Fixed Point Theory, Volume 5, No. 1, 2004, 117-129
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

FIXED POINT THEOREMS FOR OPERATORS IN SPACES OF CONTINUOUS FUNCTIONS

E. DE PASCALE[1] AND P. P. ZABREIKO[2]

[1]Dipartimento di Matematica, Universitá della Calabria
87036 Rende (Cosenza), Italy
e-mail: e.depascale@unical.it

[2]Institute of Mathematics, National Academy of Sciences
220080 Minsk, Belarus & Belarus State University
220050, Minsk, Belarus
e-mail: zabreiko@mmf.bsu.unibel.by

Abstract. Let $C([0,T],X)$ be the Banach space of continuous functions defined in $[0,T]$ and taking values in a Banach space X and let Q be a positive linear operator acting on $C([0,T],\mathbb{R})$. In this article we deal with nonlinear operators A acting on a subset $\mathbf{M} \subset C([0,T],X)$ that satisfies the operator Lipschitz condition $\|Ax_1(t) - Ax_2(t)\| \leq Q\|x_1(t) - x_2(t)\|$ $(0 \leq t \leq T, x_1, x_2 \in \mathbf{M})$. We are interested in the case when A is a nonlinear integral operator and the operator Lipschiz coefficient Q is itself a linear integral operator. If the spectral radius of Q is strictly less than 1, we can use a generalized contraction principle to achieve the existence of an attractive fixed point for A. In particular we focus on the case when Q is a linear Volterra operator, where effective estimates for the spectral radius of Q are known.
Key Words and Phrases: Fixed points, iterative sequences, K-normed spaces, positive and nonnegative operators, normal cone, Lipschitz condition, contraction operators.
2000 Mathematics Subject Classification: 47H10, 45D05.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space, $C([0,T],\mathbb{R})$ and $C([0,T],X)$ the real Banach spaces of continuous functions defined in $[0,T]$ with values respectively in $\mathbb{R}$ and X. Assume that $\mathbf{M}$ is a closed subset in $C([0,T],X)$ and that an operator A acts in $\mathbf{M}$ ($A:\ \mathbf{M} \to \mathbf{M}$). The operator A is said to satisfy the generalized Lipschitz condition in $\mathbf{M}$ if

$$\|Ax_1(t) - Ax_2(t)\| \leq Q\|x_1 - x_2\|(t) \qquad (\,0 \leq t \leq T, \quad x_1, x_2 \in \mathbf{M}), \tag{1.1}$$

where Q is a nonnegative linear operator in $C([0,T],\mathbb{R})$. The operator Q is called the *operator Lipschitz coefficient* of the operator A. We remark that the right hand side of (1.1) is a continuous function obtained from the continuous function $\|x_1(s) - x_2(s)\|$ under the application of the operator Q, and evaluated in t.

Our main tool is the following generalized contraction principle (see, e.g., [1, 3, 4, 6, 9, 16]):

Theorem 1.1. *Let an operator* $A: \ \mathbf{M} \to \mathbf{M}$ *satisfy a generalized Lipschitz condition with operator Lipschitz coefficient* Q *such that its spectral radius* $\rho(Q) < 1$. *Then* A *has a unique fixed point* $x_* \in \mathbf{M}$. *The iterative sequence* $x_{n+1} = Ax_n$ *converges to* x_* *for any initial approximation* $x_0 \in \mathbf{M}$. *Moreover*

$$\|x_n(t) - x_*(t)\| \leq (I - Q)^{-1}Q^n\|Ax_0(t) - x_0(t)\| \quad (\,0 \leq t \leq T, \quad n = 0, 1, 2, \ldots). \tag{1.2}$$

The estimates (1.2) imply the following numerical ones for every $k \in (\rho(Q), 1)$

$$|||x_n - x_*||| = o(k^n) \quad \text{as} \quad n \to \infty$$

($|||\cdot|||$ denotes the sup-norm in the function spaces $C([0,T],X)$ and $C([0,T],\mathbb{R})$). For special types of operators Q, the following more general estimate

$$|||x_n - x_*||| = O(\rho(Q)^n) \quad \text{as} \quad n \to \infty$$

holds. For a detailed study of the estimates (1.2) see [15].

