-1)t + hk)\,\dot{x}(t)\,(x \in M)\}$$

($M \subset C^1[-h,0]$ bounded).

One may then show that the shift operator U_T is condensing with respect to this measure of noncompactness, and so one may apply Darbo's fixed point principle to find T-periodic solutions of (11.1).

The second method joining T-periodic solutions of (11.1) to fixed points of a certain condensing operator goes as follows. Suppose that the right hand side of (11.1) is T-periodic in the first argument, i.e., $f(t+T,y,z) \equiv f(t,y,z)$. By $C_T(\mathbb{R})$ resp. $C_T^1(\mathbb{R})$ we denote the vector space of all continuous resp. continuously differentiable T-periodic real functions equipped with the usual maximum norm. Denoting by

$$[y] := \frac{1}{T}\int_0^T y(s)\,ds$$

the integral mean of $y \in C_T(\mathbb{R})$, we define an operator $J : C_T(\mathbb{R}) \to C(\mathbb{R})$ by

$$J(y)(t) := \int_0^t y(s)\,ds - [y]t,$$

an operator $F : \mathbb{R} \times C_T(\mathbb{R}) \to C(\mathbb{R})$ by

$$F(a, y)(t) := f(t, a + (J(y))_t, y_t) \qquad (a \in \mathbb{R}),$$

and an operator $\Phi : \mathbb{R} \times C_T(\mathbb{R}) \to \mathbb{R} \times C_T(\mathbb{R})$ by

$$\Phi(a, y) := (a - [y], F(a, y)). \tag{11.6}$$

Then the T-periodic solutions of equation (11.1) and the fixed points of the operator (11.6) are related in the following way:

Theorem 11.2. *If x is a T-periodic solution of problem* (11.1), *then the pair $(x(0), \dot{x})$ is a fixed point of the operator* (11.6) *in $\mathbb{R} \times C_T(\mathbb{R})$. Conversely, if the pair $(a, y) \in \mathbb{R} \times C_T(\mathbb{R})$ is a fixed point of the operator* (11.6), *then $x := J(y) + a$ is a T-periodic solution of problem* (11.1) *with $x(0) = a$.*

So again we have to show that the operator (11.6) is condensing with respect to a suitable measure of noncompactness, in order to guarantee the existence of T-periodic solutions of (11.1). For instance, it may be shown that

$$\tilde{\gamma}(M) := \gamma(\{y : (a, y) \in M \text{ for some } a \in \mathbb{R}\}) \qquad (M \subset C_T(\mathbb{R}) \text{ bounded})$$

is such a suitable measure of noncompactness, where γ denotes the Hausdorff measure of noncompactness in the space $C_T(\mathbb{R})$.

Finally, suppose now that $x \in C_T^1(\mathbb{R})$, and let

$$\Psi(x)(t) := x(0) + \int_0^t f(s, x_s, \dot{x}_s)\,ds - \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T f(s, x_s, \dot{x}_s)\,ds. \tag{11.7}$$

It is not hard to see that the operator (11.7) maps the space $C_T^1(\mathbb{R})$ into itself. Moreover, the fixed points of this operator are again T-periodic solutions of (11.1) in $C^1_T(\mathbb{R})$, and vice versa. In fact, for a T-periodic solution x of (11.1) we have

$$\int_0^T f(s, x_s, \dot{x}_s)\, ds = \int_0^T \dot{x}(s)\, ds = x(T) - x(0) = 0,$$

and so

$$x(t) = x(0) + \int_0^t \dot{x}(s)\, ds$$

$$= x(0) + \int_0^t f(s, x_s, \dot{x}_s)\, ds - \left(\frac{t}{T} - \frac{1}{2}\right) \int_0^T f(s, x_s, \dot{x}_s)\, ds = \Psi(x)(t).$$

Conversely, from $x = \Psi(x)$ it follows that

$$x(0) = x(T) = \Psi(x)(T) = x(0) + \frac{1}{2} \int_0^T f(s, x_s, \dot{x}_s)\, ds.$$

Consequently, the last integral vanishes, and so

$$x(t) = x(0) + \int_0^t f(s, x_s, \dot{x}_s)\, ds,$$

which upon differentiation yields the conclusion.

Now, here one may show that

$$\hat{\gamma}(M) := \gamma(\{\dot{x} : x \in M\}) \qquad (M \subset C^1_T(\mathbb{R}) \text{ bounded})$$

is a measure of noncompactness in $C^1_T(\mathbb{R})$, which makes the operator (11.7) condensing , where γ denotes again the Hausdorff measure of noncompactness in the space $C_T(\mathbb{R})$. This gives still another method to reduce the problem of T-periodic solutions to applying Darbo's fixed point principle in functions space with an appropriate measure of noncompactness. For details and proofs we refer to the papers [65, 70-73].

12. Operators without fixed points

In Section 8 we have seen that γ-condensing and, more generally, weakly γ-condensing operators which map the closed unit ball $B(X)$ of a Banach space X into itself, have fixed points. Now we want to compare these and related classes of operators from the viewpoint of their "size". To be more precise, we denote by $\Gamma_k(X)$ the set of all operators $A : B(X) \to B(X)$ with $[A]_\gamma \le k$; in particular, $\Gamma_1(X)$ consists of all γ-nonexpansive operators. Furthermore, we put

$$\Gamma_1^-(X) = \bigcup_{k<1} \Gamma_k(X),$$

and by $\Gamma(X)$ we denote the class of all weakly γ-condensing operators. So we have the trivial inclusions

$$\Gamma_1^-(X) \subseteq \Gamma(X) \subseteq \Gamma_1(X). \tag{12.1}$$

If we equip $\Gamma_1(X)$ with the natural metric

$$D(A_1, A_2) = \sup_{||x|| \leq 1} ||A_1(x) - A_2(x)||,$$

we may easily show that $(\Gamma_1(X), D)$ is a complete metric space, and $\Gamma_k(X)$ is closed in $\Gamma_1(X)$ for any $k \leq 1$.

Before studying these classes of operators, we first show by means of two examples that both inclusions in (12.1) are strict. The first example is due to Nussbaum [64], the second one to Kakutani [51].

