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PSEUDOMETRIC VERSIONS OF THE CARISTI-KIRK FIXED POINT THEOREM

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Abstract. A fixed point conversion is given for the pseudometric variational principle in Tataru [18]. The obtained facts are then used to get an improved version of the fixed point result in Ray and Walker [15].

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1. Introduction

Let X be a nonempty set; and $(\leq)$, some *order* (i.e.: reflexive, transitive and antisymmetric relation) over it. Call $z \in X$, $(\leq)$-*maximal*, when

$$w \in X, z \leq w \Longrightarrow z = w \text{ (i.e.: } z < w \text{ is false, for each } w \in X). \tag{1.1}$$

Further, let us term the selfmap T of X, *progressive* in case

$$x \leq Tx, \text{ for all } x \in X \quad \text{(cf. Manka [10]).} \tag{1.2}$$

The connection between these concepts is the one stated in Bourbaki [2]:

(a) the Zorn maximality principle is deductible from a fixed point theorem involving progressive selfmaps

(b) under the conditions of the Zorn maximality principle, any progressive selfmap has fixed points.

This fact (referred to as the Bourbaki meta–theorem; in short: BMT) goes beyond the order context; precisely, it is equally applicable to *quasi-orders* (reflexive and transitive relations) or even to *pseudo-orders* (reflexive relations).

Despite its generality, BMT has not been used in a systematic manner until now. The situation is very well illustrated by what is hapening in the realm of *metrical* structures [where, by an intensive (ab)use, "maximality principle" turned into "variational principle"]. The following "temporal" diagram is an argument for our claim:

FIXED POINT THEOREM	VARIATIONAL PRINCIPLE
Caristi-Kirk (1975)	Ekeland (1974)
Ray-Walker (1982)	Zhong (1997)
■	Tataru (1992)

Although self-contained, it necessitates a lot of technical explanations. For the sake of completeness, we shall briefly discuss them in the sequel.

(i) Let (M, d) be a complete metric space; and $x \vdash \varphi(x)$, some function from X to $R_+ = [0, \infty[$ with

$$\varphi \text{ is lsc on } M \ (\varphi(x) \le \liminf_n \varphi(x_n), \text{ whenever } x_n \to x). \tag{1.3}$$

The following 1975 fixed point theorem by Caristi and Kirk [5] (in short: CK-FPT) is our starting point.

Theorem 1. *Let the selfmap T of M be such that*

$$d(x, Tx) \le \varphi(x) - \varphi(Tx), \quad \textit{for all } x \in M. \tag{1.4}$$

Then, T has at least one fixed point in M.

The original proof of this result is by transfinite induction; see also Wong [22]. Note that, in terms of the associated (to φ) order in M

$$(x, y \in M) \ \ x \le y \quad \text{iff } d(x, y) \le \varphi(x) - \varphi(y) \tag{1.5}$$

the selfmap in the statement is progressive. So, by BMT, this result is logically equivalent with the Zorn maximality principle subsumed to the order (1.5); i.e., with Ekeland's variatonal principle [7] Hence, the sequential type argument used by the quoted author to get his principle is also working in our precised setting; see also Pasicki [13]. A proof of Theorem 1 involving the chains of the structure $(M, \le)$ may be found in Turinici [19]; and its sequential translation has been developed in Dancs, Hegedus and Medvegyev [6]. Further aspects (involving the general case) may be found in Hyers, Isac and Rassias [8, ch 5]; see also Brunner [4], Taskovic [17] and Valyi [21].

