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ON THE COMPACTNESS OF THE FIXED POINT SET

SORIN MUREŞAN

University of Oradea
Armatei Române 5
Oradea, Romania
e-mail: smuresan@uoradea.ro

Abstract. In this paper we study compactness of the fixed point set for Caristi type operators by means of regular -global-inf function technique given by Angrisani and Clavelli in [1]. The results which are obtained will be applied to Ekeland's variational principle and Takahashi's minimization theorem.
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1. Preliminaries

Let (X, d) be a metric space and $f : X \to X$ an operator. In this paper we study the compactness of the fixed points set of f:

$$F_f := \{x \in X : f(x) = x\}.$$

At least two methods are used to prove the non-emptiness of the set F_f.
Firstly is the study of the convergence to a fixed point of f of the successive approximations sequence $(f^n(x))_{n\in\mathbb{N}}$, where $x \in X$.
In connection with this method Rus has defined in [8] and [10] the notions of weakly Picard and $c-$ weakly Picard operators, respectively, as follows:

Definition 1.1. *an operator $f : X \to X$ is called:*
i) ***weakly Picard operator (w.P.o)*** *iff the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit, denoted by $f^\infty(x)$, is a fixed point of f. Moreover, if f has a unique fixed point, denoted by x^*, then f will be called Picard operator (P.o).*

ii) $c-$ ***weakly Picard operator*** *($c-$* ***w.P.o****) iff f is a w.P.o and there exists $c>0$ such that*

$$d\left(x,f^{\infty}\left(x\right)\right)\leq cd\left(x,f\left(x\right)\right),\ \textit{for all } x\in X.$$

Example 1.2. *Let (X,d) be a complete metric space. If f is an $a-$ contraction on X into itself (i.e. there exists $a\in[0,1)$ such that $d\left(f\left(x\right),f\left(y\right)\right)\leq ad\left(x,y\right)$, for all $x,y\in X$) then, from Banach's fixed point theorem, we have that f is a P.o and an $\frac{1}{1-a}-$ w.P.o. If f is a closed orbitally $a-$ contraction on X into itself (i.e. there exists $a\in[0,1)$ such that $d\left(f\left(x\right),f^{2}\left(x\right)\right)\leq ad\left(x,f\left(x\right)\right)$, for all $x\in X$) then f is not a P.o but it is an $\frac{1}{1-a}-$ w.P.o.*

Example 1.3. *The operators*

$$f:[0,\infty)\rightarrow[0,\infty),\ f(x)=\begin{cases}0,\ \textit{if } x\in[0,1]\\ n,\ \textit{if } x\in(n,n+1]\end{cases},\ \textit{where } n\in\mathbb{N}\backslash\{0\}$$

and

$$g:\mathbb{R}\rightarrow\mathbb{R},\ g\left(x\right)=\begin{cases}0,\ \textit{if } x\in\mathbb{Q}\\ 1,\ \textit{if } x\in\mathbb{R}\backslash\mathbb{Q}\end{cases}$$

are P.o but none of them is $c-$ w.P.o for any $c>0$.

Another method to find the fixed points of $f:X\rightarrow X$ is the study of the zeros for the function

$$F:(X,d)\rightarrow[0,\infty),\ F\left(x\right)=d\left(x,f\left(x\right)\right).$$

In connection with this method Angrisani and Clavelli in [1] proved the non-emptiness and compactness of the global minimum points set of a regular-global-inf function (Theorem 1.5 below).

On (X,d), for a function $F:X\rightarrow\mathbb{R}$ and for a bounded subset $A\subseteq X$ we will consider:

the global minimum points of F

$$M\left[F\right]:=\left\{x\in X:F\left(x\right)=\inf_{y\in X}F\left(y\right)\right\},$$

the $p-$ level set

$$L_{p}:=\left\{x\in X:F\left(x\right)\leq p\right\},\text{where } p\in\mathbb{R},$$

the Kuratowski's measure of non-compactness

$$\alpha_K(A) := \inf\left\{\varepsilon > 0 : A \subseteq \underset{i=\overline{1,n}}{\cup} A_i,\ diam(A_i) \le \varepsilon,\ \forall i = \overline{1,n}\right\},$$

