DATA DEPENDENCE OF FIXED POINTS FOR MEIR-KEELER TYPE OPERATORS

GABRIELA PETRUȘEL
Department of Applied Mathematics
Babeș-Bolyai University Cluj-Napoca
Kogălniceanu 1, 400084 Cluj-Napoca, Romania
E-mail: gabip@math.ubbcluj.ro

Abstract. The purpose of this note is to present data dependence results for the fixed points of Meir-Keeler type operators.

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1. Preliminaries

Throughout this paper, the standard notations and terminologies in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let $(X, d)$ be a metric space and $f : X \to X$ be an operator. Then $f^0 := 1_X$, $f^1 := f$, $f^{n+1} = f \circ f^n$, $n \in \mathbb{N}$ denote the iterate operators of $f$. A sequence of successive approximations of $f$ starting from $x \in X$ is a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ with $x_0 = x$, $x_{n+1} = f(x_n)$, for $n \in \mathbb{N}$. By $F_f := \{x \in X| x = f(x)\}$ we will denote the fixed point set of the operator $f$.

Also we will use the following symbols:

$P(X) = \{Y \subset X| Y \text{ is nonempty}\}$, $P_d(X) := \{Y \in P(X)| Y \text{ is closed}\}$.

If $T : X \to P(X)$ is a multivalued operator, then $F_T := \{x \in X| x \in T(x)\}$ denotes the fixed point set of the $T$.

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The Pompeiu-Hausdorff generalized functional will be denoted by $H$. Also, the generalized functional $\delta$, is used in the main section of the paper, i.e. $\delta : P(X) \times P(X) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$, $\delta(A, B) = \sup \{d(a, b) \mid a \in A, \ b \in B\}$.

It is well-known that $(P_{b,cl}(X), H)$ is a complete metric space provided $(X, d)$ is a complete metric space.

For more details and basic results concerning the above notions see for example [3], [7] among others.

By definition, $f : X \rightarrow X$ is a Meir-Keeler operator if it satisfies the condition:

for all $\varepsilon > 0$ there is $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(f(x), f(y)) < \varepsilon$.

Meir-Keeler operators were introduced in order to generalize the well-known Banach contraction principle. In fact Meir and Keeler [6] proved the following result.

**Theorem 1.1.** Let $(X, d)$ be a complete metric space and $f : X \rightarrow X$ be a Meir-Keeler operator. Then $\text{Fix } f = \{x^*\}$ and for each $x \in X$ the sequence $(f^n(x))_{n \in \mathbb{N}}$ of successive approximations of $f$ starting from $x$ converges to the unique fixed point.

For more details about generalizations of contractive-type conditions see Kirk [4] or Rus [9].

A function $\varphi : [0, \infty] \rightarrow [0, \infty]$ is said to be an L-function if $\varphi(0) = 0$, $\varphi(s) > 0$ for all $s > 0$, and for every $s > 0$ there exists $u > s$ such that:

$$\varphi(t) \leq s \quad \text{for } t \in [s, u].$$

Recall also the concept of modulus of uniform continuity. If $(X, d)$ is a metric space and $f : X \rightarrow X$ is an operator then, by definition, the modulus of uniform continuity of $f$ is defined by:

$$\delta(\varepsilon) := \sup \{\lambda : d(x, y) < \lambda \Rightarrow d(f(x), f(y)) < \varepsilon\}, \quad \text{for } \varepsilon > 0 \text{ and } \delta(0) = 0.$$

In [5] T.C. Lim proved the following characterization theorem:

**Theorem 1.2.** (Lim [5]) Let $X$ be a metric space. Let $f : X \rightarrow X$ and let $\delta(\varepsilon)$ be its modulus of uniform continuity. Then the following assertions are equivalent:

$(i)$ $f$ is a Meir-Keeler operator

$(ii)$ $\delta(\varepsilon) > \varepsilon$, for each $\varepsilon > 0$
(iii) there exists a nondecreasing and right continuous $L$-function $\varphi$ such that $d(f(x), f(y)) < \varphi(d(x, y))$ for each $x, y \in X$, with $x \neq y$.

