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A NOTE ON LERAY-SCHAUDER ALTERNATIVES FOR THE DECOMPOSABLE MAPS OF BADER

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Abstract. A new Leray-Schauder alternative is presented for the decomposable maps of Bader. Our proof is elementary and relies on fixed point theory of self maps and properties of the Minkowski functional.

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1. Introduction

In [2] Bader presented a fixed point index for his decomposable maps and then deduced a multivalued Leray-Schauder alternative. He then used his Leray-Schauder alternative to establish existence for some evolution inclusions. The purpose of this paper is to give a simple proof of Bader's Leray-Schauder alternative. Indeed we will present a more general result. The main idea involved is to work with a larger class of maps (the maps of Park [5] which are closed under compositions). We remark here that unfortunately no fixed point index exists for the maps of Park.

For the remainder of this section we present some definitions and some known facts. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is <u>Kakutani</u> if F is upper semicontinuous with convex values.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of

Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:
(i). $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii). for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.
Definition 1.1. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

The Kakutani maps are examples of $\mathcal{U}_c^\kappa$ maps (in fact they are examples of $\mathcal{U}_c$ maps). Indeed many other maps in the literature, for example (i). the acyclic maps, (ii). the O'Neill maps, and (iii). the maps admissible in the sense of Gorniewicz, are examples of $\mathcal{U}_c^\kappa$ maps.

Let (E,d) be a pseudometric space. For $S \subseteq E$, let $B(S,\epsilon) = \{x \in E : d(x,S) \leq \epsilon\}$, $\epsilon > 0$, where $d(x,S) = \inf_{y\in Y} d(x,y)$. The measure of noncompactness of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\epsilon > 0 : \ M \subseteq B(A,\epsilon) \ \text{for some finite subset} \ A \ \text{of} \ E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F : S \to 2^E$; here $S \subseteq E$. The map F is said to be a countably P-concentrative mapping if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \leq \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e. X is not precompact in the pseudonormed space (E,p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\,.\,)$ denotes the measure of noncompactness in the pseudonormed space (E,p).

We now recall the following definition from the literature.
Definition 1.2. A Hausdorff topological space X is said to be angelic if for every relatively countably compact set $C \subseteq X$ the following hold:
(i). C is relatively compact, and
(ii). for each $x \in \overline{C}$ there exists a sequence $\{x_n\}_{n\geq 1} \subseteq C$ such that $x_n \to x$.
Remark 1.1. All metrizable locally convex spaces equipped with the weak topology are angelic (see the Eberlein-Smulian theorem).
Theorem 1.1. *Let E be a topological space, Y a Hausdorff locally convex topological vector space which is angelic when furnished with the weak topology and let D be a weakly compact subset of Y. If $F : D \to 2^E$ (here 2^E*

denotes the family of nonempty subsets of E*) is a weakly-strongly sequentially upper semicontinuous map (i.e. for any closed set* A *of* E *we have that* $F^{-1}(A)$ *is weakly sequentially closed) then* $F : D \to 2^E$ *is weakly-strongly upper semicontinuous.*

Proof. Let A be a closed subset of E. We first show $F^{-1}(A)$ is weakly sequentially closed. Let $y_n \in F^{-1}(A)$ and $y_n \rightharpoonup y$. Then since F is a weakly-strongly sequentially upper semicontinuous map we have $y \in F^{-1}(A)$. Thus $F^{-1}(A)$ is weakly sequentially closed.

Now since D is weakly compact we have that $\overline{F^{-1}(A)}^{\,w}$ is weakly compact. Let $x \in \overline{F^{-1}(A)}^{\,w}$. Now since Y is angelic when furnished with the weak topology there exists a sequence $x_n \in F^{-1}(A)$ with $x_n \rightharpoonup x$. Also since $F^{-1}(A)$ is weakly sequentially closed we have $x \in F^{-1}(A)$. Thus $\overline{F^{-1}(A)}^{\,w} = F^{-1}(A)$, so $F^{-1}(A)$ is weakly closed. Thus $F : D \to 2^E$ is weakly-strongly upper semicontinuous. $\square$

2. Leray-Schauder Alternative

We begin by presenting a result from [1] and the proof is included since it is elementary. Let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, and $0 \in U$. Notice $int_C\, U = U$ since U is open in C and as a result $\partial U = \partial_E\, U$ (here ∂U denotes the boundary of U in C).

We will consider maps $F : \overline{U} \to K(C)$ (here $\overline{U}$ denotes the closure of U in C). We will assume the map F satisfies one of the following conditions:

(H1). F is compact;

(H2). if $D \subseteq \overline{U}$ and $D \subseteq \overline{co}\,(\{0\} \cup F(D))$ then $\overline{D}$ is compact;

or

(H3). F is countably P-concentrative and E is Fréchet (here P is a defining system of seminorms).

Fix $i \in \{1, 2, 3\}$.

Definition 2.1. We say $F \in LS^i(\overline{U}, C)$ if $F \in \mathcal{U}_c^{\kappa}(\overline{U}, C)$ satisfies (Hi).

Theorem 2.1. *Fix* $i \in \{1, 2, 3\}$ *and let* E *be a Hausdorff locally convex topological vector space,* C *a closed convex subset of* E*,* $U \subseteq C$ *convex,* U *an open subset of* E *and* $0 \in U$*. Suppose* $F \in LS^i(\overline{U}, C)$ *and assume the* *following condition hold:*

$$x \notin \lambda F x \quad \textit{for every} \quad x \in \partial U \quad \textit{and} \quad \lambda \in (0,1). \tag{2.1}$$

Then F *has a fixed point in* $\overline{U}$.