and

$$D(x,A) := \inf\{d(x,a) : a \in A\}, \text{for any } x \in X,$$

$$diam(A) := \sup\{d(a,b) : a, b \in A\}.$$

Definition 1.4. *(see* [1]*) Let* (X,d) *be a metric space. A function* $F : X \to \mathbb{R}$ *is called* ***regular-global-inf (r.g.i)*** *in* $x \in X$ *iff*

$$F(x) > \inf_{y\in X} F(y) \Rightarrow \exists p > \inf_{y\in X} F(y) \ \textit{ such that } D(x, L_p) > 0.$$

We will say that F *is r.g.i on* X *iff it is r.g.i in every element of* X.

Relating to the compactness of the set $M[F]$ we have the following theorem:

Theorem 1.5. *(see* [1]*) Let* (X,d) *be a complete metric space and* $F : X \to \mathbb{R}$ *be a r.g.i function on* X. *If* $\lim\limits_{p \searrow \inf_{y\in X} F(y)} \alpha_K(L_p) = 0$ *then* $M[F]$ *is a nonempty and compact set.*

In this paper we will study compactness of the set F_f in case of fixed point theorem due to Caristi. Also, using some results from [3], we will give similarly results in case of Ekeland's variational principle and Takahashi's minimization theorem.

2. Main results

In 1976 Caristi (see [2]) obtained the following generalization of Banach's fixed point theorem:

Theorem 2.1. *Let* (X,d) *be a complete metric space and* $\varphi : X \to \mathbb{R}$ *a lower semi-continuous function bounded from below. Let* $f : X \to X$ *be an operator such that*

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)), \ \textit{ for all } x \in X. \tag{1}$$

Then $F_f \neq \emptyset$.

We will say that $f \in C[\varphi]$ iff the operator $f : X \to X$ fulfils the relation (1). Can we say that an operator $f \in C[\varphi]$ is w.P.o. or $c-$ w.P.o ? This problem is studied under supplementary conditions imposed to φ and f in the following

Proposition 2.2. *(see [9], [11])Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$ a function, $f : X \to X$ a continuous operator and $f \in C[\varphi]$.*
Then:
(i) f is a w.P.o.
(ii) If there exists $c > 0$ such that

$$\varphi(x) \leq c \cdot d(x, f(x)), \text{ for all } x \in X \tag{2}$$

then f is $c-$ w.P.o.

Generally, even if the hypotheses of Proposition 2.2 holds, we not obtain the compactness of the set F_f as show the following

Example 2.3. *Let (X, d) be a complete metric space. For the continuous function $\varphi_0 : X \to [0, \infty), \varphi_0(x) = 0, \forall x \in X$ and the operator $1_X : X \to X, 1_X(x) = x, \forall x \in X$ we have that $1_X \in C[\varphi_0]$ and the condition (2) holds for any $c > 0$ but the set $F_{1_X} = X$ is not necessarily compact.*

Under supplementary conditions we will obtain the compactness of the set F_f but first we present a general result which characterize the compactness of F_f.

Lemma 2.4. *Let (X, d) be a complete metric space and $f : X \to X$ a continuous operator such that*

$$\inf_{x \in X} d(x, f(x)) = 0. \tag{3}$$

Then the following assertion are equivalent:
(i) F_f is nonempty and compact.
(ii) there exists $p_0 \geq 0$ such that the level set

$$L_{p_0} := \{x \in X : d(x, f(x)) \leq p_0\}$$

is nonempty and compact.

Proof. "(i) $\Rightarrow$ (ii)" Since $L_0 = F_f$ we consider $p_0 = 0$ and (ii) holds.
"(ii) $\Rightarrow$ (i)" If $p_0 = 0$ then $F_f = L_0$ is a nonempty and compact set.
In the case $p_0 > 0$ we consider

$$F : X \to \mathbb{R},\ F(x) = d(x, f(x)),\ \forall x \in X.$$

From (3) we obtain that $\inf_{x \in X} F(x) = 0$.
From the continuity of f the function F is also continuous and then, for all $p \geq 0$, we have that