Example 12.1. Let X be any infinite dimensional Banach space, and let $A : B(X) \to B(X)$ be defined by

$$A(x) = (1 - ||x||)x.$$

Then $A \notin \Gamma_1^-(X)$, as may be seen by letting $r \to 0$ in the estimate

$$\gamma(A(B_r(X))) \geq \gamma(B_{r(1-r)}(X)) = r(1-r) = (1-r)\gamma(B_r(X)).$$

However, A is weakly γ-condensing. In fact, for any $M \subseteq B(X)$ with $\gamma(M) > 0$ and $0 < r < \frac{1}{2}\gamma(M)$ we have

$$\gamma(A(M \cap B_r(X))) \leq \gamma(M \cap B_r(X))) \leq \gamma(B_r(X)) = 2r < \gamma(M),$$

on the one hand, and

$$\gamma(A(M \setminus B_r(X))) \leq \gamma(\overline{co}\,[(1-r)M \cup \{0\}]) = (1-r)\gamma(M) < \gamma(M),$$

on the other. Applying (1.1) we obtain

$$\gamma(A(M)) = \max\{\gamma(A(M \cap B_r(X))), \gamma(A(M \setminus B_r(X)))\} < \gamma(M),$$

and so $A \in \Gamma(X)$ as claimed. □

Example 12.2. Let $X = \ell^2$ be the space of all square-summable sequences, and let $A : B(X) \to B(X)$ be defined by

$$A(x) = A(\xi_1, \xi_2, \xi_3, \ldots) = (\sqrt{1 - ||x||^2}, \xi_1, \xi_2, \ldots).$$

We may write A as a sum $A = A_1 + A_2$ of the linear isometry ("right shift operator")

$$A_1(x) = A_1(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \ldots)$$

and the nonlinear compact map

$$A_2(x) = A_2(\xi_1, \xi_2, \xi_3, \ldots) = (\sqrt{1 - ||x||^2}, 0, 0, \ldots).$$

So by (1.17) we have $[A]_\gamma \leq ||A_1|| = 1$, i.e., $A \in \Gamma_1(X)$. On the other hand, A cannot belong to the class $\Gamma(X)$, since A has no fixed point in $B(X)$. $\square$

The following theorem gives an idea about the "smallness" of the class $\Gamma_1^-(X)$ in $\Gamma(X)$ and of the class $\Gamma(X)$ in $\Gamma_1(X)$. The proof of this theorem, as well as some other results of this type, may be found in the book [13].

Theorem 12.1. *The class* $\Gamma_1^-(X)$ *is both dense and of first category in* $\Gamma_1(X)$*, while the set* $\Gamma_1(X) \setminus \Gamma(X)$ *is both dense and of type* F_σ *in* $\Gamma_1(M)$*.*

Theorem 12.1 shows that Example 12.1 is "generic" from the viewpoint of category, while Example 12.2 is typical from the viewpoint of density, but not vice versa. Of course, it is no paradox that a certain set is "small" from one point of view, but "large" from another. Theorem 12.1 also means, in a certain sense, that Theorem 8.2 is an essential extension of Theorem 8.1, and the difference between the classes $\Gamma_1^-(X)$ and $\Gamma(X)$ is not as harmless as it may seem.

As we have seen, a γ-nonexpansive operator need not have fixed points. However, one may show that such an operator has "almost" fixed points. To make this precise, for an operator $A : B(X) \to B(X)$ we introduce the number

$$\eta(A) := \inf_{||x|| \leq 1} ||x - A(x)|| \tag{12.2}$$

which is usually called the *minimal displacement* of A (on $B(X)$). Of course, the condition that A maps the ball $B(X)$ into itself trivially implies that $\eta(A) \leq ||A(0)|| \leq 1$. If A has a fixed point in $B(X)$, then trivially $\eta(A) = 0$.

The converse is not true; for instance, the fixed point free operator A in Example 12.2 satisfies $\eta(A) = 0$.

The following remarkable estimate was proved first in [41] for the minimal Lipschitz constant (1.20), and afterwards in [37] for the Hausdorff measure of noncompactness (1.10). As we will show later, this estimate is in a certain sense sharp.

Theorem 12.2. *For any map $A : B(X) \to B(X)$ with $[A]_\gamma < \infty$ one has*

$$\eta(A) \leq \max \left\{ 1 - \frac{1}{[A]_\gamma}, 0 \right\}. \tag{12.3}$$

In particular, $\eta(A) = 0$ if A is γ-nonexpansive.

As a matter of fact, Theorem 12.2 is simply a consequence of Darbo's fixed point theorem. Indeed, if $[A]_\gamma < 1$ then A has a fixed point, by Theorem 8.1. On the other hand, if $[A]_\gamma \geq 1$ we may fix $\varepsilon > 0$ with $\varepsilon[A]_\gamma < 1$. Then the operator $\varepsilon A : B(X) \to B_\varepsilon(X) \subseteq B(X)$ is γ-condensing and thus has a fixed point $\hat{x}$. So we obtain

$$||A(\hat{x}) - \hat{x}|| = ||A(\hat{x}) - \varepsilon A(\hat{x})|| = (1 - \varepsilon)||A(\hat{x})|| \leq 1 - \varepsilon,$$

hence $\eta(A) \leq 1 - \varepsilon$. Since $\varepsilon \in (0, 1/[A]_\gamma)$ was arbitrary, we conclude that $\eta(A) \leq 1 - 1/[A]_\gamma$ as claimed.

Interestingly, the estimate (12.3) allows us to show the equivalence of Theorems 8.1 and 8.2. To see this, let $A : B(X) \to B(X)$ be weakly condensing. Then $\eta(A) = 0$, by (12.3), and so we may find a sequence $(x_n)_n$ in $B(X)$ with $||x_n - A(x_n)|| \to 0$ as $n \to \infty$. Now, since the set $M = \{x_1, x_2, x_3, \ldots\}$ satisfies $\gamma(A(M)) = \gamma(M)$, it must be precompact, and so the sequence $(x_n)_n$ has a convergent subsequence whose limit is of course a fixed point of A.

The discussion of this section suggests the following general question: *Which metric spaces have the "fixed point property" for one of the classes of maps discussed above?* One knows many partial answers to this question in special classes of metric (or even topological) spaces, but of course a complete answer is not known (and seems very unlikely). A discussion of this and related questions, based on the axiomatic setting of *fixed point structures*, may be found in a series of papers of Ioan A. Rus (e.g., [66-68]).

13. Some Banach space constants

Example 12.2 is typical, inasmuch as in *each* infinite dimensional Banach space X one may find a fixed point free map $A : B(X) \to B(X)$. By Darbo's fixed point theorem (Theorem 8.1), such a map necessarily satisfies $[A]_\alpha \geq 1$. On the other hand, a quite remarkable result by Lin and Sternfeld [59] (see also [17,38]) states that such a map may always be chosen Lipschitz continuous, and so $[A]_\alpha \leq Lip(A) < \infty$.

The existence of fixed point free self-maps is closely related to the existence of other "pathological" objects in infinite dimensional Banach spaces, e.g., retractable balls. Recall that a set $M \subset X$ is a *retract* of a larger set $N \supset M$ if there exists a map $\rho : N \to M$ with $\rho(x) = x$ for $x \in M$; this means that one may extend the identity from M by continuity to N. For instance, Example 1.4 shows that every ball $B_r(X)$ is a retract of the whole space, where the corresponding retraction may even be chosen α-nonexpansive.

The problem of finding a retraction of the unit ball $B(X)$ onto its boundary $S(X)$ is much more complicated and, in fact, in sharp contrast to our geometric intuition. It turns out that the existence (resp., non-existence) of fixed points of self-maps of the unit ball is closely related to the non-existence (resp., existence) of such retractions. The following Theorem 13.1 which is fundamental in topological nonlinear analysis makes this more precise.