Under the assumptions of Theorem 1.1, the original norm may be replaced by an equivalent one such that A satisfies the usual Banach – Caccioppoli contraction principle. This is due to the possibility to replace the original norm in the space X with an equivalent one $\|\cdot\|_{(k)}$ such that $\|Q\|_{(k)} \leq k$ for each $k \in (\rho(Q), 1)$ (where $\|Q\|_{(k)}$ is the norm of the operator Q computed with respect to the new norm $\|\cdot\|_{(k)}$). If we introduce in $C([0,T],X)$ the norm

$$|||x|||_{(k)} = \max_{0 \leq t \leq T} \ \|x(t)\|_{(k)},$$

then the inequality (1.1) implies

$$|||Ax_1(t) - Ax_2(t)|||_{(k)} \leq \|Q\|_{(k)}\, |||x_1(t) - x_2(t)|||_{(k)} \leq k\, |||x_1(t) - x_2(t)|||_{(k)}.$$

Thus, in the new norm A satisfies the usual scalar Lipschitz condition with a numerical constant k. Moreover, if there exists a norm $\|\cdot\|_{(\rho(Q))}$ such that $\|Q\|_{(\rho(Q))} = \rho(Q)$, then $C([0,T],X)$ can be renormed in such a way that A satisfies the Lipschitz condition with constant $k = \rho(Q)$.

Recall (see [15]) that when $\rho(Q) < 1$ the family of norms

$$\|x\|_{(k)} = \sup_{n=0,1,2,\ldots} k^{-n}\, \|Q^n x\| \qquad (0 < k < \rho(Q))$$

satisfy the following properties:

(1) $\|x\| \leq \|x\|_{(k)} \leq \nu(k)\|x\|$ $(x \in X)$ where

$$\nu(k) = \sup_{n=0,1,2,\ldots} k^{-n}\, \|Q^n\| < \infty \qquad (0 < k < \rho(Q));$$

(2) $\|Qx\|_{(k)} \leq k\, \|x\|_{(k)}$ $(x \in X)$;

(3) the formula

$$\|x\|_{(\rho(Q))} = \sup_{n=0,1,2,\ldots} \rho(Q)^{-n}\, \|Q^n x\|$$

defines a norm in X (equivalent to the original one) if and only if the operator Q satisfies the condition $\|Q^n\| = O(\rho^n(Q))$ as $n \to \infty$; if this condition is satisfied then $\|Q\|_{(\rho(Q))} = \rho(Q)$.

The operator Q is called *infra-normaloid*, if the latter property is true. This property is true in the case of nonnegative operators in $C([0,T],\mathbb{R})$ with spectral radius strictly greater than zero; in particular, it is sufficient to assume that Q is indecomposable (see e.g., [8, 9, 12, 15, 17]).

Summarizing the previous analysis we can state the following result.

Theorem 1.2. *Let $A: \mathbf{M} \to \mathbf{M}$ satisfy the conditions of Theorem 1.1. If Q is an infra-normaloid operator then for every $k \in [\rho(Q), 1)$, there exists a norm in $C([0,T],X)$ equivalent to the original one and such that A satisfies the Lipschitz condition with constant k.*

In section 2 we study the case when the operator Lipschitz coefficient Q is a Volterra-like operator in Banach space $C([0,T],\mathbb{R})$. The spectral radius of such operators can be computed or estimated without difficulties in the more common cases. In particular, the results in [10] (see also [4]) are obtained from Theorem 1.1 using estimates of the spectral radius of Volterra operators. Moreover the results regarding the behavior of the iterates of Volterra operators give better estimates on the rate of convergence for successive approximations to the fixed point of the operator A.

In section 3, again using Theorem 1.1, we obtain a version of Theorem 2.2 holding for unbounded intervals.

In section 4 we study the case when the operator coefficient Q is a Fredholm-like operator in the Banach space $C([0,T],\mathbb{R})$. In this case, seemingly, it is not possible to give effective formulas for the spectral radius of the operator coefficients. However, we describe some simple results holding for special kernels.

The K-normed linear spaces theory is a natural framework for the results of this article: We recall some of the basic definitions in section 5 for the benefit of the reader. As a survey on this interesting topic see [16].