(ii) Now, CK-FPT found (especially via Ekeland's aproach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. So, it must be not surprising that, soon after its formulation, many extensions of Theorem 1 were proposed. Here, we shall concentrate on the "functional" one obtained in 1982 by Ray and Walker [15] (referred to as the Ray-Walker fixed point theorem; in short: RW-FPT). Let the function $\varphi : M \to R_+$ be as in (1.3); and take some function $b : R_+ \to R_+$ with

$$b \text{ is decreasing and } \int_0^\infty b(\tau)d\tau = \infty. \tag{1.6}$$

Theorem 2. *Let the selfmap T of M be such that (for some $a \in M$)*

$$b(d(a,x))d(x,Tx) \leq \varphi(x) - \varphi(Tx), \quad \textit{for all } x \in M. \tag{1.7}$$

Then, conclusions of Theorem 1 are holding.
[As a matter of fact, the original result is with (1.6) substituted by

$$b \text{ is continuous, decreasing and } \int_0^\infty b(\tau)d\tau = \infty. \tag{1.8}$$

But, the authors' argument also works in this relaxed setting].

Clearly, Theorem 2 includes Theorem 1, to which it reduces when $b = 1$. On the other hand, the reciprocal inclusion also holds (cf. Park and Bae [12]). Summing up,

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 2} \quad \text{(from a logical viewpoint)}. \tag{1.9}$$

Hence, the "functional" extension of CK-FPT assured by RW-FPT has a technical significance only. This, however, may be sometimes useful for concrete applications; we refer to the quoted papers for details.

Strange enough, the variational principle attached–via BMT–to Theorem 2 was obtained fifteen years later by Zhong [23]. The argument used there is rather involved [and, naturally, it has nothing to do with the Park-Bae techniques]. In the light of these remarks, it might be possible to develop a simpler approach for deriving this result, by reducing it to Ekeland's variational principle; we shall discuss these facts in a separate paper.

(iii) It remains to say what hapens with the fixed point results which extend RW-FPT. Note that the lower left-hand corner in the above diagram is empty. It may be filled out by starting from the 1992 pseudometric variational principle in Tataru [18]; details will be given in Section 4 below. The basic tool of our investigation is a pseudometric maximality principle in Turinici [20], equivalent with the Brezis-Browder's one [3] (discussed in Section 2). The usefulness of such techniques is made clear in Section 5, where a technical extension of the RW-FPT is obtained, via normal functions (and the developments in Section 3). Further aspects will be discussed elsewhere.

2. Brezis-Browder ordering principles

Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) $(\leq)$ over M; as well as a function $x \vdash \varphi(x)$ from M to $R_+ = [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$- *maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w).$$

A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [3]:

Proposition 1. *Suppose that*

$$\textit{each ascending sequence in } M \textit{ has an upper bound} \tag{2.1}$$

$$\varphi \textit{ is } (\leq)\textit{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{2.2}$$

Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.

This principle includes the one due to Ekeland [7]. It found some useful applications to convex and nonconvex analysis (cf. the above references). So, it would be not without interest having "purely" pseudometric versions of Proposition 1. To get an appropriate statement in this direction, we need some conventions. Let again $(\leq)$ be a quasi-order on M; and $e : M \times M \to R_+$, some *pseudometric* over M, in the sense:

$$e \text{ is } \textit{reflexive} \quad (e(x, x) = 0, \text{ for all } x \in M).$$

Call the point $z \in M$, $(\leq, e)$-*maximal* if

$$u, v \in M \text{ and } z \leq u \leq v \text{ imply } e(u, v) = 0.$$

To see the usefulness of this concept, let us consider some particular cases. Firstly, note that, under the regularity condition

$$\begin{aligned}&e \text{ is } (\leq)\text{-triangular (for each } \varepsilon > 0 \text{ there exists } \delta > 0\\ &\text{such that } x \leq y \leq z \text{ and } e(x, y), e(x, z) \leq \delta \text{ imply } e(y, z) \leq \varepsilon)\end{aligned}$$

the $(\leq, e)$-maximal property may be written in the simplified form

$$w \in M \text{ and } z \leq w \quad \text{imply } e(z, w) = 0.$$

[Remark that, in general, this requirement is not comparable with

$$e \text{ is triangular } (e(x, z) \leq e(x, y) + e(y, z), \text{ for all } x, y, z \in M)$$