Theorem 13.1. *The following two statements are equivalent in a Banach space X:*

(a) *each operator $A : B(X) \to B(X)$ has a fixed point;*

(b) *$S(X)$ is <u>not</u> a retract of $B(X)$.*

It is a striking fact that both assertions of Theorem 13.1 are *true* if $\dim X < \infty$, but *false* if $\dim X = \infty$. The first example of a fixed point free self-map on $B(X)$ has been given in the sequence space $X = \ell^2$ by Kakutani [51] and is our Example 12.2 above. Moreover, it has been shown by Leray [57] that in the space $X = C[0,1]$ the identity map on the unit sphere $S(X)$ may be homotopically deformed into a constant map; this is still another property which is equivalent to (a) and (b).

We point out that the proof of Theorem 13.1 is in a certain sense constructive. In fact, if $\rho : B(X) \to S(X)$ is a retraction, then $A : B(X) \to S(X) \subset B(X)$ defined by

$$A(x) := -\rho(x) \tag{13.1}$$

is certainly fixed point free. Conversely, if one is given a fixed point free operator $A : B(X) \to B(X)$, one may use A to construct a retraction $\rho : B(X) \to S(X)$ by a geometric reasoning [38]. For example, if X is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, and $A : B(X) \to B(X)$ has no fixed point, one may define $\rho : B(X) \to S(X)$ by $\rho(x) := A(x) + \tau(x)(x - A(x))$, where $\tau = \tau(x)$ is the unique positive solution of the quadratic equation

$$\tau^2 + 2\tau \frac{\langle A(x), x - A(x) \rangle}{||x - A(x)||^2} + \frac{||x||^4 - ||x||^2}{||x - A(x)||^2} = 0. \tag{13.2}$$

Geometrically, $\rho(x)$ is the unique point where the ray starting from $A(x)$ and passing through x hits the boundary $S(X)$. Clearly, $\rho(x) = x$ if x belongs to this boundary, and so ρ is a retraction.

We will return to the equivalence stated in Theorem 13.1 below. As a first application we will state a result from [24] in the following Example 13.1. This result shows that in a separable Hilbert space one may find a retraction of the unit ball onto its boundary which is "invariant" with respect to certain finite dimensional subspaces. This fact is used in [24] to prove a Rabinowitz type saddle point theorem in critical point theory.

Example 13.1. Let X be an infinite dimensional separable Hilbert space which admits a representation as direct sum $X = U \oplus V$ with U being finite dimensional. Also let $W \neq \{0\}$ be a finite dimensional subspace of V. We claim that there exists a retraction $\rho : B(X) \to S(X)$ which satisfies the additional inclusion

$$\rho(B(U)) \subseteq S(U \oplus W).$$

To see this, we choose first a basis $\{e_1, \ldots, e_n\}$, say, in U, extend it first to a basis $\{e_1, \ldots, e_m\}$ $(m > n)$ in $U \oplus W$, and then to a Schauder basis $(e_n)_n$ in the whole Hilbert space X. In particular, by associating to each $x \in X$ its coordinates in this basis, we may identify X with the sequence space ℓ^2.

Consider now the fixed point free map $A : B(\ell^2) \to B(\ell^2)$ from Example 12.2. Since $x = (\xi_1, \ldots, \xi_n, 0, 0, \ldots) \in U$ implies $A(x) =$

$(\sqrt{1-||x||^2}, \xi_1, \ldots, \xi_n, 0, 0, \ldots) \in U \oplus \mathbb{R}e_{n+1}$, we see that $A(B(U)) \subseteq S(U \oplus W)$, provided that $W \subseteq V$ is *one-dimensional.* More generally, if W is k-dimensional, we simply replace A by its k-th iterate A^k and get $A^k(B(U)) \subseteq S(U \oplus W)$. Now, if $\rho : B(\ell^2) \to S(\ell^2)$ is the retraction constructed from A as above, the fact that $\rho(x)$ belongs to the subspace spanned by x and $A(x)$ implies that $\rho(B(U)) \subseteq S(U \oplus W)$ as well, and the assertion follows. □

In view of Theorem 13.1, the two characteristics

$$\begin{gathered} F(X) = \inf \{k > 0 : \text{ there exists a fixed point free map} \\ A : B(X) \to B(X) \text{ with } Lip(A) \leq k\} \end{gathered} \tag{13.3}$$

and

$$\begin{gathered} R(X) = \inf \{k > 0 : \text{ there exists a retraction} \\ \rho : B(X) \to S(X) \text{ with } Lip(\rho) \leq k\} \end{gathered} \tag{13.4}$$

have found some interest in the literature; we call (13.3) the *fixed point constant* and (13.4) the *retraction constant* of the space X. These constants give an idea, roughly speaking, how "well-behaved" the maps A and ρ arising in Theorem 13.1 may be.

Surprisingly, the calculation of the characteristic (13.3) is completely trivial: one has $F(X) = 1$ in each infinite dimensional space X, which is the best (or worst, depending on your point of view) value, by the classical Banach-Caccioppoli fixed point theorem. In fact, by the result from [59] mentioned above we have $F(X) < \infty$ in every infinite dimensional Banach space X. Now, if $A : B(X) \to B(X)$ satisfies $Lip(A) > 1$, we fix $\varepsilon \in (0, Lip(A) - 1)$ and consider the map $A_\varepsilon : B(X) \to B(X)$ defined by

$$A_\varepsilon(x) := x + \varepsilon \frac{A(x) - x}{Lip(A) - 1}.$$

A straightforward computation shows then that every fixed point of A_ε is also a fixed point of A, and that $Lip(A_\varepsilon) \leq 1 + \varepsilon$, hence $F(X) \leq 1 + \varepsilon$.

On the other hand, calculating or estimating the characteristic (13.4) is highly nontrivial and requires rather sophisticated individual constructions in each space X (see [18-20,22,23,35,42,53,58,63,81,82,87]). To cite a few sample results, one knows that $R(X) \geq 3$ in any Banach space, while $4.5 \leq R(X) \leq 31.45...$ if X is Hilbert. Moreover, the special upper estimates

$$R(\ell^1) < 31.64..., \quad R(c_0) < 35.18..., \quad R(L^1[0,1]) \leq 9.43...,$$
$$R(C[0,1]) \leq 23.31...$$

are known; a survey of such estimates and related problems may be found in the book [44] or, more recently, in [43].

Now let ϕ be some measure of noncompactness (for example, $\phi \in \{\alpha, \beta, \gamma\}$). Parallel to (13.3) and (13.4), we introduce the characteristics

$$\begin{aligned} F_\phi(X) = \inf\{k > 0 :\ & \text{there exists a fixed point free map} \\ & A : B(X) \to B(X) \text{ with } [A]_\phi \leq k\}, \end{aligned} \tag{13.5}$$

and

$$\begin{aligned} R_\phi(X) = \inf\{k > 0 :\ & \text{there exists a retraction} \\ & \rho : B(X) \to S(X) \text{ with } [\rho]_\phi \leq k\}. \end{aligned} \tag{13.6}$$

From Darbo's fixed point principle (Theorem 8.1) it follows immediately that $F_\phi(X) \geq 1$ in every infinite dimensional space X and $\phi \in \{\alpha, \beta, \gamma\}$. On the other hand, $F_\phi(X) \leq F(X)$, if ϕ is Lip-compatible, and so $F_\phi(X) = 1$ in every infinite dimensional space X, by what we have observed before. Similarly, $R_\phi(X) \leq R(X)$, whenever ϕ is a Lip-compatible measure of noncompactness.

