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KY FAN'S BEST APPROXIMATION THEOREM AND APPLICATIONS

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Abstract. In this paper we give Ky Fan's best approximation theorem and then illustrate a few applications where we derive results in fixed point theory, approximation theory, and variational inequalities.

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The following result given in R^n is known as Ky Fan's Best Approximation Theorem [4].

Theorem 1. *Let C be a closed bounded convex subset of R^n and $f : C \to R^n$ a continuous function. Then there is a $y \in C$ such that*

$$\|y - fy\| = d(fy, C) \qquad (*)$$

where $d(x, C) = \inf\{\|x - y\| : y \in C\}$ for $x \in R^n$ but $x \notin C$.

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Proof. We note that y is a solution of $(*)$ if and only if y is a fixed point of $P \circ f$, where P is the metric projection onto C. Let $P : R^n \to C$ be the metric projection. Then P is a continuous function. Thus, $P \circ f : C \to C$ is a continuous function and has a fixed point in C by the Brouwer fixed point theorem. On the other hand, if $P \circ f$ has a fixed point say, $P \circ fy = y, y \in C$, then $\|y - fy\| = d(fy, C)$.

This theorem has application in fixed point theory ([1, 2]) and the Brouwer fixed point theorem is a special case. Let $f : C \to C$ be a continuous function, where C is a closed bounded convex subset of R^n. Then f has a fixed point. In this case $d(fy, C) = 0$ and therefore, $fy = y$. In case $f : C \to R^n$ is a continuous function and C is a closed bounded convex subset of R^n, then f has a fixed point provided the following condition is satisfied:

If $fy \neq y$, then the line segment $[y, fy]$ has at least two points of C.

In case the additional boundary condition $f(\partial C) \subset C$ is satisfied in Theorem 1, then f has a fixed point (see [1]).

Theorem 1 has applications in approximation theory. For example, a closed bounded convex subset, C, of R^n is a set of existence; that is, for each $x \in R^n, x \notin C$, there is a $y \in C$ such that $\|y-x\| = d(x, C)$. We define $f : C \to R^n$ by $fy = x$ for all $y \in C$. Then we have $\|y - fy\| = d(fy, C)$; that is, $\|y-x\| = d(x, C)$. Note that $\|y-x\| = d(x, C)$ is the same as $< x-y, y-z > \geq 0$ for all $z \in C$.

This theorem is also applicable in deriving results of variational inequalities [2, 9].

Theorem 2. *If C is a closed bounded convex subset of R^n , and $f : C \to R^n$ is a continuous function, then there is a $y \in C$ such that $< fy, x - y > \geq 0$, for all $x \in C$.*

Proof. Let $g = I - f$. Then $g : C \to R^n$ is a continuous function and, by Theorem 1, we have $\|y - gy\| = d(gy, C)$. Thus $\|y - gy\| \leq \|gy - x\|$ for all $x \in C$; that is, $< gy - y, y - x > \geq 0$. Hence, $< fy, x - y > \geq 0$ for all $x \in C$.

We need the following preliminaries [1,2,3].

Definition 1. *Let $f : X \to X$ be a map, where X is a Banach space. Then f is said to be nonexpansive if $\|fx - fy\| \leq \|x - y\|$ for all $x, y \in X$.*

A nonexpansive map is of Lipscitz class. In case $\|fx - fy\| \leq k\|x - y\|$, where $0 < k < 1$, then f is said to be a contraction map.

We state the following [12]

Theorem 3. *Let C be a closed convex subset of a Hilbert space H and $f : C \to H$ a nonexpansive map with $f(C)$ bounded. Then there is a $y \in C$ such that $\|y - fy\| = d(fy, C)$.*

It is easy to see that $P \circ f$ has a fixed point $y \in C$ and the result follows [12], where $P : H \to C$ is a metric projection (a proximity map) and is a nonexpansive map.

Note. If in Theorem 3 , $f : C \to C$, then f has a fixed point. The following theorem, proved independently by Browder [3], Gohde [6] and Kirk [10] is derived as a corollary from Theorem 3.

If C is a closed bounded convex subset of H and $f : C \to C$ is a nonexpansive map, then f has a fixed point.

If we take $B_r = \{x \in H | \|x\| \leq r\}$, a ball of radius r and center 0, in place of C in Theorem 3, then the following result holds [1].

A. Let $f : B_r \to H$ be nonexpansive map. Then there is a $y \in B_r$ such that $\|y - fy\| = d(fy, B_r)$.

   If $fy \in B_r$ for $y \in B_r$, then f has a fixed point. (This follows from A.)

B. In addition, if in A we have one of the following boundary conditions, then f has a fixed point. For $y \in \partial B_r$, where ∂B_r denotes the boundary of B_r.

   (i) $\|fy\| \leq \|y\|$
   (ii) $< y, fy > \leq \|y\|^2$
   (iii) $\|fy\| \leq \|y - fy\|$
   (iv) If $fy = ky$, then $k \leq 1$
   (v) $\|fy\|^2 \leq \|y\|^2 + \|y - fy\|^2$.

For example, if $< y, fy > \leq \|y\|^2$, then $\|fy\| \leq \|y\| = r$, implies that $fy \in B_r$. Hence, f has a fixed point.