$$L_p = \{x \in X : F(x) \leq p\} = F^{-1}[0, p] \text{ is closed.} \tag{4}$$

Since $L_p \subseteq L_{p_0}$ for any $p \leq p_0$ we obtain (using (4) and the compactness of L_{p_0}) the compactness of L_p, for any $p \leq p_0$. Hence $\lim_{p \searrow 0} \alpha_K(L_p) = 0$.
Now we suppose that F is not r.g.i on X. The will exists $x \in X$ such that $F(x) > 0$ and $D\left(x, L_{\frac{1}{n}}\right) = 0$, for any $n \in \mathbb{N}$. Using (4) we have

$$x \in L_{\frac{1}{n}}, \forall n \in \mathbb{N} \Leftrightarrow F(x) \leq \frac{1}{n}, \forall n \in \mathbb{N} \Leftrightarrow F(x) = 0.$$

Contradiction with assumption $F(x) > 0$. Hence F is r.g.i on X.
Applying Theorem 1.5 the set $M[F] = F_f$ is nonempty and compact. □

We observe that the continuity of the operator f in the Lemma 2.4 can be change by closedeness of the graph of f and in this case we will have the following

Lemma 2.5. *Let (X, d) be a complete metric space and $f : X \to X$ an operator. Then the following assertion hold:*
(i) If F_f is nonempty and compact then there exists $p_0 \geq 0$ such that the level set $L_{p_0} := \{x \in X : d(x, f(x)) \leq p_0\}$ is nonempty and compact.
(ii) If $\inf_{x \in X} d(x, f(x)) = 0$, f has a closed graph and there exists $p_0 \geq 0$ such that the level set $L_{p_0} := \{x \in X : d(x, f(x)) \leq p_0\}$ is nonempty and compact then F_f is compact, too.

We will use Lemma 2.4 to prove the compactness of F_f in the case of a Caristi type operator, $f \in C[\varphi]$ because in the continuity conditions on f we have also that f is a w.P.o.

Theorem 2.6. *Let (X, d) be a complete metric space, $\varphi : X \to [0, \infty)$ a lower semi-continuous function and $f \in C[\varphi]$ a continuous operator. The following*

statements are equivalent:
i) F_f *is a nonempty compact set;*
ii) there exists $p_0 \geq 0$ *such that*

$$L_{p_0} = \{x \in X : d(x, f(x)) \leq p_0\}$$

is a nonempty compact set.

Proof. For any $x \in X$ the sequence $(\varphi(f^n(x)))_{n\in\mathbb{N}}$ is monotone decreasing and bounded below by 0, hence it converges and

$$d\left(f^n(x), f^{n+1}(x)\right) \leq \varphi(f^n(x)) - \varphi\left(f^{n+1}(x)\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence $\inf\limits_{x\in X} d(x, f(x)) = 0$. We apply Lemma 2.4.

□

If the operator f is not continuous then we can obtain a sufficient conditions for compactness of the set F_f if we consider that the condition (2) hold, as follows:

Theorem 2.7. *Let* (X, d) *be a complete metric space,* $\varphi : X \to [0, \infty)$ *a lower semi-continuous function and* $f \in C[\varphi]$ *an operator (not necessarily continuous) such that the condition (2) holds. If* φ *is an open operator then* F_f *is a nonempty compact set.*

Proof. We consider $F : X \to \mathbb{R}$, $F(x) = \varphi(x), \forall x \in X$.
From the lower continuity of φ we infer that F is r.g.i on X.
From $f \in C[\varphi]$ we obtain that the sequence $(f^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and then, from (2) we have that

$$\varphi(f^n(x)) \leq c \cdot d\left(f^n(x), f^{n+1}(x)\right) \overset{n\to\infty}{\longrightarrow} 0 \Rightarrow \inf_{y\in X} F(y) = 0.$$

From $f \in C[\varphi]$ we have that

$$d\left(f^n(x), f^{n+1}(x)\right) \leq \varphi(f^n(x)) \leq \varphi(x) \leq p, \forall x \in L_p.$$