The theorems obtained in this paper cover and improve some results in [4, 7, 10, 11].

2. Volterra operator Lipschitz coefficients: The bounded interval

Theorems 1.1 and 1.2 can be used in the analysis of nonlinear integral equation of the type

$$x(t) = A\left(t, x(t), \int\limits_0^t K(t,s,x(s))\,ds\right).$$

In fact, if $A(t,u,v)$ satisfy a Lipschitz condition with respect to u, v and $K(t,s,u)$ with respect to u , than an integral linear operator of the following type

$$Q\xi(t) = c(t)\xi(t) + \int\limits_0^t k(t,s)\xi(s)\,ds \tag{2.1}$$

with a nonnegative coefficient $c(t)$ and a nonnegative kernel $k(t,s)$ is a useful operator Lipschitz coefficient. The literature on the spectral properties of this type of operators is very extensive. For example we use the following result from [18] (see also [13, 14]).

Theorem 2.1. *Let the operator (2.1) act in the space $C([0,T],\mathbb{R})$. Then its spectral radius satisfies the inequality*

$$\rho(Q) \le |||c||| + \lim_{b-a\to 0} \sup_{a\le t\le b} \int_a^t k(t,s)\,ds. \tag{2.2}$$

In particular,

$$\rho(Q) = |||c||| \tag{2.3}$$

provided that the linear integral operator K with the kernel $k(t,s)$ acts in $C([0,T],\mathbb{R})$ and is weakly compact.

We remark that the operator in 2.1 can act in the space $C([0,T],\mathbb{R})$ even if its summands (i.e. operators $\xi(t) \to c(t)\xi(t)$ and K) do not (see e.g. [2],page 38).

Theorems 1.1 and 2.1 imply the following result.

Theorem 2.2. *Let an operator $A:\ \mathbf{M} \to \mathbf{M}$ satisfy the generalized Lipschitz condition*

$$\|Ax_1(t) - Ax_2(t)\| \le c\|x_1(t) - x_2(t)\| + \int_0^t k(t,s)\|x_1(s) - x_2(s)\|\,ds \tag{2.4}$$

$$(0 \le t \le T, \quad x_1, x_2 \in \mathbf{M}),$$

and

$$c + \lim_{b-a\to 0} \sup_{a\le t\le b} \int_a^t k(t,s)\,ds < 1. \tag{2.5}$$

Then the assertion of Theorem 1.1 holds.

Now we consider some examples of operator Lipschitz coefficients of the type (2.1).

First, consider the operator

$$Q\xi(t) = c\xi(t) + u(t)\int_0^t \xi(s)\,ds \tag{2.6}$$

with a continuous function $u(t)$ $0 < t \le T$. This operator acts in the space $C([0,T],\mathbb{R})$ and is compact if

$$\lim_{t\to 0} tu(t) = 0.$$

Theorem 2.2 states that the condition

$$\|Ax_1(t) - Ax_2(t)\| \leq c\|x_1(t) - x_2(t)\| + u(t) \int_0^t \|x_1(s) - x_2(s)\| \, ds$$

$$(0 \leq t \leq T, \quad x_1, x_2 \in \mathbf{M}),$$

with $c < 1$ implies the existence the unique fixed point x_* of the operator A in $\mathbf{M} \subseteq C([0,T], X)$.

In particular, the case $u(t) = t^{-\alpha}$, $0 \leq \alpha < 1$ is considered in [10] and in [4].

In similar way we can consider the following operator

$$Q\xi(t) = c\xi(t) + u(t) \int_0^t v(s)\xi(s) \, ds \tag{2.7}$$

with continuous functions $u(t), v(t)$ $0 < t \leq T$. This operator acts in the space $C([0,T], \mathbb{R})$ and is compact if

$$\lim_{t \to 0} u(t) \int_0^t v(s) \, ds = 0.$$

Theorem 2.2 states that the condition

$$\|Ax_1(t) - Ax_2(t)\| \leq c\|x_1(t) - x_2(t)\| + u(t) \int_0^t v(s)\|x_1(s) - x_2(s)\| \, ds$$

$$(0 \leq t \leq T, \quad x_1, x_2 \in \mathbf{M}),$$

again with $c < 1$ implies the existence of a unique fixed point x_* of the operator A in $\mathbf{M} \subseteq C([0,T], X)$.