unless $(x, y) \vdash e(x, y)$ is *symmetric* ($e(x, y) = e(y, x)$, for all $x, y \in M$). But, in the following, this will be not accepted]. Secondly, if our pseudometric $e : M \times M \to R_+$ is *sufficient* [$e(x, y) = 0$ implies $x = y$], the $(\leq, e)$-maximal property becomes the familiar one of (1.1). So, existence results involving such points may be viewed as "metrical" versions of the Zorn maximality principle (cf. Moore [11, ch 4, Sect 4]). To get sufficient conditions for these, one may proceed as below. Call the (ascending) sequence (x_n) in M, *e-Cauchy*, when

$$\forall \varepsilon > 0, \exists n(\varepsilon) \text{ such that } n(\varepsilon) \leq p \leq q \Longrightarrow e(x_p, x_q) < \varepsilon;$$

and *e-asymptotic*, provided

$$e(x_n, x_{n+1}) \to 0, \quad \text{as } n \to \infty.$$

Clearly, each (ascending) e-Cauchy sequence is e-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions

$$\text{each ascending sequence is } e\text{-Cauchy} \tag{2.3}$$

$$\text{each ascending sequence is } e\text{-asymptotic} \tag{2.4}$$

are equivalent each other. [Since the proof is immediate, we do not give details].

The following answer to the posed question, obtained in Turinici [20] may be stated.

Proposition 2. *Let the pseudometric $e : M \times M \to R_+$ be such that (2.1) is valid, as well as either of the (mutually equivalent) conditions (2.3)/(2.4). Then, for each $u \in M$ there exists a $(\leq, e)$-maximal $v \in M$ with $u \leq v$.*

This result extends Proposition 1, to which it reduces when

$$e(x, y) = |\varphi(x) - \varphi(y)|, \quad x, y \in M \quad \text{(where } \varphi \text{ is the above one).}$$

Further enlargements of Proposition 2 were obtained in Altman [1]; see also Kang and Park [9]. However, all these are equivalent with the Brezis-Browder ordering principle (subsumed to Proposition 1); we refer to the paper by Turinici [op. cit.] for details.

3. Normal functions

Let $b : R_+ \to R_+$ be some function; call it *normal* in case

$$b \text{ is decreasing and strictly positive on } R_+ \tag{3.1}$$

$$B(\infty) = \infty, \text{ where } B(t) = \int_0^t b(\tau)d\tau, \quad t \geq 0. \tag{3.2}$$

[As a matter of fact, the strict positivity follows from (3.2); but this is not essential for us]. Assume that the function $\tau \vdash b(\tau)$ is endowed with such a property. Some basic facts involving the couple (b, B) are being collected in the lemma below. (The proof being evident, we omit the details).

Lemma 1. *The following are true*

$$sb(t+s) \leq B(t+s) - B(t) \leq sb(t), \text{ for all } t, s \in R_+ \tag{3.3}$$

$$\begin{array}{l}\tau \vdash B(\tau) \text{ is strictly increasing, continuous} \\ \text{and maps } R_+ \text{ onto itself; hence, so is } B^{-1}\end{array} \tag{3.4}$$

$$\begin{array}{l}\tau \vdash [B(\tau+t) - B(\tau+s)] \text{ is decreasing on } R_+, \\ \text{for all } t, s \in R_+ \text{ with } t \geq s\end{array} \tag{3.5}$$

$$B \text{ is sub-additive and } B^{-1} \text{ is super-additive (on } R_+). \tag{3.6}$$

Having these precised, take some nonempty set M; and let $d : M \times M \to R_+$ stand for a certain metric over it. Further, take some function $\Gamma : M \to R_+$ with

$$|\Gamma(x) - \Gamma(y)| \leq d(x, y), \text{ for all } x, y \in M \quad \text{(nonexpansive)}. \tag{3.7}$$