So, for all $x \in L_p$, the orbit $O_f(x) := \left\{x, f(x), f^2(x), ...\right\} \subseteq L_p$.
For any $x \in L_p$ and any $k, q \geq 1$ we have $d\left(f^k(x), f^{k+q}(x)\right) \leq \varphi(x) \leq p$ and $d\left(x, f^{k+q}(x)\right) \leq d(x, f(x)) + d\left(f(x), f^{k+q}(x)\right) \leq 2p$.
Hence $diamO_f(x) \leq 2p$.
Denoting by $B(x, r)$ the open ball with center $x \in X$ and radius $r > 0$ we have

that $L_p = \varphi^{-1}[0,p] \Rightarrow [0,p] = \varphi[L_p] = \varphi\left[\bigcup_{x\in L_p} O_f(x)\right] = \bigcup_{x\in L_p}\varphi(O_f(x)) \subseteq \bigcup_{x\in L_p}\varphi(B(x,3p))$ which is an open covered of $[0,p]$. Will exists $x_1, x_2, ..., x_m \in L_p$ such that $[0,p] \subseteq \bigcup_{i=1,m}\varphi(B(x_i,3p)) = \varphi\left(\bigcup_{i=1,m} B(x_i,3p)\right) \Rightarrow L_p \subseteq \bigcup_{i=1,m} B(x_i,3p) \Rightarrow \alpha_K(L_p) \leq diam B(x,3p) \leq 6p$.
Hence $\lim\limits_{p\searrow 0}\alpha_K(L_p) = 0$.
Applying Theorem 1.5, the set $M[F] = \{x \in X : \varphi(x) = 0\}$ is nonempty and compact.
Let $x \in M[F]$. We have

$$f \in C[\varphi] \Rightarrow d(x, f(x)) \leq \varphi(x) = 0 \Rightarrow x \in F_f.$$

Let $x \in F_f$. By (2) we have

$$0 \leq \varphi(x) \leq c \cdot d(x, f(x)) = 0 \Rightarrow x \in M[F].$$

Hence $M[F] = F_f$ and so F_f is a nonempty and compact set. □

3. Applications

3.1. **Ekeland's variational principle and Takahashi's minimization theorem cases.** It is known (see [3]) that Caristi's fixed point theorem (Theorem 2.1) is equivalent with Ekeland's variational principle and Takahashi's minimization theorem (Theorem 3.1 and 3.2 from below). Using Theorems 2.6 and 2.7 we will obtain in this section compactness results in this cases.

Theorem 3.1. *(Variational Principle of Ekeland) Let (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}$ a lower semi-continuous function bounded from below. Then, for any real number ε with $\varepsilon > 0$, the set*

$$\phi_\varepsilon[\varphi] := \{x \in X : \varphi(x) \leq \inf_{y\in X}\varphi(y) + \varepsilon,\ \varphi(x) < \varphi(y) + \varepsilon d(x,y), \forall y \in X\backslash\{x\}\}$$

is nonempty.

Theorem 3.2. *(Minimization Theorem of Takahashi) Let (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}$ a lower semi-continuous function bounded from* *below. If the condition*

$$\forall x \in X \text{ with } \varphi(x) > \inf_{y \in X} \varphi(y) \Rightarrow \exists x' \in X \setminus \{x\} \text{ with } \varphi(x') + d(x, x') \leq \varphi(x) \tag{5}$$

holds then $M[\varphi]$ *is a nonempty set.*

In [3] are studied different relations between the sets $\phi_\varepsilon[\varphi]$, $M[\varphi]$ and F_f, where $f \in C[\varphi]$. We summarize some of them in

Proposition 3.3. *Let* (X, d) *be a complete metric space,* $\varphi : X \to \mathbb{R}$ *a lower semi-continuous function bounded from below and* $f : X \to X$ *an operator. We have that:*
(i) $f \in C[\varphi] \Rightarrow M[\varphi] \subset \phi_\varepsilon[\varphi] \subset F_f$*, for* $\varepsilon \in (0, 1]$.
(ii) If condition (5) holds then there exists $\varepsilon > 0$ *such that* $\phi_\varepsilon[\varphi] \subset M[\varphi]$.
(iii) If condition (5) holds, then, denoting by $\varepsilon_0 = \sup_{\varepsilon > 0}\{\phi_\varepsilon[\varphi] \subset M[\varphi]\} \in [0, \infty]$*, we have*

$$\varepsilon_0 > 0 \text{ and } f \in C[\varphi] \Rightarrow M[\varphi] = \phi_\varepsilon[\varphi] \subset F_f, \text{ for } 0 < \varepsilon < \min\{1, \varepsilon_o\}.$$

We can obtain compactness results for the sets $\phi_\varepsilon[\varphi]$ and $M[\varphi]$ using Theorems 2.6, 2.7 proving first that these sets are equal with F_f.