Now we consider the operator

$$Q\xi(t) = c\xi(t) + u(t) \int_0^t v(s)\xi(s) \, ds \tag{2.8}$$

with continuous functions $u(t), v(t)$ $0 < t \leq T$ satisfying the condition

$$\lim_{t \to 0} u(t) \int_0^t v(s) \, ds = k.$$

This operator acts in the space $C([0,T],\mathbb{R})$ and is not compact for k different from zero. It is easy to verify that

$$\lim_{b-a\to 0} \sup_{a\le t\le b} \int_a^t u(t)v(s)\,ds = k.$$

Consequently, applying Theorem 2.2 , we obtain that the condition

$$\|Ax_1(t)-Ax_2(t)\| \le c\|x_1(t)-x_2(t)\| + u(t)\int_0^t v(s)\|x_1(s)-x_2(s)\|\,ds$$

$$(0\le t\le T,\quad x_1,x_2\in \mathbf{M}),$$

with $c+k<1$ implies the existence of a unique fixed point x_* of the operator A in $\mathbf{M}\subseteq C([0,T],X)$. As example of this situation is given by the operator

$$Q\xi(t) = c\xi(t) + \frac{k}{t}\int_0^t \xi(s)\,ds$$

with $c+k<1$.

3. Volterra operator Lipschitz coefficients: The unbounded interval

Theorem 2.2 can be used in some cases (suitable acting conditions are required on A) to obtain a continuous solution defined in $[0,\infty)$. In fact we can apply this theorem for an interval $[0,\tau]\subset[0,\infty)$ with τ arbitrarily large.

If we are interested in continuous bounded solutions, Theorem 2.2 does not help. However, as remarked in [4], the technique used for a bounded interval can be modified in such a way, that the result of Theorem 2.2 holds in the unbounded case as well.

Theorem 3.1. *Let the operator (2.1) act in the space $BC([0,\infty),\mathbb{R})$, the space of bounded and continuous function defined in $[0,\infty)$. Then its spectral radius satisfies the inequality*

$$\rho(Q) \le |||c||| + \max\left\{ \lim_{b-a\to 0} \sup_{a\le t\le b} \int_a^t k(t,s)\,ds,\ \lim_{a\to\infty} \sup_{a\le t<\infty} \int_a^t k(t,s)\,ds, \right\}. \tag{3.1}$$

Theorems 1.1 and 3.1 imply the following result.

Theorem 3.2. *Let an operator* $A: \mathbf{M} \to \mathbf{M}$ *satisfy the generalized Lipschitz condition*

$$\|Ax_1(t) - Ax_2(t)\| \le c\|x_1(t) - x_2(t)\| + \int_0^t k(t,s)\|x_1(s) - x_2(s)\|\,ds \tag{3.2}$$

$$(0 \le t < \infty, \quad x_1, x_2 \in \mathbf{M}),$$

and

$$c + \max\left\{ \lim_{b-a\to 0} \sup_{a\le t\le b} \int_a^t k(t,s)\,ds,\ \lim_{a\to\infty} \sup_{a\le t<\infty} \int_a^t k(t,s)\,ds, \right\} < 1. \tag{3.3}$$

Then the assertion of Theorem 1.1 holds.

We again consider some examples of operator Lipschitz coefficients of the type (2.1).

The spectral radius of the operator (2.6) with a continuous function $u(t),\ 0 < t < \infty$, satisfying the conditions

$$\lim_{t\to 0} tu(t) = 0, \quad \lim_{t\to\infty} tu(t) = 0,$$

is given by the formula $\rho(Q) = c$. More generally the same expression holds for the spectral radius of the operator (2.7) with bounded continuous functions $u(t), v(t),\ 0 < t < \infty$, satisfying the conditions

$$\lim_{t\to 0} u(t) \int_0^t v(s)\,ds = 0, \quad \lim_{t\to\infty} u(t) \int_0^t v(s)\,ds = 0.$$

Theorem 3.2 states that the condition

$$\|Ax_1(t) - Ax_2(t)\| \le c\|x_1(t) - x_2(t)\| + u(t) \int_0^t v(s)\|x_1(s) - x_2(s)\|\,ds$$

$$(0 \le t < \infty, \quad x_1, x_2 \in \mathbf{M}),$$

again with $c < 1$ implies the existence of a unique fixed point x_* of the operator A in $\mathbf{M} \subseteq BC([0,\infty[, X)$.