We point out that the paper [84] is concerned with characterizing some classes of spaces X in which the infimum $F_\phi(X) = 1$ is actually *attained*, i.e., there exists a fixed point free ϕ-*nonexpansive* selfmap of $B(X)$. This is a nontrivial problem which is solved in Theorem 13.2 below.

In [83] it was shown that $R_\alpha(X) \leq 6$, $R_\gamma(X) \leq 6$, and $R_\beta(X) \leq 4 + \beta(B(X))$. (Recall that the lattice measure of noncompactness of the unit ball $B(X)$ is highly sensitive with respect to the geometry of the space X and has to be calculated in every space individually.) It was also shown in [83] that $R_\phi(X) \leq 4$ for separable or reflexive spaces X, and that even $R_\phi(X) \leq 3$ if X contains an isometric copy of ℓ^p with $p \leq (2 - (\log 3/\log 2))^{-1} \approx 2.41$. Observe that, by our construction of A through ρ in (13.1), we always have the estimate

$$1 = F_\phi(X) \leq R_\phi(X) \qquad (\phi \in \{\alpha, \beta, \gamma\}). \tag{13.7}$$

Later (see Theorem 13.3 below) we will discuss a class of spaces in which the inequality sign in (13.7) also turns into equality.

The following Theorem 13.2 [10] shows that the fixed point free map $A : B(X) \to B(X)$ may always been chosen with $[A]_\phi = 1$ and $[A]^-_\phi$ arbitrarily close to 1, see (1.23). This means, in particular, that the infimum in (13.5) is actually a minimum, and that fixed point free ϕ-nonexpansive operators may be chosen proper.

Theorem 13.2. *Let X be an infinite dimensional Banach space and $\varepsilon > 0$. Then there exists a fixed point free map $A : B(X) \to B(X)$ satisfying $[A]_\phi = 1$ and $[A]^-_\phi \geq 1 - \varepsilon$ for $\phi \in \{\alpha, \beta, \gamma\}$. Moreover, if X contains a complemented infinite dimensional subspace with a Schauder basis, it may be arranged in addition that $Lip(A) \leq 1 + \varepsilon$.*

We give now more precise estimates for the characteristic (13.6) in a special class of spaces. Recall that a Banach space X with Schauder basis $E = (e_n)_n$ is said to have a *monotone norm* (with respect to E) if

$$|\xi_k| \leq |\eta_k| \text{ for all } k \in \mathbb{N} \text{ implies } \left\|\sum_{k=1}^{\infty} \xi_k e_k\right\| \leq \left\|\sum_{k=1}^{\infty} \eta_k e_k\right\| \tag{13.8}$$

for all sequences $(\xi_k)_k$ and $(\eta_k)_k$ for which the two series on the right hand side of (13.8) converge.

For example, the sequence spaces ℓ^p $(1 < p < \infty)$ and c have a monotone norm with respect to the usual bases. Interestingly, one may characterize this property by an intrinsic property [10]: a basis $E = (e_n)_n$ on a Banach space X is *unconditional* (i.e., any rearrangement of E is also a basis) if and only if one may pass to an equivalent norm on X which is monotone with respect to E. This shows that there are many spaces which do not have a monotone norm with respect to any basis. For instance, no space with the so-called *Daugavet property* has a monotone norm [49,50], and even does not imbed into a space with a monotone norm. In particular, $C[0, 1]$ and $L^1[0, 1]$ (and all spaces into which they imbed) do not possess a monotone norm.

Theorem 13.3. *Let X be an infinite dimensional Banach space X whose norm is monotone with respect to some basis. Then the equality*

$$R_\phi(X) = 1 \qquad (\phi \in \{\alpha, \beta, \gamma\}) \tag{13.9}$$

holds true.

Theorem 13.3 is of course in sharp contrast to the fact that, as mentioned above, one has $R(X) \geq 3$ in *every* Banach space X. This shows that in many spaces it is possible to find retractions of the unit ball onto its boundary which are "almost ϕ-nonexpansive", but there are no retractions which are "almost nonexpansive". The problem whether or not there exists a space X in which actually $[\rho]_\phi = 1$ for some retraction $\rho : B(X) \to S(X)$ is still open.

Let us return now to the minimal displacement (12.2) of an operator $A : B(X) \to B(X)$. Taking into account the estimate (12.3), it seems reasonable to introduce the characteristic

$$\begin{aligned} D(X) = \inf \Big\{ \tfrac{k}{k\delta+1} :\ & k \geq 1,\ \delta \geq 0,\ \text{there exists an operator} \\ & A : B(X) \to B(X) \text{ with } \eta(A) > \delta \text{ and } Lip(A) \leq k \Big\}, \end{aligned} \tag{13.10}$$

which we call the *displacement constant* of X, and the parallel characteristic

$$\begin{aligned} D_\phi(X) = \inf \Big\{ \tfrac{k}{k\delta+1} :\ & k \geq 1,\ \delta \geq 0,\ \text{there exists an operator} \\ & A : B(X) \to B(X) \text{ with } \eta(A) > \delta \text{ and } [A]_\phi \leq k \Big\} \end{aligned} \tag{13.11}$$

for some Lip-compatible measure of noncompactness ϕ. Observe that the function $(k, \delta) \mapsto \frac{k}{k\delta+1}$ is increasing in k, but decreasing in δ. Clearly, for $\delta = 0$ the numbers (13.10) and (13.11) simply reduce to the characteristics (13.3) and (13.5), respectively. On the other hand, for $\delta > 0$ the estimate (12.3) shows then that $D_\phi(X) \geq 1$ in every Banach space X. Conversely, in [83] it was shown that, given any infinite dimensional space X, $k > 1$, and $\varepsilon > 0$, one may find $A : B(X) \to B(X)$ with $[A]_\phi \leq k$ and

$$\eta(A) \geq \frac{1}{2} - \frac{1}{k} - \varepsilon. \tag{13.12}$$

Since $\varepsilon > 0$ is arbitrary, this gives the upper estimate $D_\phi(X) \leq 2$. Moreover, in spaces X with the so-called "separable retraction property" (e.g., reflexive or separable spaces), the constant $\frac{1}{2}$ in (13.12) may be replaced by 1 for $\phi = \gamma$, and so one even has $D_\gamma(X) = 1$. A similar result holds for spaces X which contain an isometric copy of ℓ^p or c_0; in this case, one may also for $\phi = \alpha$ and $\phi = \beta$ replace the constant $\frac{1}{2}$ in (13.12) at least by $2^{(1-p)/p}$ and obtains $D_\alpha(X), D_\beta(X) \leq 2^{(p-1)/p}$.