Define another map $e = e(B, \Gamma; d)$ from $M \times M$ to R_+ by

$$e(x, y) = B(\Gamma(x) + d(x, y)) - B(\Gamma(x)), \quad x, y \in M. \tag{3.8}$$

This may be viewed as an "explicit" formula; the implicit version of it is

$$d(x, y) = B^{-1}(B(\Gamma(x)) + e(x, y)) - \Gamma(x), \quad x, y \in M. \tag{3.9}$$

We shall be interested in establishing some qualitative/quantitative properties of the couple (d, e), useful for our developments

(i) First, we have to precise the "metrical" nature of the map $(x, y) \vdash e(x, y)$. An appropriate answer to this is contained in

Lemma 2. *The map* $e : M \times M \to R_+$ *is a sufficient triangular pseudometric (under the conventions in Section 2).*

Proof. The reflexivity is clear; as well as sufficiency. It remains to establish the triangular property. Let $x, y, z \in M$ be arbitrary fixed. The triangular property of $d : M \times M \to R_+$ and (3.4) give

$$e(x,z) = B(\Gamma(x) + d(x,z)) - B(\Gamma(x)) \leq$$
$$B(\Gamma(x) + d(x,y) + d(y,z)) - B(\Gamma(x)); \quad \text{wherefrom}$$

$$e(x,z) \leq B(\Gamma(x) + d(x,y) + d(y,z)) - B(\Gamma(x) + d(x,y))$$
$$+B(\Gamma(x) + d(x,y)) - B(\Gamma(x)).$$

On the other hand, the nonexpansiveness condition (3.7) gives

$$\Gamma(y) \leq \Gamma(x) + d(x,y); \quad \text{so, by (3.5) above}$$

$$B(\Gamma(x) + d(x,y) + d(y,z)) - B(\Gamma(x) + d(x,y))$$
$$\leq B(\Gamma(y) + d(y,z)) - B(\Gamma(y)).$$

Combined with a previous relation yields the desired conclusion. ■

In addition, the following properties of the couple (d, e) are available [via (3.3)+(3.6)]:

Lemma 3. *Under the above conventions, one has* $(\forall x, y \in M)$

$$b(\Gamma(x) + d(x,y)) \leq e(x,y) \leq b(\Gamma(x))d(x,y) \tag{3.10}$$

$$e(x,y) \leq B(d(x,y)) \quad [\text{ hence } B^{-1}(e(x,y)) \leq d(x,y) \text{ }]. \tag{3.11}$$

(ii) We now have two "parallel" (pseudo) metric structures on M; namely, the ones introduced by the *metric* $d : M \times M \to R_+$ and the *pseudometric* $e : M \times M \to R_+$. It is natural then to ask whether

(A) the specific d-metrical concepts are being transformed into (corresponding) e-metrical ones

(B) to what extent is the reciprocal implication retainable.

The basic metrical concepts to be considered are: *convergence*, *boundedness* and *Cauchy property*. Concerning the former of these, we must emphasize that the "natural" e-convergence structure over M is that introduced as

$$x_n \xrightarrow{e} x \text{ iff } e(x_n, x) \to 0 \text{ as } n \to \infty. \tag{3.12}$$

[Note that, another convergence structure to be considered is the "dual" of this one:

$$x \xleftarrow{e} x_n \text{ iff } e(x, x_n) \to 0 \text{ as } n \to \infty. \tag{3.13}$$

But, in the following, this will be not effectively used]. By Lemma 3, and the representation formulae (3.8)/(3.9), one derives at once

Lemma 4. *The following is true*

$$x_n \xrightarrow{d} x \text{ implies } x_n \xrightarrow{e} x \text{ (and } x \xleftarrow{e} x_n). \tag{3.14}$$

The reciprocal is not in general valid for the convergence structure (3.12) (but valid for its dual, (3.13)).

