Fixed Point Theory, Volume 5, No. 1, 2004, 81-85
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

NOTE ON A FIXED POINT THEOREM

DOREL MIHEŢ

West University of Timişoara
Faculty of Mathematics
Bv. V. Parvan 4, 1900 Timişoara, Romania
e-mail: mihet@math.uvt.ro

Abstract. A result from [4] is improved.

Key Words and Phrases: Menger space, probabilistic contraction.
2000 Mathematics Subject Classification: 54E70, 54H25.

1. Preliminaries

In this section we recall some definitions and results that will be used in in the sequel. The other notions concerning theory of fixed points in probabilistic metric spaces are classical ones and follow the books [2], [8].

Definition 1.1. ([1]) We say that a t-norm T is *of Hadžić-type,* if the family $\{T^n\}$ of its iterates is equicontinuous at $x = 1$, that is

$$\forall \varepsilon \in (0,1) \exists \delta \in (0,1) \ : \ x > 1 - \delta \Longrightarrow T^n(x) > 1 - \varepsilon \forall n \geq 1.$$

We will denote by $\mathcal{H}$ the class of all t-norms of Hadžić-type. There is a nice characterization of continuous t norms T of the class $\mathcal{H}$ ([6]):

i) If there exists a strictly increasing sequence $(b_n)_{n \in N}$ *in* $[0,1]$ *such that* $\lim_{n\to\infty} b_n = 1$ *and* $T(b_n, b_n) = b_n$ $\forall n \in N$*, then* T *is of Hadžić-type.*

ii) If T *is continuous and* $T \in \mathcal{H}$*, then there exists a sequence* (b_n) *as in* i).

Definition 1.2. ([2]) If T is a t-norm and $(x_{1,}\ x_2, ..., x_n) \in [0,1]^n$ $(n \geq 1)$ $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_{i.}, x_n) \forall n \geq 2$. $T_{i=1}^\infty x_i$ is defined as $\lim_{n\to\infty} T_{i=1}^n x_i$.

In this paper, the case when $\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1$, with x_i of the form $1-\varphi^i(\eta)$, where φ is a map from $(0,1)$ to $(0,1)$and the series $\sum_{n=1}^{\infty}\varphi^n(\eta)$ is convergent for every $\eta \in (0,1)$) will be considered.

It is worth noting that if the series $\sum_{n=1}^{\infty}\varphi^n(\eta)$ is convergent and $T \in \mathcal{H}$, then $\lim_{n\to\infty} T_{i=n}^{\infty}(1-\varphi^n(\eta)) = 1$.

If $\varphi(t) = kt$ where $k \in (0,1)$ is given, then, obviously, $\sum_{n=1}^{\infty}\varphi^n(\delta)$ is convergent for each $\delta \in (0,1)$. The condition $\lim_{n\to\infty} T_{i=n}^{\infty}(1-\varphi^n(\delta)) = 1$ for all $\delta \in (0,1)$ is satisfied for example by *Sugeno-Weber family,* defined by $T_{\lambda}^{SW} = \max(0, \frac{x+y-1+\lambda xy}{1+\lambda}), \lambda \in (-1,\infty)$ and by the Lukasiewics T-norm $T_L(a,b) = \max\{a+b-1, 0\}$. Note that all members of the family T_{λ}^{SW}, $\lambda \in (-1,0)$ are weaker than T_L (see [2]).

2. Main results

The following theorem has been proved in [4] as an answer to an open problem of Radu ([7]):

Theorem 2.1. ([4, Th. 3.11.]) *Let (X,F,T) be a complete generalized Menger with $\sup\limits_{a<1} T(a,a) = 1$ and $f : X \to X$ be a mapping with the property*

$$F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)),\ \forall x,y \in X, \forall t > 0$$

where $\alpha : [0,\infty) \to [0,\infty)$ is (strictly) increasing, $\alpha(s) < s\ \forall s \in (0,1)$, $\beta : [0,1] \to [0,1]$ is (strictly) increasing, $\beta(u) > u, \forall u \in (0,1)$, and the mapping $\varphi : [0,1] \to [0,1]$, $\varphi(s) := \max\{\alpha(s), 1-\beta(1-s)\}$ satisfies $\sum_n \varphi^n(s) < \infty$ for all $s < 1$. If $\lim_{n\to\infty} T_{i=1}^{\infty}(1-\varphi^{n+i}(s)) = 1$ for all $s \in (0,1)$ and there exists $x \in X$ such that $F_{xf(x)}(1) > 0$, then f has a fixed point.