However, we can do much better. From all maps occuring in our definitions, the retraction $\rho : B(X) \to S(X)$ is the most "powerful" map. In fact, each such retraction can be used to construct a continuous operator $A : B(X) \to B(X)$ with minimal displacement $\eta(A) = \delta < 1$ as close to 1 as we want:

Example 13.2. Given an infinite dimensional Banach space X, choose a retraction $\rho : B(X) \to S(X)$ with $Lip(\rho) < \infty$. Fix $\delta \in (0,1)$ and define $A : B(X) \to B(X)$ by

$$A(x) := \begin{cases} -\rho\left(\dfrac{x}{r}\right) & \text{if} \quad ||x|| \leq r, \\ -\dfrac{x}{||x||} & \text{if} \quad ||x|| > r, \end{cases} \tag{13.13}$$

where $r := 1 - \delta$. This operator satisfies $[A]_\phi \leq [\rho]_\phi / r = [\rho]_\phi/(1-\delta)$, because

$$\begin{aligned} \phi(A(M)) &= \max\left\{\phi(A(M \cap B_r(X))), \phi(A(M \setminus B_r(X)))\right\} \\ &\leq \max\left\{\phi\left(\rho(\tfrac{1}{r}M)\right), \phi\left([0,1] \cdot \tfrac{1}{r}M\right)\right\}. \end{aligned}$$

Moreover, since ρ is Lipschitz continuous, A is Lipschitz continuous as well. More precisely, we have the estimate

$$Lip(A) \leq \max\left\{\frac{Lip(\rho)}{r}, \frac{2}{r}\right\} = \frac{Lip(\rho)}{r} = \frac{Lip(\rho)}{1-\delta},$$

since $Lip(\rho) \geq 3$, as mentioned before. In fact, in case $||x|| < r < ||y||$, let $z \in S_r(X)$ be a convex combination of x and y and observe that

$$||f(x) - f(y)|| \leq \frac{Lip(\rho)}{r}(||x - z|| + ||z - y||) = \frac{Lip(\rho)}{r}||x - y||.$$

Using the shortcut $k := Lip(A)$ and $c := Lip(\rho)$, with A given by (13.13), we have, in particular,

$$\frac{k}{k\delta + 1} = \frac{1}{\delta + \frac{1-\delta}{c}} \to 1 \qquad (\delta \uparrow 1),$$

and so we get the surprising consequence that

$$D(X) = D_\phi(X) = 1$$

in *every* infinite dimensional normed space! This means in a sense that the estimate (12.3) in Theorem 12.2 becomes "arbitrarily sharp" in each space if $\eta(A)$ is sufficiently close to 1, even if we replace $[A]_\gamma$ by $Lip(A)$.

14. Nonlinear spectral theory

In view of the importance of spectral theory in functional analysis, operator theory and quantum mechanics, it is not surprising at all that various attempts have been made to define and study spectra also for *nonlinear operators*. A detailed account of nonlinear spectral theory and its applications may be found in the recent monograph [9]. In this final section we discuss a particularly interesting nonlinear spectrum which was introduced by Väth in [85,86] and is closely related to fixed point theory for condensing operators.

First some definition are in order. An operator $A : B_r(X) \to X$ is called 0-*epi* [39] if $A(x) \neq 0$ on $S_r(X)$ and, given any compact map $C : B_r(X) \to X$ which vanishes on $S_r(X)$, one may find a solution $x \in B_r(X)$ of the coincidence equation

$$A(x) = C(x). \tag{14.1}$$

More generally, if this equation still has a solution for every C satisfying $[C]_\phi \leq k$ for some measure of noncompactness ϕ, then F is called k-epi [80]. In this terminology, Schauder's fixed point principle states that the identity operator is 0-epi on every ball, while Darbo's fixed point principle states that the identity operator is k-epi on every ball for $k < 1$, see Theorem 8.1 and the following remark.

A trivial example of a k-epi map (for any k) is every scalar function $f : [-r, r] \to \mathbb{R}$ satisfying $f(-r)f(r) < 0$, by the intermediate value theorem (or, equivalently, Brouwer's fixed point theorem). It was an open problem for some time to find a Banach space X and a map A which is 0-epi on $B(X)$, but not k-epi for any $k > 0$. (The authors of [80] claim to give an example of this type, but their example does not have the required properties.) This problem was solved only quite recently by Furi [36]; we report Furi's example in a slightly more general setting in the following

Example 14.1. Let X be an infinite dimensional Banach space which has a monotone norm with respect to some basis. We claim that the operator A

from Example 1.6 is 0-epi on $B(X)$, but not k-epi for any $k > 0$. In fact, since A is a homeomorphism from $B(X)$ onto itself, the coincidence equation (14.1) is equivalent to the fixed point equation $x = A^{-1}(C(x))$, and this equation is solvable for all *compact* operators $C : B(X) \to X$ with $C(x) \equiv 0$ on $S(X)$, by Schauder's theorem.

However, there is no $k > 0$ such that A is k-epi on $B(X)$ for $\phi = \alpha$, say. To see this, suppose that A is k-epi on $B(X)$ for some $k > 0$, and let $\rho : B(X) \to S(X)$ be a retraction with $[\rho]_\alpha \le 1 + \varepsilon$, whose existence is guaranteed by Theorem 13.3. Fix $r \in (0, 1)$ such that $r(2+\varepsilon) < k$ and consider the operator

$$C(x) := \begin{cases} A(x) - A\left(r\rho(x/r)\right) & \text{if } \ ||x|| \le r, \\ 0 & \text{if } \ r < ||x|| \le 1. \end{cases}$$

This operator is continuous, since $\rho(\frac{x}{r}) = \frac{x}{r}$ for $x \in S_r(X)$, but not compact. Moreover, for each set $M \subseteq B_r(X)$ we have

$$\alpha(C(M)) \le \alpha(A(M)) + \alpha(A(r\rho(\tfrac{1}{r}M))) \le r\alpha(M) + r[\rho]_\alpha \alpha(M) \le (2+\varepsilon) r\alpha(M),$$

since $||A(x)|| = ||x||^2$. This shows that $[C]_\alpha \le (2+\varepsilon)r < k$. So, by assumption we find a solution x^* of equation (14.1). But this implies that $||x^*|| \le r$ and $A\left(r\rho(\frac{x^*}{r})\right) = 0$, hence $\rho(\frac{x^*}{r}) = 0$ which is impossible. □

Of course, an analogous reasoning shows that the operator A from Example 14.1 is 0-epi, but not k-epi on *every* ball $B_r(X)$.

