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WEAK CONVERGENCE OF RESOLVENTS OF MAXIMAL MONOTONE OPERATORS AND MOSCO CONVERGENCE

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Abstract. We consider a function defined on a subset of a reflexive Banach space with some conditions, which is weaker than that of Bregman functions, and apply it to a resolvent of a maximal monotone operator. We obtain a weak convergence theorem with respect to a sequence of maximal monotone operators, which also implies weak convergence of a sequence of certain projections that generalizes the metric projections.

Key Words and Phrases: Monotone operator, resolvent, Mosco convergence, Bregman function, Bregman projection.

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1. INTRODUCTION

Let E be a reflexive Banach space and $\{C_n\}$ a sequence of weakly closed subsets of E. $\{C_n\}$ is said be Mosco convergent[12] to C_0 if it coincides with both the set of limit points of $\{C_n\}$ and the set of weak limit points of all subsequences of $\{C_n\}$.

In 1984, Tsukada[16] studied relations between Mosco convergence of a sequence of closed convex subsets and its associated sequence of metric projections, and pointwise convergence theorems were established. In 2003, Ibaraki, Kimura, and Takahashi[7] proved an analogous result to Tsukada's by using Alber's generalized projections. This result is extended[11] to that used the

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Bregman projection, whose theory was established by Bregman[3] and has been studied by many researchers.

On the other hand, finding zeros of maximal monotone operators is one of the most important problems since it can be applied to many problems such as optimization, variational inequalities, and so forth.

It is known that if the underlying space is a Hilbert space, then the strong limit of the resolvent $\{J_tx\}$ of a maximal monotone operator A as $t \to \infty$ belongs to zeros of A, where t is a positive real number and $J_t = (I + tA)^{-1}$. Reich[14] and Kido[8] proved analogous results for an maximal monotone operator on Banach spaces with different types of resolvent each other.

These results show that each type of resolvents of a maximal monotone operator is closely connected with the associated type of projections to a closed convex subset. Namely, Kido's theorem is related with the metric projection, whereas Reich's theorem is related with the Alber's generalized projection.

In this paper, we consider weaker conditions of the Bregman function and apply it to the resolvents used by Kido. We obtain a weak convergence theorem with respect to monotone operators. It also implies weak convergence of a sequence of projections that generalizes the metric projections.

2. Preliminaries

Throughout this paper, we will always consider a real Banach space. Its norm is denoted by $\|\cdot\|$ and the dual space of a Banach space E is denoted by E^* . The dual pair of $x \in E$ and $x^* \in E^*$ is denoted by $\langle x, x^* \rangle$.

Let E be a Banach space and $\{C_n\}$ a sequence of weakly closed subsets of E. We denote by s-Li_n C_n the set of limit points of $\{C_n\}$, that is, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \in C_n$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges strongly to x. We denote by w-Ls_n C_n the set of subsequential weak limit points of $\{C_n\}$, that is, $y \in \text{w-Ls}_n C_n$ if and only if there exists $\{y_{n_i}\}$ such that $y_{n_i} \in C_{n_i}$ for all $i \in \mathbb{N}$ and $\{y_{n_i}\}$ converges weakly to y, where $\{C_{n_i}\}$ is a subsequence of $\{C_n\}$.

Using these definitions, we define the Mosco convergence [12] of $\{C_n\}$. If C_0 satisfies that

$$\operatorname{s-Li}_n C_n = C_0 = \operatorname{w-Ls}_n C_n,$$

we say that $\{C_n\}$ is Mosco convergent to C_0 and write

$$C_0 = \operatorname{M-lim}_{n \to \infty} C_n.$$

Notice that the inclusion s-Li_n $C_n \subset$ w-Ls_n C_n is always true, so that the inclusion w-Ls_n $C_n \subset$ s-Li_n C_n implies the existence of M-lim_{$n\to\infty$} C_n . It is easy to see that the Mosco limit is always closed and, if each C_n is convex, M-lim_{$n\to\infty$} C_n is also convex. For more details, see [2].

Let A be a set-valued mapping of E into E^* . We say that A is monotone if it satisfies that

$$\langle x - y, x^* - y^* \rangle \ge 0$$

for all $x, y \in E$ and $x^*, y^* \in E^*$ with $x^* \in Ax$ and $y^* \in Ay$. A monotone operator A is said to be maximal if the graph of A is not a proper subset of the graph of any other monotone operator.