(iii) In the following, the transfer of sequential boundedness will be discussed. Let (x_n) be a sequence in M; we say that it is *e-bounded*, provided

$$e(x_i, x_j) \leq \mu, \text{ for all } i, j \text{ with } i < j \text{ and some } \mu \geq 0. \tag{3.15}$$

[Note that, this is distinct from the *strong* e-boundedness of (x_n), expressed as

$$e(x_i, x_j) \leq \nu, \text{ for all } i, j \text{ and some } \nu \geq 0; \tag{3.16}$$

because $(x, y) \vdash e(x, y)$ is not in general symmetric. But the d-counterparts of these are identical].

Lemma 5. *For each sequence (x_n) in M, one has*

$$(x_n) \text{ is } d\text{-bounded} \Longleftrightarrow (x_n) \text{ is } e\text{-bounded.} \tag{3.17}$$

In other words: the sequential boundedness is transferrable from d to e and viceversa.

Proof. The left to right implication is clear, in view of the evaluation (cf. Lemma 3)

$$e(x_i, x_j) \leq B(d(x_i, x_j)), \text{ for all ranks } i, j \tag{3.18}$$

and the regularity properties (3.4). So, it remains to prove the converse implication. By the implicit formula (3.9),

$$d(x_0, x_i) = B^{-1}(B(\Gamma(x_0)) + e(x_0, x_i)) - \Gamma(x_0) \leq$$
$$B^{-1}(B(\Gamma(x_0)) + \mu) - \Gamma(x_0), \quad \text{for all } i > 0.$$

This, along with the nonexpansiveness property (3.7), yields

$$0 \leq \Gamma(x_i) \leq \Gamma(x_0) + d(x_0, x_i) \leq B^{-1}(B(\Gamma(x_0)) + \mu), \text{ for all } i > 0.$$

Putting these facts together yields (again via (3.9))

$$d(x_i, x_j) = B^{-1}(B(\Gamma(x_i)) + e(x_i, x_j)) - \Gamma(x_i) \leq$$
$$B^{-1}(B(\Gamma(x_0)) + 2\mu), \quad \text{for all } i, j \text{ with } i < j;$$

and the conclusion follows. ■

(iv) We are now in position to discuss the transfer of the Cauchy property between these (pseudo) metric structures.

Lemma 6. *For each sequence* (x_n) *in* M

$$(x_n) \text{ is } d\text{-Cauchy} \Longleftrightarrow (x_n) \text{ is } e\text{-Cauchy.} \tag{3.19}$$

In other words; the Cauchy property is retainable in passing from d to e (and viceversa).

Proof. The left to right implication is (again) clear, by (3.18) and the regularity properties (3.4). It remains to establish the converse implication. Assume that the sequence (x_n) in M is e-Cauchy (cf. Section 2). In particular, this yields

$$(x_n) \text{ is } e\text{-bounded (hence } d\text{-bounded, via Lemma 5)}$$

if we take into account the triangular property of $(x,y) \vdash e(x,y)$ (cf. Lemma 2). As a consequence, the evaluation (3.15) is true (with d in place of e); wherefrom (cf. (3.7))

$$\begin{aligned} &\Gamma(x_i) \leq \Gamma(x_0) + d(x_0, x_i) \leq \Gamma(x_0) + \mu, \quad \text{for all } i \geq 0; \text{ hence} \\ &\Gamma(x_i) + d(x_i, x_j) \leq \Gamma(x_0) + 2\mu, \quad \text{for all } i, j \text{ with } i < j. \end{aligned}$$

Taking Lemma 3 into account, one derives (via (3.1))

$$b(\Gamma(x_0) + 2\mu)d(x_i, x_j) \leq e(x_i, x_j), \quad \text{for all } i, j \text{ with } i < j.$$

It suffices now remembering the working assumption about (x_n) to establish the desired implication. ■

(v) An immediate consequence of these developments is the following. Call the introduced pseudometric $e : M \times M \to R_+$, *complete* when each e-Cauchy sequence (x_n) in M is e-convergent to some $x \in M$. [Here, as precised, e-convergence is taken in the sense of (3.12)].