Furi's example shows that being 0-epi on $B_r(X)$ and different from zero on $S_r(X)$ is not a stable property of a nonlinear operator. Therefore a more suitable property is the following. We say that an operator $A : B_r(X) \to X$ is *strictly epi* on $B_r(X)$ if A is k-epi on $B_r(X)$ for some $k > 0$ and, in addition, if

$$\text{dist }(0, A(S_r(X))) = \inf_{||x||=r} ||A(x)|| > 0. \tag{14.2}$$

Of course, condition (14.2) is stronger than just $A(x) \neq 0$ on $S_r(X)$, at least in infinite dimensional spaces. However, a remarkable theorem due to Väth [86] states that, whenever A is just 0-epi and satisfies $[A]^-_\alpha > 0$, where $[A]^-_\alpha$ denotes the lower α-norm (1.23) of A, then A is actually strictly epi. Indeed, from $[A]^-_\alpha > 0$ it follows that A maps closed bounded sets into closed sets. In particular, $A(S_r(X))$ is closed, and so (14.2) holds true whenever $A(x) \neq 0$ on $S_r(X)$. It is therefore not accidental that the operator A from Example 14.1 satisfies $[A]_\alpha^- = 0$, as we already observed in Example 1.6.

From Väth's theorem it follows, in particular, that being epi and being strictly epi is *equivalent* for operators of the form $A = I - K$ with $[K]_\alpha < 1$, i.e., for condensing perturbations of the identity. In fact, this is an immediate consequence of the simple estimate $[A]_\alpha^- \geq 1 - [K]_\alpha > 0$.

There is a perturbation result of "Rouché type" which illustrates the usefulness of the notion of strictly epi operators. Suppose that $A : B_r(X) \to X$ is strictly epi on $B_r(X)$, and $B : B_r(X) \to X$ is another operator which satisfies the two conditions

$$\sup_{||x||=r} ||B(x)|| < \inf_{||x||=r} ||A(x)||, \qquad [B]_\alpha < [A]_\alpha^-. \tag{14.3}$$

Then one may show that $A+B$ is strictly epi on $B_r(X)$ as well. In particular, if $\lambda I - A$ is strictly epi on $B_r(X)$ for some $\lambda \in \mathbb{K}$, and $\mu \in \mathbb{K}$ is sufficiently close to λ in the sense that

$$|\lambda - \mu| < \min \left\{ \frac{1}{r} \inf_{||x||=r} ||\lambda x - A(x)||, [\lambda I - A]_\alpha^- \right\}, \tag{14.4}$$

then $\mu I - A$ is also strictly epi on $B_r(X)$.

We are now ready for defining an important spectrum for nonlinear operators. Given a Banach space X over the field $\mathbb{K}$ and $A : X \to X$, we put

$$\rho(A) := \{\lambda \in \mathbb{K} : \lambda I - A \text{ is strictly epi on } B_\rho(X) \text{ for some } \rho > 0\} \tag{14.5}$$

and

$$\sigma(A) := \mathbb{K} \setminus \rho(A), \tag{14.6}$$

and call (14.6) the *Väth spectrum* of A ([85], see also [75]). The most important properties of this spectrum are summarized in the following

Theorem 14.2. *The Väth spectrum is $\sigma(A)$ closed and, in case $[A]_\alpha < \infty$, also bounded, hence compact. Moreover, for linear operators it coincides with the familiar spectrum.*

The closedness of the spectrum $\sigma(A)$ is essentially a consequence of the above Rouché type perturbation result which implies that, if $\lambda \in \rho(A)$ for some $\lambda \in \mathbb{K}$, then also $\mu \in \rho(A)$ for μ as in (14.4). If $A : B_r(X) \to X$ satisfies

$[A]_\alpha < \infty$, then the spectrum $\sigma(A)$ is bounded in the complex plane by the number $\frac{1}{r} \sup \{||A(x)|| : ||x|| \leq r\}$, hence compact.

The last assertion of Theorem 14.2 may be proved as follows. If $L : X \to X$ is linear and strictly epi on some ball $B_\rho(X)$, condition (14.2) implies that L has a trivial nullspace and thus is injective. But L is also surjective, as can easily be seen as follows. Suppose that there is some $y \in X$ which is not in the range $R(L)$ of L. Then $R(L)$ does not contain the whole ray $\{\mu y : \mu \in \mathbb{K}\}$ either. In particular, equation (14.1) (with $A = L$) has no solution for the compact operator $C(x) := \mathrm{dist}\,(x, S_\rho(X))y$ which clearly vanishes on $S_\rho(X)$. This contradicts the hypothesis on L to be strictly epi on $B_\rho(X)$.

Conversely, if L is a linear isomorphism on X, then clearly $[L]_\alpha^- > 0$, since X is infinite dimensional. Moreover, let $C : X \to X$ be any compact map satisfying $C(x) \equiv 0$ on $S_\rho(X)$ for some $\rho > 0$. Then the coincidence equation (14.1) is equivalent to the fixed point problem $x = L^{-1}C(x)$, and the latter equation has a solution in $B_\rho(X)$, by Schauder's fixed point theorem. From Väth's theorem it follows that L is strictly epi on $B_\rho(X)$.

Replacing L by $\lambda I - L$, we have proved that $\lambda \in \rho(L)$, with $\rho(L)$ given by (14.5), if and only if $\lambda I - L$ is a linear isomorphism, and so $\rho(L)$ and $\sigma(L)$ coincide with the familiar resolvent set and spectrum of L, respectively.

Example 14.2. Let us calculate the Väth spectrum for the nonlinear operator A from Example 1.6, i.e., $A(x) = ||x||x$. We claim that

$$\sigma(A) = \{0\}.$$

To see this, suppose first that $\lambda \neq 0$. Then $\lambda I - A : B_{|\lambda|/2}(X) \to X$ is injective and open with $0 \in (\lambda I - A)(B_{|\lambda|/2}(X))$ and $[\lambda I - A]_\alpha^- > 0$, and so $\lambda I - A$ is strictly epi on $B_{|\lambda|/2}(X)$, hence $\lambda \in \rho(A)$. On the other hand, the operator A itself cannot be strictly epi on any ball $B_\rho(A)$, as we have shown in Furi's Example 14.1. $\square$

Let us make some remarks on the definition of the spectrum (14.6) in connection with Example 14.2. In case of a bounded linear operator L we have $0 \in \rho(L)$ if and only if L is an isomorphism. So it seems to be a tempting idea to define the resolvent set and spectrum for nonlinear operators A in precisely the same way: just let $\lambda \in \rho(A)$ if $\lambda I - A$ is a homeomorphism on

X, and $\lambda \in \sigma(A)$ otherwise. However, this definition is too "naive" to be of any use. First, such a spectrum does not share any of the familiar properties with the usual spectrum, as simple examples show [9], and so there is no analogue to Theorem 14.2. Second, in contrast to Väth's spectrum (14.6), or similar spectra in the literature (see Chapters 6-7 in [9]), it does not admit significant applications. Third, such a definition does not take into account important *compactness properties*, as Väth's spectrum does. For instance, the operator A from Example 14.1 and 14.2 is a homeomorphism on *every* infinite dimensional space, and so quite "regular" from the viewpoint of continuity. Nevertheless, the fact that $[A]_\alpha^- = 0$ shows that A behaves badly from the viewpoint of compactness, and this is one of the reasons why 0 belongs to the Väth spectrum of this operator.