Proposition 3.4. *Let* (X, d) *be a complete metric space,* $\varphi : X \to [0, \infty)$ *a lower semi-continuous function and* $f \in C[\varphi]$.
(i) If condition (2) holds then $F_f = \phi_\varepsilon[\varphi]$*, for any* $\varepsilon \in (0, 1]$.
(ii) If conditions (2) and (5) hold then $F_f = M[\varphi]$.

Proof. (i) For any $x \in F_f$, from (2), we have that $\varphi(x) = 0$ hence $x \in \phi_\varepsilon[\varphi]$, for any $\varepsilon > 0$. Hence, from Proposition (3.3, i) we obtain the equality between $\phi_\varepsilon[\varphi]$ and F_f, for $\varepsilon \in (0, 1]$.
(ii) We use Proposition (3.3, iii). □

We study now the compactness of the sets $\phi_\varepsilon[\varphi]$ and $M[\varphi]$ using again Propositions 3.3 and 3.4.

Theorem 3.5. *Let* (X, d) *be a complete metric space,* $\varphi : X \to [0, \infty)$ *a lower semi-continuous function and* $f \in C[\varphi]$ *such that condition (2) holds. If*
(α) f *is a continuous function and exists* $p_0 \geq 0$ *such that*

$$\{x \in X : d(x, f(x)) \leq p_0\}$$

is a nonempty compact set
or
(δ) φ is an open map
then
(i) $\phi_\varepsilon[\varphi]$ is a compact set for any $\varepsilon \in (0,1]$.
(ii) If the condition(5) holds and $\varepsilon_0 > 0$ then $M[\varphi]$ is a compact set.

Proof. (i),(ii) From Proposition 3.4 we have that

$$\phi_\varepsilon[\varphi] = F_f, \text{ for all } \varepsilon \in (0,1] \text{ and}$$

$$\phi_\varepsilon[\varphi] = M[\varphi] = F_f, \text{ for all } 0 < \varepsilon < \min\{1, \varepsilon_0\}.$$

Using now (α) and Theorem 2.6 or (δ) and Theorem 2.7 we obtain the compactness of the sets $\phi_\varepsilon[\varphi], M[\varphi]$. □

References

[1] M. Angrisani, M. Clavelli, *Synthetic Approaches to Problems of Fixed Points in Metric Space*, Annali di Matematica pure ed applicata, vol. **CLXX**(1996), 1-12.

[2] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. A.M.S., **215**(1976), 241-251.

[3] P. Daffer, H. Kaneko, *On a variational principle of Ekeland*, Functional Analysis-Selected Topics (ed. P. K. Jain), Narosa Publishing House, New Delhi, 1998, 152-157.

[4] I. Ekeland, *On the Variational Principle*, Journal of Math. Anal. and Appl., **47**(1971), 324-353.

[5] ———, *Nonconvex minimization problems*, Bull. A.M.S., **1**(1979), 443-474.

[6] K. Goebel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press, 1990.

[7] W. A. Kirk, L. M. Saliga, *Some results on existence and approximation in metric fixed point theory*, J. Computational and Applied Math., **113**(2000), 141-152.

[8] I. A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, **34**(1993), 769-773.

[9] I. A. Rus, *Picard operators and applications*, Scientiae Math. Japonicae, **58**(2003), 191-219.

[10] I. A. Rus, S. Muresan, *Data dependence of the fixed points set of weakly Picard operators*, Studia Univ. "Babes-Bolyai", **43**(1998), 79-84.

[11] ———, *Data dependence of the fixed points set of some weakly Picard operators*, "Tiberiu Popoviciu" Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 2000.