Lemma 7. *The following is valid:*

$$d \text{ is complete} \Longrightarrow e \text{ is complete.} \tag{3.20}$$

That is: the completeness may be transferred from d to e (but not viceversa).

Proof. Let (x_n) be an e-Cauchy sequence in M. By Lemma 6, (x_n) is d-Cauchy too; wherefrom, by hypothesis, $x_n \xrightarrow{d} x$, for some $x \in X$. This, along with Lemma 4, gives $x_n \xrightarrow{e} x$. Hence the conclusion. ■

In particular, when

$$\Gamma(x) = d(a, x),\ x \in M, \quad \text{for some } a \in M, \tag{3.21}$$

these developments are comparable with the ones in Suzuki [16]. This, however, is not the only possible choice for the function in question. Further aspects will be discussed elsewhere.

4. Main result

With these informations at hand, we may now return to the question of the introductory part. Let M be some nonempty set; and $e : M \times M \to R_+$, a triangular pseudometric over it (cf. Section 2). Further, let $\varphi : M \to R \cup \{\infty\}$ be some function with

$$\varphi \text{ is proper } (\mathrm{Dom}(\varphi) \neq \emptyset) \text{ and bounded below } (\inf[\varphi(M)] > -\infty). \tag{4.1}$$

For the last regularity condition to be used, we need a convention. Call the sequence (x_n) in M, *strongly* e-*asymptotic* in case

$$\text{the series } \sum_n e(x_n, x_{n+1}) \text{ converges (in the usual sense).}$$

We note the generic implication

$$\text{strongly } e\text{-asymptotic} \Longrightarrow e\text{-asymptotic [cf. Section 2];}$$

the reciprocal is not in general valid. The announced condition may now be phrased as

$$\begin{array}{l} M \text{ is } (e;\varphi)\text{-complete: for each strongly } e\text{-asymptotic sequence} \\ (x_n) \text{ in } \mathrm{Dom}(\varphi) \text{ with } (\varphi(x_n)) \text{ descending} \\ \text{there exists } x \in M \text{ with } x_n \stackrel{e}{\longrightarrow} x \text{ and } \lim_n \varphi(x_n) \geq \varphi(x). \end{array} \tag{4.2}$$

The following variational type result involving these data is available.

Theorem 3. *Let* $\gamma > 0$ *be arbitrary fixed, as well as* $u \in \mathrm{Dom}(\varphi)$. *There exists then* $v = v(\gamma; u) \in \mathrm{Dom}(f)$ *with*

$$\gamma e(u, v) \leq \varphi(u) - \varphi(v) \quad \textit{(hence } \varphi(u) \geq \varphi(v)\textit{)} \tag{4.3}$$

$$\gamma e(v, x) > \varphi(v) - \varphi(x), \quad \textit{whenever } e(v, x) > 0. \tag{4.4}$$

Proof. Denote for simplicity

$$M_u = \{x \in M; \varphi(x) \leq \varphi(u)\} \text{ (where } u \text{ is the above one).}$$

Clearly, $M_u \neq \emptyset$ (because it contains u). Let $(\leq)$ stand for the relation (over M_u)

$$(x, y \in M_u):\ \ x \leq y \text{ iff } \gamma e(x, y) \leq \varphi(x) - \varphi(y). \tag{4.5}$$

By the triangular property of the pseudometric $(x, y) \vdash e(x, y)$, it is clear that $(\leq)$ acts as a quasi-order on M_u. We now claim that the couple $(\leq, e)$ fulfils conditions of Proposition 2 over M_u. Let (x_n) be an ascending sequence in M_u:

$$\gamma e(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m), \quad \text{whenever } n \leq m. \tag{4.6}$$