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References

[1] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, B. N. Sadovskij, *Condensing Operators* [in Russian], Itogi Nauki Tekhniki, **18** (1980), 185-250.

[2] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, A. E. Rodkina, B. N. Sadovskij, *Theory of Equations of Neutral Type* [in Russian], Itogi Nauki Tekhniki, **19** (1982), 55-126.

[3] R. R. Akhmerov, M. I. Kamenskij, A. S. Potapov, A. E. Rodkina, B. N. Sadovskij, *Measures of Noncompactness and Condensing Operators* [in Russian], Nauka, Novosibirsk 1986; Engl. transl.: Birkhäuser, Basel, 1992.

[4] A. Ambrosetti, *Un teorema di esistenza per le equazioni differenziali negli spazi di Banach*, Rend. Sem. Mat. Univ. Padova, **39** (1967), 349-361.

[5] A. Ambrosetti, *Proprietà spettrali di certi operatori lineari non compatti*, Rend. Sem. Mat. Univ. Padova, **42** (1969), 189-200.

[6] C. J. Amick, *Some remarks on Rellich's theorem and the Poincaré inequality*, J. London Math. Soc., 18, 1 (1978), 81-93.

[7] J. Appell, *Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator*, J. Math. Anal. Appl., **83, 1** (1981), 251-263.

[8] J. Appell, E. De Pascale, *Lipschitz and Darbo conditions for the superposition operator in non-ideal spaces*, Annali Mat. Pura Appl., **158** (1991), 205-217.

[9] J. Appell, E. De Pascale, A. Vignoli, *Nonlinear Spectral Theory*, de Gruyter, Berlin, 2004.

[10] J. Appell, N. A. Erzakova, S. Falcón Santana, M. Väth, *On some Banach space constants arising in nonlinear fixed point and eigenvalue theory*, Fixed Point Theory Appl., **4** (2004), 317-336.

[11] J. Appell, P. P. Zabrejko, *Nonlinear Superposition Operators*, Cambridge Univ. Press, Cambridge, 1990.

[12] J. Arias de Reyna, T. Domínguez Benavides, *On a measure of noncompactness in Banach spaces with Schauder basis*, Boll. Unione Mat. Ital., **7-A** (1993), 77-86.

[13] J. M. Ayerbe Toledano, T. Domínguez Benavides, G. López Azedo, *Measures of Noncompactness in Metric Fixed Point Theory*, Birkhäuser, Basel, 1997.

[14] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, M. Dekker, New York, 1980.

[15] J. Banaś, A. Martinón, K. Sadarangani, *Set quantities related to the Hausdorff distance in Banach spaces*, Indian J. Pure Appl. Math., **28, 10** (1997), 1421-1433.

[16] J. Banaś, L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math., **41** (2001), 13-23.

[17] Y. Benyamini, Y. Sternfeld, *Spheres in infinite dimensional normed spaces are Lipschitz contractible*, Proc. Amer. Math. Soc., **88, 3** (1983), 439-445.

[18] K. Bolibok, *Construction of a Lipschitzian retraction in the space* c_0, Ann. Univ. M. Curie-Skłodowska, **51** (1997), 43-46.

[19] K. Bolibok, *Minimal displacement and retraction problems in the space* l^1, Nonlinear Anal. Forum, **3** (1998), 13-23.

[20] K. Bolibok, K. Goebel, *A note on minimal displacement and retraction problems*, J. Math. Anal. Appl., **206, 1** (1997), 308-314.

[21] J. Caballero Mena, *Existencia y monotonía de soluciones de ecuaciones integrales no lineales sobre dominios acotados y no acotados*, Ph.D. thesis, Universidad de Las Palmas de Gran Canaria, 2005.

[22] D. Caponetti, G. Trombetta, *On proper k-ball retractions in the Banach space* $BC([0,\infty))$, Nonlin. Funct. Anal. Appl. (to appear).

[23] D. Caponetti, G. Trombetta, *An note on the measure of solvability*, Bull. Polish Acad. Sci. Math., **52, 2** (2004), 179-183.

[24] R. Chiappinelli, *Nonlinear homogeneous perturbation of the discrete spectrum of a self-adjoint operator and a new constraint saddle point theorem*, J. Convex Anal. (to appear).

[25] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto*, Rend. Sem. Mat. Univ. Padova, **24** (1955), 84-92.

[26] F. S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. Roumanie, **21** (1977), 259-262.

[27] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[28] J. Dieudonné, *Deux exemples d'équations differentielles*, Acta Sci. Math., **12 B** (1950), 38-40.

[29] T. Domínguez Benavides, personal communication, 2005.

[30] N. Dunford, J. T. Schwartz, *Linear operators I*, Int. Publ., Leyden, 1963.

[31] D. E. Edmunds, W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Math. Monographs, Oxford, 1987.

[32] N. A. Erzakova, *On the measure of noncompactness β of linear operators in spaces of summable function* [in Russian], VINITI **6133-82** (1982).

[33] N. A. Erzakova, *A measure of noncompactness* [in Russian], in: Approx. Methods Invest. Differ. Equ. Appl., Univ. of Kuybyshev (1982), 58-60.

[34] S. Falcón Santana, K. Sadarangani, *Convexity and reflexivity*, Matem. Vesnik, **52** (2000), 71-78.

[35] C. Franchetti, *Lipschitz maps and the geometry of the unit ball in normed spaces*, Arch. Math., **46, 1** (1986), 76-84.

[36] M. Furi, *Stably solvable maps are unstable under small perturbations*, Zeitschr. Anal. Appl., **21** (2002), 203-208.

[37] M. Furi, M. Martelli, *On the minimal displacement of points under α-Lipschitz maps in normed spaces*, Boll. Unione Mat. Ital., **IV-9** (1974), 791-799.

[38] M. Furi, M. Martelli, *On α-Lipschitz retractions of the unit closed ball onto its boundary*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., **57, 1** (1974), 61-65.

[39] M. Furi, M. Martelli, A. Vignoli, *On the solvability of nonlinear operator equations in normed spaces*, Annali Mat. Pura Appl., **128** (1980), 321-343.

[40] A. N. Godunov, *The Peano theorem in Banach spaces* [in Russian], Funkts. Anal. Prilozh., **9, 1** (1974), 59-60.

[41] K. Goebel, *On the minimal displacement of points under Lipschitzian mappings*, Pacific J. Math., **45** (1973), 151-163.

[42] K. Goebel, *A way to retract balls onto spheres*, J. Nonlin. Convex Anal., **2, 1** (2001), 47-51.

[43] K. Goebel, *On the problem of retracting balls onto their boundary*, Abstr. Appl. Anal., **2** (2003), 101-110.

[44] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[45] L. S. Gol'denshtejn, A. S. Markus, *On the measure of noncompactness of bounded sets and linear operators* [in Russian], Trudy Inst. Mat. Akad. Nauk Moldav. SSR Kishinjov, (1965), 45-54.