We cannot use the function $u(t) = t^{-\alpha},\ 0 < \alpha < 1$ in (2.6). However the function $u(t) = t^{-\alpha}(1+t)^{-\beta},\ 0 < 1 - \alpha < \beta$, allows us to apply Theorem 3.1.

Now we consider the operator (2.8) with continuous functions $u(t), v(t),\ 0 < t < \infty$ satisfying the conditions

$$\lim_{t\to 0} u(t)\int_0^t v(s)\,ds = k_0, \quad \lim_{t\to\infty} u(t)\int_0^t v(s)\,ds = k_\infty.$$

This operator acts in the space $BC([0,\infty),\mathbb{R})$. It is easy to verify that

$$\lim_{b-a\to 0}\ \sup_{a\le t\le b}\int_a^t u(t)v(s)\,ds = k_0$$

and

$$\lim_{a\to\infty}\ \sup_{a\le t<\infty}\int_a^t u(t)v(s)\,ds \le k_\infty$$

Therefore, Theorem 2.2 states that the condition

$$\|Ax_1(t) - Ax_2(t)\| \le c\|x_1(t) - x_2(t)\| + u(t)\int_0^t v(s)\,\|x_1(s) - x_2(s)\|\,ds$$

$$(0 \le t < \infty, \quad x_1, x_2 \in \mathbf{M}),$$

with $c + \max k_0, k_\infty < 1$ implies the existence of a unique fixed point x_* of the operator A in $\mathbf{M} \subseteq BC([0,\infty), X)$. An example of this situation is given by the operator

$$Q\xi(t) = c\xi(t) + \frac{k}{t}\int_0^t \xi(s)\,ds$$

with $c + k < 1$.

We conclude this section with a glimpse on the possibility to sharpen, and extend to more general situations, the results obtained so far. For example, the estimates (2.2), (3.1) can be sharpened in the following one:

$$\rho(Q) \le \inf \max_{1\le j\le m}\ \sup_{t_{j-1}\le t<t_j}\left(c(t) + \int_{t_{j-1}}^t k(t,s)\,ds\right), \tag{3.4}$$

where the infimum is computed over all finite subdivisions $\{0 = t_0 < t_1 < \ldots < t_m = T\}$ of the interval $[0,T]$ (T finite or infinite). In this paper we prefer not to investigate this further.

4. Fredholm operator Lipschitz coefficients

Let Ω be a compact subset in a finite dimensional space. In analogy to the Volterra case, analysis of nonlinear integral equation of the type

$$x(t) = A\left(t, x(t), \int\limits_{\Omega} K(t, s, x(s))\, ds\right),$$

is reduced, under suitable hypotheses, to the study of spectral properties of an integral linear operator

$$Q\xi(t) = c(t)\xi(t) + \int\limits_{\Omega} k(t, s)\xi(s)\, ds \tag{4.1}$$

with a nonnegative coefficient $c(t)$ and a nonnegative kernel $k(t, s)$. Spectral properties of such operators are less known then in the Volterra case. We confine ourselves to present only the following simple results.

Theorem 4.1. *Let the operator (4.1) act in the space $C(\Omega, \mathbb{R})$. Then its spectral radius satisfies the inequality*

$$\rho(Q) \leq |||c||| + \rho(K) \tag{4.2}$$

where K is a linear integral operator with the kernel $k(t, s)$. In particular,

$$\rho(Q) = a + \rho(K) \tag{4.3}$$

provided that $c(t)$ is a constant: $c(t) = a$; moreover, the operator Q in this case is infra-normaloid provided that the operator K is compact.