As we noted in [4, Remark 3.12.], the conditions $\alpha(s) < s\ \forall s \in (0,1)$ and $\beta(u) > u, \forall u \in (0,1)$ can be replaced with $\lim_{t\searrow 0}\alpha(t) = 0$ and $\lim_{t\nearrow 1}\beta(t) = 1$.

We are going to prove (see Theorem 2.6. below) that actually even these last conditions can be dropped. Furthermore, the monotonicity of α plays no role and also, as easily can be seen, the condition $\sup\limits_{a<1} T(a,a) = 1$ follows from $\lim_{n\to\infty} T_{i=1}^{\infty}(1-\varphi^{n+i}(s)) = 1$.

Lemma 2.2. If $F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)),\ \forall x,y \in X, \forall t > 0$ *and β is strictly increasing, then f satisfies the contractivity condition*

$$x,y \in X,\ t \in (0,1),\ F_{xy}(t) > 1-t \Longrightarrow\ F_{f(x)f(y)}(\varphi t) > 1-\varphi(t)$$

with $\varphi(s) := \max\{\alpha(s), 1 - \beta(1-s)\}$.

Proof. $F_{xy}(t) > 1 - t \Longrightarrow \beta(F_{xy}(t)) > \beta(1-t) \Longrightarrow F_{f(x)f(y)}(\alpha(t)) > \beta(1-t)$.

Since $\varphi(s) \geq \alpha(s)$ and $\beta(1-s) \geq 1 - \varphi(s)$ for all $s \in (0,1)$, from $F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t))$ it follows that $F_{f(x)f(y)}(\varphi(t)) > 1 - \varphi(t)$. Therefore, $F_{xy}(t) > 1 - t \Longrightarrow F_{f(x)f(y)}(\varphi(t)) > 1 - \varphi(t)$.

Definition 2.3. Let Φ be the class of all mappings $\varphi : (0,1) \to (0,1)$. If F is a probabilistic distance on X and $\varphi \in \Phi$, we say that the mapping $f : X \to X$ is a φ-H *contraction* if

$$(\varphi\text{-}H): \ x, y \in X, \ t \in (0,1), \ F_{xy}(t) > 1 - t \Longrightarrow F_{f(x)f(y)}(\varphi(t)) > 1 - \varphi(t).$$

Lemma. 2.4. *If there exists $\eta \in (0,1)$ such that $\lim_{n\to\infty} \varphi^n(\eta) = 0$, then every φ-H contraction is (uniformly) continuous that is, for every $\varepsilon > 0$ and $\delta \in (0,1)$ there exist $\varepsilon' > 0$ and $\delta' \in (0,1)$ such that $F_{xy}(\varepsilon') > 1 - \delta' \Longrightarrow F_{f(x)f(y)}(\varepsilon) > 1 - \delta$.*

Proof. Let $\varepsilon > 0$ and $\delta \in (0,1)$ be given. There is $k_0 \in N$ such that $\varphi^k(\eta) < \min\{\varepsilon, \delta\}$ for all $k \geq k_0$. Choosing $\varepsilon' = \delta' = \varphi^{k_0}(\eta)$ we have $F_{xy}(\varepsilon') > 1 - \delta' \Longrightarrow F_{f(x)f(y)}(\varphi(\varepsilon')) > 1 - \varphi(\delta') \Longrightarrow F_{f(x)f(y)}(\varepsilon) > 1 - \delta$.

Theorem 2.5. *Let (X, F, T) be a complete generalized Menger space and $\varphi \in \Phi$ such that the series $\sum_1^\infty \varphi^n(\lambda)$ is convergent for every $\lambda \in (0,1)$. If $\lim_{n\to\infty} T_{i=n}^\infty (1 - \varphi^n(\delta)) = 1$ for all $\delta \in (0,1)$ then every φ-H contraction f on X with the property that $F_{xf(x)}(1) > 0$ for some $x \in X$ has a fixed point.*

Proof. Let (X, F, T) and x, f be as in the statement of the theorem.