[46] A. I. Gusejnov, Kh. Sh. Mukhtarov, *Introduction to the Theory of Nonlinear Singular Integral Equations* [in Russian], Nauka, Moscow, 1980.

[47] V. I. Istrăţescu, *On a measure of noncompactness*, Bull. Math. Soc. Math. R. S. Roumanie, **16, 2** (1972), 195-197.

[48] R. K. Juberg, *The measure of noncompactness in L^p for a class of integral operators*, Indiana Univ. Math. J., **23** (1974), 925-936.

[49] V. M. Kadets, *Some remarks concerning the Daugavet equation*, Quaestiones Math., **19** (1996), 225-235.

[50] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc., **352** (2000), 855-873.

[51] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J., **8** (1941), 457-459.

[52] M. I. Kamenskij, *On the shift operator along the trajectories of equations of neutral type* [in Russian], Sbornik Rabot Asp. Teor. Funktsii Diff. Uravn., (1974), 19-24.

[53] T. Komorowski, J. Wośko, *A remark on the retracting of a ball onto a sphere in an infinite dimensional Hilbert space*, Math. Scand., **67, 2** (1990), 223-226.

[54] M. A. Krasnosel'skij, E. A. Lifshits, A. V. Sobolev, *Positive Linear Systems* [in Russian], Nauka, Moscow, 1985; Engl. transl.: Heldermann, Berlin, 1989.

[55] K. Kuratowski, *Sur les espaces complètes*, Fund. Math., **15** (1934), 301-335.

[56] R. W. Leggett, *Remarks on set contractions and condensing maps*, Math. Z., **132** (1973), 361-366.

[57] J. Leray, *Théorie des points fixes, indice total et nombre de Lefschetz*, Bull. Soc. Math. France, **87** (1959), 221-233.

[58] G. Lewicki, G. Trombetta, *Almost contractive retractions in Orlicz spaces*, Bull. Austral. Math. Soc., **68** (2003), 353-369.

[59] P. K. Lin, Y. Sternfeld, *Convex sets with the Lipschitz fixed point property are compact*, Proc. Amer. Math. Soc., **93, 4** (1985), 633-639.

[60] W. McLean, D. Elliot, *On the p-norm of the truncated Hilbert transform*, Bull. Austral. Math. Soc., **38** (1988), 413-420.

[61] E. Müller-Pfeiffer, *Spektraleigenschaften singulärer gewöhnlicher Differentialoperatoren*, Teubner, Leipzig, 1977.

[62] E. Nordgren, *Composition operators*, Canad. J. Math., **20** (1968), 442-449.

[63] B. Nowak, *On the Lipschitzian retraction of the unit ball in infinite dimensional Banach spaces onto its boundary*, Bull. Acad. Polon. Sci. Sér. Sci. Math., **27, 11** (1979), 861-864.

[64] R. D. Nussbaum, *The radius of the essential spectrum*, Duke Math. J., **37** (1970), 473-478.

[65] A. E. Rodkina, B. N. Sadovskij, *On the connectedness principle of Krasnosel'skij-Perov* [in Russian], Trudy Mat. Fak. Voronezh Gos. Univ., **4** (1971), 89-103.

[66] I. A. Rus, *Fixed point structures*, Rev. Anal. Numér. Théor. Approx., **28 (51)** (1986), 59-64.

[67] I. A. Rus, *On a general fixed point principle for* (θ, φ)*-contractions*, Studia Univ. Babeş-Bolyai Math., **34, 1** (1989), 65-70.

[68] I. A. Rus, *Fixed point theorems for* θ*-condensing mappings*, Studia Univ. Babeş-Bolyai Math., **35, 2** (1990), 71-80.

[69] B. N. Sadovskij, *On a fixed point principle* [in Russian], Funkts. Analiz Prilozh., **1 (2)** (1967), 74-76.

[70] B. N. Sadovskij, *Measures of noncompactness and condensing operators* [in Russian], Problemy Mat. Anal. Slozhnykh Sistem, **2** (1968), 89-119.

[71] B. N. Sadovskij, *On the local solvability of ordinary differential equations in Banach spaces* [in Russian], Probl. Mat. Anal. Slozhn. Sistem, **3** (1968), 232-243.

[72] B. N. Sadovskij, *Some remarks on condensing operators and measures of noncompactness* [in Russian], Trudy Mat. Fak. Voronezh Gos. Univ., **1** (1970), 112-124.

[73] B. N. Sadovskij, *Limit-compact and condensing operators* [in Russian], Uspekhi Mat. Nauk, **27** (1972), 81-146; Engl. transl.: Russian Math. Surveys, **27** (1972), 85-155.

[74] J. Saint-Raymond, *Une équation differentielle sans solution*, Publ. Math. Univ. P. et M. Curie (Paris VI), **46** (1981), 1-7.

[75] P. Santucci, M. Väth, *Grasping the phantom: a new approach to nonlinear spectral theory*, Annali Mat. Pura Appl., **180** (2001), 255-284.

[76] A. A. Sedaev, *The structure of certain linear operators* [in Russian], Mat. Issled., **5** (1970), 166-175.

[77] J. H. Shapiro, *The essential norm of a composition operator*, Ann. Math., **125** (1987), 375-404.

[78] W. Smith, *Composition operators between Hardy and Bergman spaces*, Trans. Amer. Math. Soc., **348** (1996), 2331-2348.

[79] C. A. Stuart, *The measure of noncompactness of some linear integral operators*, Proc. Royal Soc. Edinburgh, **A-71** (1973), 167-179.

[80] E. U. Tarafdar, H. B. Thompson, *On the solvability of nonlinear noncompact operator equations*, J. Austral. Math. Soc., **43** (1987), 103-114.

[81] G. Trombetta, *k-set contractive retractions in spaces of continuous functions*, Sci. Math. Japon., **59, 1** (2004), 121-128.

[82] A. Trombetta, G. Trombetta, *On the existence of (γ_p)-k-set contractive retractions in $L_p[0,1]$ spaces,* $1 \le p < \infty$, Sci. Math. Japon., **56, 2** (2002), 327-335.

[83] M. Väth, *On the minimal displacement problem of γ-Lipschitz maps and γ-Lipschitz retractions onto the sphere*, Zeitschr. Anal. Anw., **21, 4** (2001), 901-914.

[84] M. Väth, *Fixed point free maps of a closed ball with small measures of noncompactness*, Collectanea Math., **52, 2** (2001), 101-116.

[85] M. Väth, *The Furi-Martelli-Vignoli spectrum vs. the phantom*, Nonlin. Anal. TMA, **47** (2001), 2237-2248.

[86] M. Väth, *Coincidence points of functions pairs based on compactness properties*, Glasgow Math. J., **44** (2002), 209-230.

[87] J. Wośko, *An example related to the retraction problem*, Ann. Univ. M. Curie-Skłodowska, **45** (1991), 127-116.

[88] H.-K. Xu, *Diametrically contractive mappings*, Bull. Austral. Math. Soc., **70, 3** (2004), 463-468.

[89] K. Zhu, *Operator Theory in Function Spaces*, M. Dekker, New York, 1990.

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