In the case when the operator K acts in the space $C(\Omega, \mathbb{R})$, the proof of Theorem 4.1 is straightforward. In deed, formula (4.3) is a consequence. However, formula (4.2) is a consequence of formula (4.3) since $c(t) \leq a = |||c|||$ and $\rho(Q) \leq \rho(Q_0)$ where

$$Q_0\xi(t) = c\xi(t) + \int\limits_{\Omega} k(t, s)\xi(s)\, ds.$$

As remarked in section 2, the operator (4.1) can act in the space $C(\Omega, \mathbb{R})$ however its summands (i.e. operators $\xi(t) \to c(t)\xi(t)$ and K) do not (see s.g.[2]). Therefore the previous arguments cannot be used without restrictive hypotheses on the operator K. To avoid this difficulty we can consider the operator (4.1) as acting in the space $L_\infty(\Omega, \mathbb{R})$, where the above pathology cannot occur. To this purpose it is useful to remark that each operator (4.1) acting in the space $C(\Omega, \mathbb{R})$ also acts in the space $L_\infty(\Omega, \mathbb{R})$ and with the same spectral radius.

Theorem 4.1 is only apparently similar to Theorem 2.1. In fact formulas (2.2)-(2.3) allow us to compute or to estimate the spectral radius of operator (2.1) in terms of simple characteristics of the kernel $k(t,s)$. Meanwhile, to compute or to estimate $\rho(K)$ in the formulas (4.2)-(4.3), is not a simple problem. In practice we try to solve the equation

$$\lambda\xi(t) = \int\limits_\Omega k(t,s)\xi(s)\,ds$$

in the cone of nonnegative functions from $C(\Omega, \mathbb{R})$ (or $L_\infty(\Omega, \mathbb{R})$). If the kernel $k(t,s)$ is indecomposable and, for some $\lambda > 0$, there exists a nonzero nonnegative solution $\xi(t)$ then $\rho(K) = \lambda$. Sometimes we can find a nonzero nonnegative function $\xi \in C(\Omega, \mathbb{R})$ such that

$$\int\limits_\Omega k(t,s)\xi(s)\,ds \le \lambda\xi(t).$$

In this case , if the kernel $k(t,s)$ is indecomposable, we have the estimate $\rho(K) \le \lambda$. More detailed informations about these problems can be found in [17] and in the subsequent literature.

There are some simple cases where we can calculate the spectral radius of K directly. For example, if $k(t,s) = u(t)v(s)$ (the so-called one-dimensional kernel) then

$$\rho(Q) = \int\limits_\Omega u(s)v(s)\,ds.$$

A similar formula can be written in the case when the kernel $k(t,s)$ is degenerated (or, in other words, the operator K is finite-dimensional).

Theorems 1.1 and 4.1 imply the following result.

Theorem 4.2. *Let an operator* $A:\ \mathbf{M} \to \mathbf{M}$ *satisfy the generalized Lipschitz condition*

$$\|Ax_1(t) - Ax_2(t)\| \le c\|x_1(t) - x_2(t)\| + \int\limits_\Omega k(t,s)\|x_1(s) - x_2(s)\|\,ds \qquad (4.4)$$

$$(0 \le t \le T, \quad x_1, x_2 \in \mathbf{M}),$$

and

$$c + \rho(K) < 1. \tag{4.5}$$

Then the thesis of Theorem 1.1 holds.

A particular case of this theorem ($c = 0$) was used in [5]. An analysis of convergence for approximation methods for finding nonlinear oscillations in systems described by ordinary differential equations, using spectral properties of the linear integral operators K with special kernels, can be found in the same paper.

5. The space $C([0,T],X)$ and K-normed linear spaces

The space $C([0,T],X)$ can be considered as an example of a K-normed linear space. Recall that a K-norm on a space $\mathfrak{X}$ is a mapping $[]\cdot[]$ defined on $\mathfrak{X}$ and taking values in an ordered (by a cone $\mathbf{K}$) real linear space B satisfying the following conditions:

(K1) $[]x[] \geq 0,\ x \in \mathfrak{X}$,
(K2) $[]x[] = 0,\ x \in \mathfrak{X}$, if and only if $x = 0$
(K3) $[]\lambda x[] = |\lambda|\,[]x[],\ x \in \mathfrak{X},\ \lambda \in \mathbb{R}$ or $\mathbb{C}$
(K4) $[]x_1 + x_2[] \leq []x_1[] + []x_2[],\ x_1, x_2 \in \mathfrak{X}$.