Let f be a function of E into $]-\infty, +\infty]$. A function f is said to be Gâteaux differentiable at a point x of the interior of the domain of f, or $x \in$ int dom f, if $f'(x,y) = \lim_{t\to 0} (f(x+ty)-f(y))/t$ exists for all $y \in E$. If f is proper, convex, lower semicontinuous, and Gâteaux differentiable at $x \in C \subset$ int dom f, then $f'(x, \cdot)$ is linear continuous function of E into \mathbb{R} . We denote it by $\nabla f(x) \in E^*$, that is, $\langle y, \nabla f(x) \rangle = f'(x,y)$ for $x \in C$ and $y \in E$.

For a proper lower semicontinuous convex function f of E into $]-\infty, +\infty]$, a subdifferential ∂f is defined by

$$\partial f(x) = \{x^* \in E : f(y) \ge \langle y - x, x^* \rangle + f(x) \text{ for all } y \in E\}$$

for all $x \in E$. It is an important result that ∂f is maximal monotone[15]. The function f^* on E^* defined by

$$f^*(y) = \sup_{x \in E} (\langle x, y \rangle - f(x))$$

is called the conjugate of f. It is known that, if f is a proper, lower semicontinuous, convex, and Gâteaux differentiable function on $C \subset E$, then $(\nabla f)^{-1} = \nabla f^*$. For more details, see, for example, [1].

Suppose that a Banach space E is reflexive. Let $f: E \to]-\infty, +\infty]$ be a lower semicontinuous convex and Gâteaux differentiable function such that int dom f is nonempty. Let C be a subset of int dom f. We consider the following conditions for the function f with the set C.

(1) f is strictly convex on C;

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- (2) The set $L^f_{\alpha}(x, C) = \{y \in C : D_f(y, x) \le \alpha\}$ is bounded for any $\alpha \ge 0$;
- (3) The set $R^f_{\alpha}(y, C) = \{x \in C : D_f(y, x) \le \alpha\}$ is bounded for any $\alpha \ge 0$,

where D_f is a Bregman distance[5], that is, $D_f(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x) \rangle$. The Bregman distance has the following important property called the three point identity[6]: for $x, y, z \in \text{int dom } f$, it follows that

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle.$$

In [4], Butnariu and Iusem assumed the total convexity of f with the condition (3) above and called it a Bregman function. Since the conditions (1) and (2) are implied from the total convexity of f, a Bregman function f satisfies all of our assumptions. We will call the function satisfying (1), (2), and (3) a Bregman function in a weaker sense.

The following theorem is a slightly modified version of the theorem proved by Otero and Svaiter[13]. It plays an important role to define the resolvent operator we deal with. See also [10].

Theorem 1 (Otero-Svaiter[13]). Let E be a reflexive Banach space, C a nonempty closed convex subset of E, and f a strictly convex, lower semicontinuous and Gâteaux differentiable function on C satisfying that $R^f_{\alpha}(y,C) =$ $\{x \in C : D_f(y,x) \leq \alpha\}$ is bounded for any $\alpha \geq 0$. Let A be a maximal monotone operator of E into E^* satisfying dom $A \subset C$ and $A^{-1}0 \neq \emptyset$. Then, for any $x \in C$ and $\lambda > 0$, there exists a unique element $y \in C$ such that $\nabla f(x) \in \nabla f(y) + \lambda Ay$.

3. Lemmas

In this section, we prove some lemmas used to show our main theorems. The following lemma guarantees well-definedness of the resolvent operator we define below.

Lemma 1. Let E be a reflexive Banach space, C a nonempty closed convex subset of E, $a \in C$, and f a Bregman function on C in a weaker sense. Let Abe a maximal monotone operator of E into E^* such that $A^{-1}0$ is nonempty. Suppose that $C - \text{dom } A \subset C$. Then, for each $\lambda > 0$ and $x \in C$, there exists a unique point $w \in \text{dom } A$ such that

$$x \in w + \nabla f^*(\lambda Aw + \nabla f(a)).$$

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Proof. Fix $\lambda > 0$ and $x \in C$ arbitrarily. Define a multivalued mapping B of E into E^* by

$$By = -A(x - y)$$

for all $y \in E$. Since A is maximal monotone, B is also a maximal monotone operator that satisfies

$$\operatorname{dom} B = \{x\} - \operatorname{dom} A \subset C - \operatorname{dom} A \subset C.$$