**Remark 1.3.** From the proof of the above theorem, it follows that the mapping $\varphi$ can be represented as follows: $\varphi(t) := \sup\{s : \frac{s + \delta(s)}{2} \leq t\}$, where $\delta(s)$ is the modulus of uniform continuity of $f$.

For the multivalued case, the following notion was introduced by Reich [8].

By definition, $F : X \to P_{cl}(X)$ is called a multivalued Meir-Keeler operator if

for all $\varepsilon > 0$ there is $\delta > 0$ such that $\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow H(F(x), F(y)) < \varepsilon$.

Lim also proved that the above characterization of Meir-Keeler operators remains true in the multivalued case (Theorem 2 in [5]).

In the main section of the paper, we will need the notion of $L$-space in the sense of Fréchet.

Let $X$ be a nonempty set and $s(X) := \{(x_n)_{n \in \mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$.

Let $c(X) \subset s(X)$ a subset of $s(X)$ and $\text{Lim} : c(X) \to X$ an operator. By definition the triple $(X, c(X), \text{Lim})$ is called an $L$-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}(x_n)_{n \in \mathbb{N}} = x$.

(ii) If $(x_n)_{n \in \mathbb{N}} \in c(X)$ and $\text{Lim}(x_n)_{n \in \mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i \in \mathbb{N}}$, of $(x_n)_{n \in \mathbb{N}}$ we have that $(x_{n_i})_{i \in \mathbb{N}} \in c(X)$ and $\text{Lim}(x_{n_i})_{i \in \mathbb{N}} = x$.

By definition an element of $c(X)$ is a convergent sequence and $x := \text{Lim}(x_n)_{n \in \mathbb{N}}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$.

In what follows we will denote an $L$-space by $(X, \to)$.

Actually, an $L$-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces (in Perov’s sense: $d(x, y) \in \mathbb{R}^m_+$, in Luxemburg-Jung’s sense: $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$, $d(x, y) \in K$, $K$ a cone in an ordered Banach space, $d(x, y) \in E$, $E$ an ordered linear space with a notion of linear convergence, etc.), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such $L$-spaces. For more details see Fréchet [2], Blumenthal [1].
2. Main results

Definition 2.1. Let $(X, \to)$ be an L-space. Then, $f : X \to X$ is called a weakly Picard operator if the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on $x$) is a fixed point of $f$. If the fixed point is unique, then $f$ is said to be a Picard operator.

Example 2.2. Let $(X, d)$ be a complete metric space and $f : X \to X$ an a-contraction, i.e. $a \in [0, 1]$ and $d(f(x), f(y)) \leq a \cdot d(x, y)$, for each $x, y \in X$. Then the operator $f$ is Picard. (Banach-Caccioppoli)

Example 2.3. Let $(X, d)$ be a complete generalized metric space $(d(x, y) \in \mathbb{R}^m)$ and $A \in M_{mm}(\mathbb{R}_+)$, such that, $A^n \to 0$ as $n \to \infty$. If $f : X \to X$ is an A-contraction, i.e., $d(f(x), f(y)) \leq Ad(x, y)$, for all $x, y \in X$, then it is Picard operator. (Perov)

Example 2.4. Let $(X, d)$ be a complete metric space and $f : X \to X$ be a Meir-Keeler type operator, i.e. for each $\eta > 0$ there exists $\delta > 0$ such that $x, y \in X, \eta \leq d(x, y) < \eta + \delta$ we have $d(f(x), f(y)) < \eta$. Then $f$ is a Picard operator. (Meir-Keeler)

In [10] the basic theory of Picard operators is presented. Some data dependence results for several classes of singlevalued and multivalued (weakly) Picard operators are proved in A. Petrușel [7], Rus [10], I. A. Rus and S. Mureșan [11], Rus, A. Petrușel and A. Sîntamărian [13] and Sîntamărian [14]. The data dependence of the fixed points for singlevalued and multivalued Meir-Keeler type operators was announced as an open problem in I. A. Rus, A. Petrușel and G. Petrușel [12]. A partial answer to this question is the main purpose of this note.