The space $\mathfrak{X} = C([0,T],X)$ turns out to be a K-normed linear space if we define a K-norm $[]x(\cdot)[] :\ C([0,T],X) \to C([0,T],\mathbb{R})$ by the formula

$$[]x(\cdot)[] = \|x(t)\|, \quad x \in C([0,T],X); \tag{5.1}$$

thus, in this case $B = C([0,T],\mathbb{R})$. The condition (1.1) can be written as

$$[]Ax_1 - Ax_2[] \leq Q[]x_1 - x_2[], \quad x \in C([0,T],X). \tag{5.2}$$

Theorem 1.1 is a special case of the Banach-Caccioppoli fixed point principle modification (see s.g. [1, 3, 6, 9, 16]). All the inequalities about the operator A used in this paper are particular cases of (5.2).

References

[1] M. Altman, *Contractors and equations in pseudometric spaces,* Boll. Un. Mat. Ital., **6**(1972), 376-384.

[2] J. Appell, A. S. Kalitvin, P. P. Zabreĭko, *Partial Integral Operators and Integro-Differential Equations* (Pure and Applied Mathematics: A Series of Monographs and Textbooks, **230**). Marcel Dekker, Inc., 2000.

[3] L. Collatz, *Functional Analysis and Numerical Mathematics,* Academic Press: Boston, 1966.

[4] E. De Pascale, L. De Pascale, *Fixed points for some non obviously contractive operators,* Proc. Amer. Math. Soc., **130**(2002), 3249-3254.

[5] N. A. Evkhuta, P. P. Zabrejko, *The Poincarè method and Samojlenko method for the construction of periodic solutions to ordinary differential equations,* Mathematische Nachrichten, **153**(1991), 85-99.

[6] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones,* Academic Press, Boston, 1988.

[7] L. V. Kantorovich, B.Z. Vulikh, A.G. Pinsker, *Functional analysis in semi-ordered spaces,* (Russian), M.-L.: Gostekhizdat, 1950.

[8] M. A. Krasnosel'skiĭ, E. A. Lifshits, A. V. Sobolev, *Positive linear systems. The method of positive operators,* Sigma Series in Applied Mathematics, **5**. Heldermann Verlag, Berlin, 1989.

[9] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Ya. B. Rutitskij, V. Ya. Stetsenko, *Approximate solution of operator equations,* Nauka, Fitmatgiz,Moskow (1969); English translation: Wolters-Noordhoff Publishing, Groningen (1972).

[10] B. Lou, *Fixed points for operators in a space of continuous functions and applications,* Proc. Amer. Math. Soc., **127**(1999), 2259-2264.

[11] A. I. Perov, *The Cauchy problem for systems of ordinary differential equations,* In: *Approximate methods of solving differential equations*, Kiev, Naukova Dumka, 1964, 115-134.

[12] H. Schaefer, *Banach Lattices and Positive Operators* (Die Grundlehren der Mathematischen Wissenschaften, **215**). - Springer-Verlag, Berlin - Heidelberg - New York, 1974. 1-376.

[13] P. P. Zabreiko *Volterra integral operators* (Russian), Uspekhi Mat. Nauk, **22**(1967), 167-168.

[14] P. P. Zabreiko *A formula for the spectral radius of integral Volterra operators* (Russian), Litov. Mat. Sb., **7**(1967), 281-287.

[15] P. P. Zabreiko *Error estimates for successive approximations and spectral properties of linear operators,* Numer. Funct. Anal. Optim., **11**(1990), 823-838.

[16] P. P. Zabreiko, *K-metric and K-normed linear spaces: a survey*, Collect. Math., **48**(1997), 825-859.

[17] P. P. Zabreiko, M. A. Krasnosel'skiĭ, V. Y. Stetsenko *Estimates of the spectral radius of positive linear operators* (Russian), Mat. Zametki, **1**(1967), 461-468.

[18] P. P. Zabreiko, A. N. Lomakovich *Generalization of a theorem of Volterra,* Ukr. Math. J., **39**(1987), 530-533.