Hartman and Stampacchia [8] proved the following interesting result of variational inequalities that is applicable in mathematical, physical and economic problems.

Theorem 4. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Then there is a $y \in C$ such that $< fy, x - y > \geq 0$ for all $x \in C$.*

The goal of the study of variational inequalities is to find a solution of the variational inequality problem (VIP).

Note that the variational inequality problem (VIP) has a solution if and only if $P(I - f) : C \to C$ has a fixed point.

A variant of Theorem 4 is given below, where a continuous monotone map is taken in Hilbert space.

Theorem 5. *Let C be a closed bounded convex subset of H and $f : C \to H$ a monotone, continuous map. Then there is a $y \in C$ such that $< fy, x - y > \geq 0$ for all $x \in C$.*

Recall that $f : C \to H$ is monotone if $< fx - fy, x - y > \geq 0$ for all $x, y \in C$. A very elegant proof of Theorem 5 is given by Granas [7] by using the KKM-map principle.

An application of Theorem 5 is to prove the following [7].

Theorem 6. *Let C be a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map. Then f has a fixed point.*

Proof. Since f is a nonexpansive map, it is continuous. Consider $g : C \to H$, where $g = I - f$. Then g is a continuous map. It is easy to see that g is monotone, that is, $< gx - gy, x - y > \geq 0$. In fact, $\|fx - fy\| \leq \|x - y\|$, so $\|fx - fy\|^2 \leq \|x - y\|^2 + \|gx - gy\|^2$.

Now,

$$\begin{aligned} \|fx - fy\|^2 &= \|(I - g)x - (I - g)y\|^2 \\ &= \|gx - gy\|^2 + \|x - y\|^2 - 2 < gx - gy, x - y > . \end{aligned}$$

But $\|fx - fy\|^2 \leq \|x - y\|^2 + \|gx - gy\|^2$ only if $< gx - gy, x - y > \geq 0$, that is, $g = I - f$ is monotone.

Now, g is continuous and monotone therefore by Theorem 5 there is a $y \in C$ such that $< gy, x - y > \geq 0$ for all $x \in C$, that is, $< y - fy, x - y > \geq 0$ for all $x \in C$. Since $f : C \to C$ so by taking $x = fy$ we get that $< y - fy, fy - y > \geq 0$, that is, $< y - fy, y - fy > \leq 0$. But $\|y - fy\|^2 \geq 0$ so $y = fy$.

The following is an application of Theorem 5 in [7].

Theorem 7. *A closed bounded convex subset C of a Hilbert space H is a set of existence, that is, for each $y \notin C$, there is an $x \in C$ such that $\|x-y\| = d(y, C)$.*

Recall that if C is closed convex subset of a Hilbert space H and $y \notin C$, then there is an $x \in C$ such that $< y - x, x - z >\geq 0$ for all $z \in C$; that is, x is a nearest point to y.

Proof. Let $f : C \to H$ be defined by $fz = z - y$ for all $z \in C$.

Then f is a continuous function and is monotone. It is easy to see that $< fz - fx, z - x >\geq 0$. Hence by Theorem 5, there is an $x \in C$ such that $< fx, w - x >\geq 0$ for all $w \in C$. Thus, $< x - y, w - x >\geq 0$ for all $w \in C$. That is, $< y - x, x - w >\geq 0$ for all $w \in C$ and x is nearest to y. It follows easily that x is unique.

In the end we discuss results on maximal elements in mathematical economics [2, 5].

The existence of maximal elements in mathematical economics is very useful in the study of fixed point theory, variational inequalities, approximation theory and complementarity problems.

Definition 2. *A binary relation F on a set C is a subset of $C \times C$ or a mapping of C into itself. It is written yFx or $y \in Fx$ to mean that y stands in relation F to x.*

Definition 3. *A maximal element of F is a point x such that no point y satisfies $y \in Fx$; that is, $Fx = \emptyset$.*

The following result is due to Ky Fan [5].

Theorem 8. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that (i) $x \notin Fx$, (ii) Fx is convex for each x, and (iii) F has an open graph.*

Then F has a maximal element.

The following example illustrates the application of the maximal elements to approximation theory.

C. Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Then there is a $y \in C$ such that $\|y - fy\| = d(fy, C)$.

Proof. Define F on C such that for each $x \in C$, $\|y - fx\| < \|x - fx\|$. Then by Theorem 8 F has a maximal element since $x \notin Fx$, F has an open graph and Fx is convex for each $x \in C$. Hence $Fx = \phi$ for some $x \in C$ and $\|x - fx\| \leq \|fx - y\|$ for all $y \in C$.

Now an application of approximation result is given to find a fixed point and in turn to find zero of a polynomial equation.

D. Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Let $x - rfx \in C$ for all $x \in C$ and for some $r > 0$. Then $fx = 0$ has a solution, that is, there is a $z \in C$ such that $fz = 0$.

Proof. Let $gx = x - rfx$ for all $x \in C$. Then $g : C \to R^n$ is a continuous function and therefore there is a $z \in C$ such that $\|z - gz\| = d(gz, C)$ by part C. Since for all $x \in C, gx \in C$, therefore g has a fixed point, say $gz = z$ and consequently, $z - rfz = z$, that is, $fz = 0$ since $r > 0$.

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