Proof. Let μ be the Minkowski functional on $\overline{U}$ and let $r : E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for} \quad x \in E.$$

Let $G = r\,F$. Now $G \in \mathcal{U}_c^\kappa(\overline{U}, \overline{U})$ since $\mathcal{U}_c^\kappa$ is closed under compositions. We claim

$$G \in LS^i(\overline{U}, \overline{U}). \tag{2.2}$$

If $i = 1$ then (2.2) is immediate. Next suppose $i = 2$, and let $D \subseteq \overline{U}$ with $D \subseteq \overline{co}\,(\{0\} \cup G(D))$. Now since $r(B) \subseteq co\,(\{0\} \cup B)$ for any subset B of E, we have

$$D \subseteq \overline{co}\,(\{0\} \cup co\,(\{0\} \cup F(D))\,) = \overline{co}\,(\{0\} \cup F(D)\,).$$

Thus $\overline{D}$ is compact since $F \in LS^i(\overline{U}, C)$, and so (2.2) is true if $i = 2$. A similar argument establishes (2.2) if $i = 3$.

If $i = 1$ (respectively $i = 2$, respectively $i = 3$) then we know from [5] (respectively [3], respectively [4]) that there exists $x \in \overline{U}$ with $x \in G(x) = r\,F(x)$ Thus $x = r(y)$ for some $y \in F(x)$ with $x \in \overline{U} = U \cup \partial U$ (note $int_C\, U = U$ since U is also open in C). Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$ so $x = y \in F(x)$, and we are finished. If $y \notin \overline{U}$ then $r(y) = \frac{y}{\mu(y)}$ with $\mu(y) > 1$. Then $x = \lambda\, y$ (i.e. $x \in \lambda\, F(x)$) with $0 < \lambda = \frac{1}{\mu(y)} < 1$; note $x \in \partial U$ since $\mu(x) = \mu(\lambda\, y) = 1$ (note $\partial U = \partial_E U$ since $int_C\, U = U$). This of course contradicts (2.1). □

Let Y be a Hausdorff locally convex topological vector space. We let Y_w be the space Y furnished with the weak topology and we let $CK(Y)$ denote the family of nonempty, convex, weakly compact subsets of Y. Let X be a subset of a Hausdorff locally convex topological vector space. Suppose the map $A : X \to CK(Y)$ is upper semicontinuous from X into Y_w. Then for Section 1 we know that $A \in \mathcal{U}_c(X, Y_w)$. Suppose the map $B : Y \to X$ is weakly-strongly continuous (i.e. continuous from Y_w into X). Then $B \in \mathcal{U}_c(Y_w, X)$. Similarly if the map $C : Y \to CC(X)$ (here $CC(X)$ denotes the family of nonempty, convex, compact subsets of X) is weakly-strongly upper semicontinuous then $C \in \mathcal{U}_c(Y_w, X)$.

Suppose in addition Y is angelic when furnished with the weak topology. Assume the map $D : Y \to X$ is weakly-strongly sequentially continuous map (i.e. $D : Y \to X$ is completely continuous i.e. if $x_n, x \in Y$ with $x_n \rightharpoonup x$ then $D\,x_n \to D\,x$). Then Theorem 1.1 guarantees that $D \in \mathcal{U}_c^\kappa(Y_w, X)$ since if Ω is a weakly compact subset of Y then $D : \Omega \to X$ is weakly-strongly continuous. Similarly if $H : Y \to CC(X)$ is a weakly-strongly sequentially upper semicontinuous map (i.e. for any closed set A of X we have that $F^{-1}(A)$ is weakly sequentially closed) then $H \in \mathcal{U}_c^\kappa(Y_w, X)$

Theorem 2.2. *Let E be a Hausdorff locally convex topological vector space, Y a Hausdorff locally convex topological vector space which is angelic when furnished with the weak topology, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E and $0 \in U$. Suppose $G : \overline{U} \to CK(Y)$ is upper semicontinuous from $\overline{U}$ into Y_w and $T : Y \to C$ a weakly-strongly sequentially continuous map. Also assume $F = T\,G$ satisfies* $(H1)$, $(H2)$ *or* $(H3)$ *and suppose* (2.1) *holds. Then F has a fixed point in $\overline{U}$.*

PROOF: Note $G \in \mathcal{U}_c^\kappa(\overline{U}, Y_w)$ and $T \in \mathcal{U}_c^\kappa(Y_w, C)$. As a result $F \in \mathcal{U}_c^\kappa(\overline{U}, C)$ since $\mathcal{U}_c^\kappa$ is closed under compositions. The result now follows from Theorem 2.1. □

Remark 2.1. In Theorem 2.2 we could remove the assumption that Y is angelic when furnished with the weak topology if $T : Y \to C$ weakly-strongly sequentially continuous is replaced by $T : Y \to C$ weakly-strongly continuous.

Of course we could consider a multivalued T in Theorem 2.2 also.

Theorem 2.3. *Let E be a Hausdorff locally convex topological vector space, Y a Hausdorff locally convex topological vector space which is angelic when furnished with the weak topology, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E and $0 \in U$. Suppose $G : \overline{U} \to CK(Y)$ is upper semicontinuous from $\overline{U}$ into Y_w and $T : Y \to CC(C)$ a weakly-strongly sequentially upper semicontinuous map. Also assume $F = T\,G$ satisfies* $(H1)$, $(H2)$ *or* $(H3)$ *and suppose* (2.1) *holds. Then F has a fixed point in $\overline{U}$.*

Remark 2.2. In Theorem 2.3 we could remove the assumption that Y is angelic when furnished with the weak topology if $T : Y \to CC(C)$ weakly-strongly sequentially upper semicontinuous is replaced by $T : Y \to CC(C)$ weakly-strongly upper semicontinuous.

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