The sequence $(\varphi(x_n))$ is descending and bounded from below in R; wherefrom, the series $\sum_n[\varphi(x_n) - \varphi(x_{n+1})]$ converges (in R). As a consequence of this, $\sum_n e(x_n, x_{n+1})$ converges too [i.e., (x_n) is strongly e-asymptotic]. So, by simply combining with a previous (generic) remark, one gets (2.4). Moreover, by (4.2), there must be some $y \in M$ with

$$x_n \xrightarrow{e} y \text{ and } \lim_n \varphi(x_n) \geq \varphi(y). \tag{4.7}$$

This firstly gives $\varphi(y) \leq \varphi(u)$ (i.e., $y \in M_u$); because $(x_n) \subseteq M_u$. Secondly, fix some rank n. By (4.6) and the triangular property of our pseudometric $e : M \times M \to R_+$,

$$\gamma e(x_n, y) \leq \gamma e(x_n, x_m) + \gamma e(x_m, y) \leq \varphi(x_n) - \varphi(x_m) + \gamma e(x_m, y), \forall m \geq n.$$

This, along with (4.7), gives by a limit process (relative to m)

$$\gamma e(x_n, y) \leq \varphi(x_n) - \lim_m \varphi(x_m) \leq \varphi(x_n) - \varphi(y) \ \ \text{(i.e.: } x_n \leq y).$$

As n was arbitrary, one deduces that y is an upper bound (modulo $(\leq)$) of (x_n); i.e., (2.1) holds too; hence the claim. By the quoted result, it then follows that, for the starting point $u \in M_u$ there exists $v \in M_u$ with

$$u \leq v \quad \text{and} \ \ v \text{ is } (\leq, e)\text{-maximal in } M_u.$$

The former of these is just (4.3) (by the very definition of $(\leq)$). And the latter one gives

$$x \in M_u, \gamma e(v, x) \leq \varphi(v) - \varphi(x) \Rightarrow e(v, x) = 0.$$

But, from this, (4.4) is clear. The proof is thereby complete. ■

In particular, the regularity condition (4.2) holds under

$$\begin{array}{l}\text{for each strongly } e\text{-asymptotic sequence } (x_n) \text{ in } \mathrm{Dom}(\varphi) \\ \text{there exists } x \in M \text{ with } x_n \xrightarrow{e} x \text{ and } \liminf_n \varphi(x_n) \geq \varphi(x).\end{array} \tag{4.8}$$

This shows that Theorem 3 also comprises the variational principle obtained by Tataru [18]. But, as explicitly shown in that paper, this principle generalizes Ekeland's [7]; hence, so does our statement. Further aspects may be found in Suzuki [16].

Let us now return to the setting of (4.2). Assume that (in addition to being transitive) the pseudometric $(x, y) \vdash e(x, y)$ is sufficient (cf. Section 2). The following fixed point result is directly obtainable via BMT (cf. Section 1) from Theorem 3 above.

Theorem 4. *Let the selfmap T of M be such that*

$$e(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad \textit{for all } x \in \mathrm{Dom}(\varphi); \tag{4.9}$$

and let $u \in \mathrm{Dom}(\varphi)$ be arbitrary fixed. There exists then $v = v(T; u)$ in $\mathrm{Dom}(\varphi)$ with

$$e(u, v) \leq \varphi(u) - \varphi(v) \textit{ and } v = Tv \tag{4.10}$$

$$e(v, x) > \varphi(v) - \varphi(x), \quad \textit{for all } x \in M \setminus \{v\}. \tag{4.11}$$

Proof. By the admitted hypotheses, Theorem 3 applies to these data. So, for the starting point $u \in \mathrm{Dom}(\varphi)$ there exists $v = v(T; u)$ in $\mathrm{Dom}(\varphi)$ fulfiling (4.3)+(4.4) [with $\gamma = 1$]. This, along with (4.9) (and the sufficiency of $(x, y) \vdash e(x, y)$) gives the desired conclusion. ∎

In particular, when $e : M \times M \to R_+$ is a complete metric over M, the key condition (4.2) is fulfilled under (103) [with e in place of d]. This tells us that Theorem 4 above includes the Caristi-Kirk fixed point theorem (cf. Section 1). The question of the reciprocal inclusion being also true remains open; we conjecture that the answer is negative. An interesting problem to be stated is that of the regularity conditions imposed upon the ambient pseudometric $e : M \times M \to R_+$ being minimal so as to retain the fixed point conclusion. For example, the reflexivity condition (cf. Section 2) may be removed without affecting in a significant way the relations (4.10) above. We shall discuss these facts in a separate paper.