Let $w_0 \in A^{-1}0$. Then we have $0 \in -Aw_0 = -A(x - (x - w_0)) = B(x - w_0)$ and hence $B^{-1}0$ is nonempty. By Theorem 1, there exists a unique point $z \in C$ such that

$$\nabla f(a) \in \nabla f(z) + \lambda Bz$$

which is equivalent to that $\nabla f(a) \in \nabla f(z) - \lambda A(x-z)$. Putting $w = x - z \in \text{dom } A$, we have

$$\nabla f(a) \in \nabla f(x - w) - \lambda Aw,$$

or $\nabla f(x-w) \in \lambda Aw + \nabla f(a)$. Since $(\nabla f)^{-1} = \nabla f^*$, we obtain $x \in w + \nabla f^*(\lambda Aw + \nabla f(a))$, which completes the proof.

From this lemma, we obtain that a resolvent operator $(I + \nabla f^*(\lambda A + \nabla f(a)))^{-1}$ is defined as a single-valued mapping of C into dom A.

For a subset K of E, the indicator function i_K is defined by

$$i_K(x) = \begin{cases} 0 & \text{for } x \in K, \\ +\infty & \text{for } x \notin K \end{cases}$$

for all $x \in E$. If D is a nonempty closed and convex set, then i_K is a proper, lower semicontinuous and convex function with dom $i_K = K$. For such a set K, we consider the subdifferential ∂i_K of i_K . It is known that ∂i_K is a maximal monotone operator[15]. The following lemma shows the relation between the resolvent operator $(I + \nabla f^*(\lambda A + \nabla f(a)))^{-1}$ and the Bregman distance using the subdifferential of the indicator function.

Lemma 2. Let E be a reflexive Banach space, C a nonempty closed convex subset of E, $a \in C$, and f a Bregman function on C in a weaker sense. Let K be a nonempty closed convex subset of E such that $C - K \subset C$. Then, for any

 $\lambda > 0$ and $x \in C$, $w = (I + \nabla f^*(\lambda \partial i_K + \nabla f(a)))^{-1}x$ is a unique minimizer of $D_f(x - \cdot, a)|_K$, that is,

$$w = \operatorname*{argmin}_{y \in K} D_f(x - y, a).$$

Proof. Since $w = (I + \nabla f^*(\lambda \partial i_K + \nabla f(a)))^{-1}x$ is equivalent to

$$\frac{1}{\lambda}(\nabla f(x-w) - \nabla f(a)) \in \partial i_K w,$$

we have $w \in K$ and

$$\langle y - w, \nabla f(x - w) - \nabla f(a) \rangle \le 0$$
, for all $y \in K$.

Using the three point identity, we have

$$D_f(x - y, a) - D_f(x - w, a) = D_f(x - y, x - w)$$
$$+ \langle w - y, \nabla f(x - w) - \nabla f(a) \rangle$$
$$\geq - \langle y - w, \nabla f(x - w) - \nabla f(a) \rangle$$
$$\geq 0$$

for all $y \in K$, and hence w is a minimizer of $D_f(x - \cdot, a)|_K$. Uniqueness is easily obtained from strict convexity of f.

Let K be a nonempty closed convex subset of E such that $C - K \subset C$ and $a \in C$. We define a projection $P_K^{f,a}$ of C onto K by

$$P_K^{f,a}x = \operatorname*{argmin}_{y \in K} D_f(x - y, a) = (I + \nabla f^*(\partial i_K + \nabla f(a)))^{-1}x$$

for all $x \in C$.

The following lemma is an essential part of our main theorems.

Lemma 3. Let *E* be a reflexive Banach space, *C* a nonempty closed convex subset of *E*, and *f* a Bregman function on *C* in a weaker sense. Let *a* and *x* be points of *C*. Suppose that a sequence $\{A_n\}$ of maximal monotone operators satisfies $C - \text{dom } A_n \subset C$ and $A_n^{-1}0 \neq \emptyset$ for all $n \in \mathbb{N}$. For a sequence $\{\lambda_n\}$ of positive real numbers, we let

$$x_n = (I + \nabla f^* (\lambda_n A_n + \nabla f(a)))^{-1} x$$

for all $n \in \mathbb{N}$. If $C_0 = \text{M-lim}_{n\to\infty} A_n^{-1}0$ is not empty, then $\{x_n\}$ is bounded. Further, if all subsequential weak limit points of $\{x_n\}$ belong to C_0 , then $\{x_n\}$ converges weakly to $P_{C_0}^{f,a}x$.