Let us prove that there exists $\delta \in (0,1)$ such that $F_{xf(x)}(\delta) > 1 - \delta$. Indeed, if we suppose that $F_{xf(x)}(\delta) \leq 1 - \delta$ for all $\delta \in (0,1)$ then, by the left continuity of $F_{xf(x)}$, we deduce that $F_{xf(x)}(1) = 0$, which is a contradiction. From this inequality and (φ-H) it follows by induction that

$$F_{f^n(x)f^{n+1}(x)}(\varphi^n(\delta)) > 1 - \varphi^n(\delta), \forall n \in N.$$

Next, we will prove that the sequence $(f^n(x))_{n\in N}$ is a *Cauchy sequence,* that is

$$\forall \varepsilon > 0 \forall \lambda \in (0,1) \exists n_0 (= n_0(\varepsilon, \lambda)) : F_{f^n(x)f^{n+m}(x)}(\varepsilon) > 1 - \lambda, \forall n \geq n_0, \forall m \in N.$$

Let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given. Since the series $\sum_{n=1}^{\infty} \varphi^n(\delta)$ is convergent, there exists $n_1(= n_1(\varepsilon))$ such that $\sum_{n=n_1}^{\infty} \varphi^n(\delta) < \varepsilon$.

Then, for $n \geq n_1$ and $m \in N$, we have

$$\begin{aligned} F_{f^n(x)f^{n+m}(x)}(\varepsilon) &\geq F_{f^n(x)f^{n+m}(x)}(\sum_{i=n_1}^{\infty} \varphi^i(\delta)) \geq \\ &\geq F_{f^n(x)f^{n+m}(x)}(\sum_{i=n}^{n+m-1} \varphi^i(\delta)) \geq T_{i=1}^m x_{n+i-1} \end{aligned}$$

where $x_j := F_{f^j(x)f^{j+1}(x)}(\varphi^j(\delta)) \geq 1 - \varphi^j(\delta) \forall j \geq n$.

Let $n_2(= n_2(\lambda))$ be such that $T_{i=1}^m(1 - \varphi^{n_2+i-1}(\delta)) > 1 - \lambda$ for all m.

Then, for $n \geq n_0 := \max\{n_1, n_2\}$ and all $m \in N$, $F_{f^n(x)f^{n+m}(x)}(\varepsilon) \geq T_{i=1}^m(1 - \varphi^{n+i-1}(\delta)) \geq T_{i=1}^m(1 - \varphi^{n_2+i-1}(\delta)) > 1 - \lambda$.

Thus we proved that $(f^n(x))_{n \in N}$ is a Cauchy sequence. By the completeness of (X, F, T) and the continuity of f it follows that $(f^n(x))_{n \in N}$ converges to a fixed point of f.

From Theorem 2.5. and Lemma 2.2. we obtain the following theorem, which better answers the open problem of Radu:

Theorem 2.6. *Let (X, F, T) be a complete generalized Menger and $f : X \to X$ be a mapping with the property*

$$F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)), \forall x, y \in X, \forall t > 0$$

where α is a mapping from $[0, \infty)$ to $[0, \infty)$, $\beta : [0,1] \to [0,1]$ is strictly increasing and the mapping $\varphi : [0,1] \to [0,1]$, $\varphi(s) := \max\{\alpha(s), 1 - \beta(1-s)\}$ satisfies $\sum_n \varphi^n(s) < \infty$ for all $s < 1$. If $\lim_{n\to\infty} T_{i=1}^{\infty}(1 - \varphi^{n+i}(s)) = 1$ for all $s \in (0,1)$ and there exists $x \in X$ such that $F_{xf(x)}(1) > 0$, then f has a fixed point.

As it has been noted in the preliminary part, if the series $\sum_1^{\infty} \varphi^n(\delta)$ is convergent for every $\delta \in (0,1)$ and T is a t-norm of Hadžić-type, then $\lim_{n\to\infty} T_{i=n}^{\infty}(1 - \varphi^n(\delta)) = 1$ for all $\delta \in (0,1)$. Therefore, the following theorem holds:

Corollary 2.7. *Let (X, F, T) be a complete generalized Menger and $f : X \to X$ be a mapping with the property*

$$F_{f(x)f(y)}(\alpha(t)) \geq \beta(F_{xy}(t)), \forall x, y \in X, \forall t > 0$$

where α is a mapping from $[0,\infty)$ to $[0,\infty)$, $\beta : [0,1] \to [0,1]$ is a strictly increasing mapping and $\varphi : [0,1] \to [0,1]$, $\varphi(s) := \max\{\alpha(s), 1-\beta(1-s)\}$ satisfies $\sum_{n=1}^{\infty} \varphi^n(s) < \infty$ for all $s \in (0,1)$. If $T \in \mathcal{H}$ and there exists $x \in X$ such that $F_{xf(x)}(1) > 0$, then f has a fixed point.

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