5. An improved version of RW-FPT

The obtained fixed point statement may now be used to get an improved version of the Ray-Walker fixed point theorem (cf. Section 1). The basic elements of this are those developed in Section 3.

Let M be some nonempty set; and $d : M \times M \to R_+$, a complete metric over it. Further, let $b : R_+ \to R_+$ stand for some normal function (in the sense of (3.1)+(3.2)); and $\Gamma : M \to R_+$, a nonexpansive function (cf. (3.7)). Finally, take the function $\varphi : M \to R \cup \{\infty\}$ according to (4.1)+(13). The following fixed point result is available.

Theorem 5. *Let the selfmap T of M be such that*

$$B(\Gamma(x) + d(x,Tx)) - B(\Gamma(x)) \leq \varphi(x) - \varphi(Tx),\ \forall x \in \mathrm{Dom}(\varphi); \tag{5.1}$$

and let $u \in \mathrm{Dom}(\varphi)$ be arbitrary fixed. There exists then $v = v(T;u)$ in $\mathrm{Dom}(\varphi)$ with

$$B(\Gamma(u) + d(u,v)) - B(\Gamma(u)) \leq \varphi(u) - \varphi(v) \quad \text{and } v = Tv \tag{5.2}$$

$$B(\Gamma(v) + d(v,x)) - B(\Gamma(v)) > \varphi(v) - \varphi(x), \quad \text{for all } x \in M \setminus \{v\}. \tag{5.3}$$

Proof. Let $e = e(B,\Gamma;d)$ stand for the mapping (from $M \times M$ to R_+) associated to d under the model of (3.8)/(3.9); remember that (cf. Lemma 2) $(x,y) \vdash e(x,y)$ is a sufficient triangular pseudometric over M. We claim that Theorem 4 is applicable to (M,e) and φ. The only point to be clarified is that involving the key condition (4.2). So, let (x_n) be a strongly e-asymptotic sequence in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$ descending. By the very definition of this concept

$$(x_n) \text{ is } e\text{-Cauchy}; \quad \text{hence } d\text{-Cauchy (cf. Lemma 6).}$$

This, along with the d-completeness assumption, yields (for some $y \in M$)

$$x_n \xrightarrow{d} y; \quad \text{hence } x_n \xrightarrow{e} y \text{ and } \lim_n \varphi(x_n) \geq \varphi(y);$$

if we take Lemma 4, the assumption (103) and the descending property of $(\varphi(x_n))$ into account. Putting these together, (4.2) holds. And, from this, one derives all the conclusions we need. The proof is thereby complete. ■

Now, by Lemma 1, it is clear that a sufficient condition for (5.1) is

$$b(\Gamma(x))d(x,Tx) \leq \varphi(x) - \varphi(Tx), \quad \text{for all } x \in \mathrm{Dom}(\varphi). \tag{5.4}$$

This, along with the particular choice (3.21) for Γ, shows that Theorem 5 includes the Ray-Walker fixed point theorem. Consequently, the statement we just derived may be also viewed as an extension of the Caristi-Kirk fixed point theorem. A natural question is that of such an extension being or not effective. The answer is negative, in the sense

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 5 (from a logical viewpoint).} \tag{5.5}$$

The verification is, in fact, the one given by Park and Bae [12]. Hence, the improvement of CK-FPT assured by Theorem 5 is technical in nature. However, in different concrete situations, this may be useful to get variational interpretations of (5.3) above. Further aspects may be found in Penot [14] and the references therein.

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