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Proof. From the definition of $\{x_n\}$, we have

$$\nabla f(a) - \nabla f(x - x_n) \in -\lambda_n A_n x_n$$

for each $n \in \mathbb{N}$. Thus there exists $w_n^* \in A_n x_n$ such that

$$\nabla f(a) - \nabla f(x - x_n) = -\lambda_n w_n^*.$$

Fix $u \in C_0$ arbitrarily. Then, since $C_0 = \text{M-lim}_{n\to\infty} A_n^{-1}0$, there exists a sequence $\{u_n\} \subset E$ such that $u_n \in A_n^{-1}0$ for each $n \in \mathbb{N}$ and that $\{u_n\}$ converges strongly to u. Using the three point identity and monotonicity of each A_n , we have

$$D_f(x - x_n, a) = D_f(x - u_n, a) - D_f(x - u_n, x - x_n)$$
$$+ \langle x_n - u_n, \nabla f(a) - \nabla f(x - x_n) \rangle$$
$$\leq D_f(x - u_n, a) + \langle x_n - u_n, -\lambda_n w_n^* \rangle$$
$$= D_f(x - u_n, a) - \lambda_n \langle x_n - u_n, w_n^* - 0 \rangle$$
$$\leq D_f(x - u_n, a)$$

for each $n \in \mathbb{N}$. Since $\{D_f(x - u_n, a)\}$ converges to $D_f(x - u, a)$, it follows that $\{D_f(x - x_n, a)\}$ is bounded. Hence we have $\{x - x_n\}$ is bounded and so is $\{x_n\}$.

Suppose that all subsequential weak limit points of $\{x_n\}$ belong to C_0 . Then, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to x_0 , we have $x_0 \in C_0$. Since $D_f(\cdot, a)$ is a weakly lower semicontinuous function, we have

$$D_f(x - x_0, a) \le \liminf_{i \to \infty} D_f(x - x_{n_i}, a) \le \liminf_{i \to \infty} D_f(x - u_{n_i}, a) = D_f(x - u, a)$$

Since $u \in C_0$ is arbitrarily chosen, x_0 is the unique minimizer of $D_f(x - \cdot, a)$ on C_0 . Hence we have $x_0 = P_{C_0}^{f,a} x$, which implies that $\{x_n\}$ converges weakly to $P_{C_0}^{f,a} x$.

4. Main results

Using the lemmas in the previous section, we show the following weak convergence theorem of a sequence of resolvents of maximal monotone operators.

Theorem 2. Let E be a reflexive Banach space, C a nonempty closed convex subset of E, and f a Bregman function on C in a weaker sense which satisfies that f is bounded on any bounded set of dom f. Let a and x be points of

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C. Suppose that a sequence $\{A_n\}$ of maximal monotone operators satisfies $C-\operatorname{dom} A_n \subset C \text{ and } A_n^{-1} 0 \neq \emptyset$ for all $n \in \mathbb{N}$. Let $\{\lambda_n\}$ be a sequence of positive real numbers with $\lim_{n\to\infty} \lambda_n = \infty$ and $x_n = (I + \nabla f^*(\lambda_n A_n + \nabla f(a)))^{-1}x$ for all $n \in \mathbb{N}$. Suppose

$$C_0 = \operatorname{M-lim}_{n \to \infty} A_n^{-1} 0 \neq \emptyset$$

and

w-Ls
$$A_n^{-1}y_n^* \subset C_0$$

for any $\{y_n^*\} \subset E^*$ converging to 0. Then $\{x_n\}$ converges weakly to $P_{C_0}^{f,a}x$.