We start the main section of the paper by presenting a data dependence result for a singlevalued Meir-Keeler operator.

Theorem 2.5. Let $(X, d)$ be a complete metric space and $f_i : X \to X$ be Meir-Keeler operators, for $i \in \{1, 2\}$. Denote by $\varphi$ the mapping associated with $f_1$ from Lim’s theorem. Assume that:

(a) $t - \varphi(t) \to +\infty$, as $t \to +\infty$.

(b) there exists $\eta > 0$ such that $d(f_1(x), f_2(x)) \leq \eta$, for all $x \in X$.

Then $d(x_1^*, x_2^*) \leq t_\eta$, where $t_\eta := \sup\{t > 0 | t - \varphi \leq \eta\}$ and $x_i^*$ denotes the unique fixed point of $f_i$, for $i \in \{1, 2\}$.
Proof. Using Lim's characterization theorem, we have successively:
\[ d(x_1^*, x_2^*) = d(f_1(x_1^*), f_2(x_2^*)) \leq d(f_1(x_1^*), f_1(x_2^*)) + d(f_1(x_2^*), f_2(x_2^*)) < \varphi(d(x_1^*, x_2^*)) + \eta. \] Then \( d(x_1^*, x_2^*) \leq \eta. \) □

For the multivalued case we have:

**Theorem 2.6.** Let \((X, d)\) be a complete metric space and \(T_i : X \to P_{cl}(X)\) be multivalued Meir-Keeler operators, for \(i \in \{1, 2\}\). Denote by \(\varphi_i (i \in \{1, 2\})\) the corresponding mapping from Lim’s characterization result.

Suppose that:

i) there exists \(\eta_1, \eta_2 > 0\) such that 
\[ \delta(x, T_2(x)) \leq \eta_1, \text{ for all } x \in F_{T_1} \]
and 
\[ \delta(y, T_1(y)) \leq \eta_2, \text{ for all } y \in F_{T_2}. \]

Then \(H(F_{T_1}, F_{T_2}) \leq \eta + \max\{s_1(\eta + \frac{1}{v}), s_2(\eta + \frac{1}{u})\}\), where \(\eta := \max\{\eta_1, \eta_2\}\) and \(s_i(t)\) denotes the sum of the series \(\varphi_i^k(t)\).

**Proof.** The conclusion follows from the multivalued version of Lim’s characterization theorem for a multivalued Meir-Keeler operator and a slightly modified version of Theorem 2 from Sîntâmărian [14]. □

**Appendix.** For the convenience of the reader, we recall here the modified version of Theorem 2 in Sîntâmărian [14].

**Theorem 2.** Let \((X, d)\) be a complete metric space and \(T_i : X \to P_d(X)\) be a \(\varphi_i\)-contraction, for \(i \in \{1, 2\}\). Assume \(\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+\) is strictly increasing and \(\sum_{k=1}^{n} \varphi^k(t) < +\infty, \text{ for all } t \in \mathbb{R}_+\) and for \(i \in \{1, 2\}\).

Suppose that:

i) there exists \(\eta_1, \eta_2 > 0\) such that 
\[ \delta(x, T_2(x)) \leq \eta_1, \text{ for all } x \in F_{T_1} \]
and 
\[ \delta(y, T_1(y)) \leq \eta_2, \text{ for all } y \in F_{T_2}. \]

Then 

a) \(F_{T_1}\) and \(F_{T_2}\) are nonempty and closed
b) \( H(F_{T_1}, F_{T_2}) \leq \eta + \max\{s_1(\eta + \frac{1}{u}), s_2(\eta + \frac{1}{u})\} \), where \( \eta := \max\{\eta_1, \eta_2\} \) and \( s_i(t) \) denotes the sum of the series \( \varphi^k_i(t) \).

References