Proof. From Lemma 3, it is sufficient to show that any weak subsequential limit of $\{x_n\}$ is a point of C_0 . From the definition of $\{x_n\}$, we have

$$x_n \in A_n^{-1}\left(\frac{\nabla f(x-x_n) - \nabla f(a)}{\lambda_n}\right)$$

for $n \in \mathbb{N}$. Since f is bounded on any bounded set of C, ∇f is also bounded on any bounded set. Therefore a sequence $\{(\nabla f(x-x_n) - \nabla f(a))/\lambda_n\}$ converges strongly to 0 since $\{x_n\}$ is bounded by Lemma 3. Let $y_n^* = (\nabla f(x-x_n) - \nabla f(a))/\lambda_n$ for all $n \in \mathbb{N}$. Then, using the assumption, we have $y_n^* \to 0$ strongly and that the all weak subsequential limits of $\{x_n\}$ are contained by C_0 . Hence $\{x_n\}$ converges weakly to $P_{C_0}^{f,a}x$.

We also obtain weak convergence of a sequence of projections to closed convex subsets of a reflexive Banach space.

Theorem 3. Let E be a reflexive Banach space, C a nonempty closed convex subset of E, and f a Bregman function on C in a weaker sense. Let a and x be points of C. Suppose that a sequence $\{K_n\}$ of nonempty closed convex subsets of C satisfies $C - K_n \subset C$ for all $n \in \mathbb{N}$ and $K_0 = \text{M-lim}_{n \to \infty} K_n \neq \emptyset$. Let

$$x_n = P_{K_n}^{f,a} x$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to $P_{K_0}^{f,a}x$.

Proof. Consider the sequence $\{i_{K_n}\}$ of indicator functions associated with $\{K_n\}$. Then, for each $n \in \mathbb{N}$, a subdifferential ∂i_{K_n} is a maximal monotone operator. It is easy to see that $(\partial i_{K_n})^{-1}0 = K_n$ for each $n \in \mathbb{N}$. Thus we

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have

$$\emptyset \neq K_0 = \operatorname{M-lim}(\partial i_{K_n})^{-1}0$$

We also have that dom $\partial i_{K_n} = K_n$ and hence $C - \operatorname{dom} \partial i_{K_n} \subset C$ for all $n \in \mathbb{N}$. Since

$$x_n = P_{K_n}^{f,a} x = (I + \nabla f^* (\partial i_{K_n} + \nabla f(a)))^{-1} x$$

for each $n \in \mathbb{N}$, we have $x_n \in K_n$ so that any subsequential weak limit point of $\{x_n\}$ is contained by w-Ls_n $K_n = K_0$. Therefore, by Lemma 3, we obtain that $\{x_n\}$ converges weakly to $P_{K_0}^{f,a}x$, which completes the proof. \Box

Suppose that a Banach space E is smooth, strictly convex and reflexive. Then, the function $f = \|\cdot\|^2/2$ is a Bregman function on E in a weaker sense and ∇f is the normalized duality mapping J of E onto E^* . In this case, a projection $P_K^{f,0}$ from E onto a nonempty closed convex subset K coincides with the metric projection P_K onto K. Indeed, for $x \in E$, $P_K x$ is a unique minimizer of $D_f(x - \cdot, 0)$ on K and

$$D_f(x-y,0) = \frac{1}{2} ||x-y||^2 - \frac{1}{2} ||0||^2 - \langle x-y-0, J(0) \rangle = \frac{1}{2} ||x-y||^2.$$

Letting $f = \|\cdot\|^2/2$, we obtain the following results shown in [9] and [16].

Corollary 1 (Kimura[9]). Let E be a strictly convex, reflexive, and smooth Banach space and J the duality mapping of E onto its dual E^* . Let $\{A_n\}$ be a sequence of maximal monotone operators of E into E^* and suppose

$$\operatorname{M-lim}_{n \to \infty} A_n^{-1} 0 = C_0 \neq \emptyset$$

and

w-Ls
$$A_n^{-1}y_n^* \subset C_0$$

for any sequence $\{y_n^*\} \subset E^*$ converging strongly to 0. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \lambda_n = \infty$. Then, for an arbitrary $x \in E$, a sequence of resolvents $x_n = (I + \lambda_n J^{-1} A_n)^{-1} x$ converges weakly to $P_{C_0}x$, where P_{C_0} is the metric projection onto C_0 .

Corollary 2 (Tsukada[16]). Let E be a smooth, reflexive, and strictly convex Banach space and C a nonempty closed convex subset of E. Let $\{K_n\}$ be a sequence nonempty closed convex subsets of C. If $K_0 = \text{M-lim}_{n\to\infty} K_n$ exists and nonempty, then $P_{K_n}x$ converges weakly to $P_{K_0}x$ for each $x \in C$.

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