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SOME SOLVABILITY THEOREMS FOR NONLINEAR EQUATIONS

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Abstract. Let E be a locally convex space and $f : E \to E$ a mapping. We say that the equation $f(x) = 0$ is *almost solvable* on $A \subset E$ if $0 \in \overline{f(A)}$. In this paper some results about the solvability and almost solvability are given. Our results are based on some classical fixed point theorems and on some geometrical conditions.

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1. Introduction

In the proof of Theorem 2 presented in our paper [5] we used the semi-inner-product $[\cdot,\cdot]$ in the sense defined by G. Lumer [8] and studied by J. R. Giles [4]. We indicated also another kind of semi-inner-product, this is the semi-inner-product defined in a Banach space by

$$[x, y] = \|y\| \cdot \lim_{x \to 0_+} \frac{\|y + tx\| - \|y\|}{t}$$

and presented in the K. Deimling's book [3].

We note that this semi-inner-product is not linear in the first variable, as the Lumer's semi-inner-product, but it is only sub-linear.

However, our Theorem 2 proved in [5] is also valid when we replace the Lumer's semi-inner-product by the Deimling's semi-iiner-product, because in the proof of this theorem we used only two properties, which are valid for both semi-inner-products.

Now, in the section 2 of this paper we give a new variant of Theorem 2 proved in [5]; some consequences of this result are also indicated. This result is generalized in the section 3 for other kind of operators, which are not compacts.

2. Solvability in Banach spaces

2.1. **Preliminaries.** Let $(E, \|\cdot\|)$ be a Banach space, $r > 0$ a real number and $f : E \to E$ a completely continuous mapping, i.e. f is continuous and for any bounded set $D \subset E$ we have that $f(D)$ is a relatively compact set.

We denote:

$$\overline{B}_r = \{x \in E | \ \|x\| \le r\}; \quad S_r = \{x \in E | \ \|x\| = r\}$$

We will consider in $\overline{B}_r$ the equation

$$f(x) = 0 \tag{2.1}$$

Our goal is to study the almost and the solvability of equation (2.1). We say that equation (2.1) is almost solvable if

$$0 \in \overline{f(\overline{B_r})}. \tag{2.2}$$

The notion of almost solvability is justified by the fact that, the condition (2.2) is equivalent with the equality

$$\inf \|f(x)\| = 0, \quad x \in \overline{B}_r \tag{2.3}$$

2.2. **Function** G**.** Suppose given a mapping $G : E \times E \to \mathbb{R}$, satisfying the following properties:

(g_1) $G(x, x) \ge 0$ for all $x \in S_r$

(g_2) $G(\lambda x, y) \ge \lambda G(x, y)$, for all $\lambda > 0$ and all $x, y \in S_r$.

We give some examples of functions G satisfying (g_1) and (g_2).

Example 1. If $(E, \langle\cdot,\cdot\rangle)$ is a Hilbert space, then $G(\cdot,\cdot) = \langle\cdot,\cdot\rangle$.

Example 2. If $(E, \|\cdot\|)$ is a Banach space, then $G(\cdot,\cdot) = [\cdot,\cdot]_l$ where $[\cdot,\cdot]_l$ is the semi-inner-product in the Lumer's sense, or $G(\cdot,\cdot) = [\cdot,\cdot]_d$ where $[\cdot,\cdot]_d$ is the semi-inner-product in the Deimling's sense.

Example 3. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $E = C([0,1], H)$ the normed vector space with the norm $\|x\| = \sup\limits_{t\in[0,1]} \|x(t)\|_H$, where $\|\cdot\|_H$ is the norm of the space H. Then we may take

$$G(x,y) = \sup_{t\in[0,1]} \langle x(t), y(t)\rangle$$

or

$$G(x,y) = \inf_{t\in[0,1]} \langle x(t), y(t)\rangle$$

or

$$G(x,y) = \int_0^1 \langle x(t), y(t)\rangle dt.$$

Notice that for these examples, the condition (g_2) can be replaced by

(g_2') $G(\lambda x, y) = \lambda G(x,y),\ \lambda > 0,\ x, y \in S_r.$

If the function G satisfies the following property: there exists $k > 0$ such that

$$|G(x,y)| \leq k\|x\|\|y\|, \text{ for all } x, y \in \overline{B}_r, \tag{2.4}$$

then in this case, we say that the function G is subordinated to the norm $\|\cdot\|$.

Remark that all particular functions G defined above are subordinated to the norm.

2.3. **Main result.** The main result of this section is contained in the following theorem.

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a Banach space. Suppose that*

i) $f : E \to E$ is a completely continuous mapping,

ii) $G : E \times E \to \mathbb{R}$ is a mapping such that the properties (g_1) and (g_2) are satisfied for a particular $r > 0$,

iii) the following inequality is satisfied

$$G(f(x), x) < 0, \text{ for any } x \in S_r. \tag{2.5}$$

Then the equation (2.1) is almost solvable in $\overline{B}_r$.

Proof. If there is $x \in \overline{B}_r$ such that $f(x) = 0$, then the equation (2.1) is solvable and, consequently, almost solvable.

Suppose that $0 \notin f(\overline{B}_r)$, we will show that $0 \in \overline{f(\overline{B}_r)}$. Indeed, we suppose the contrary, that is, we suppose that $0 \notin \overline{f(\overline{B}_r)}$. In this case we can consider the mapping

$$F(x) = \frac{r}{\|f(x)\|} f(x), \quad x \in \overline{B}_r.$$

We have that $F : \overline{B}_r \to E$ is well defined and continuous. We have also the inclusion $F(\overline{B}_r) \subset S_r \subset \overline{B}_r$. As in the proof of Theorem 2 ([5]), we can show that $F(\overline{B}_r)$ is relatively compact. Therefore, applying the Schauder Fixed Point Theorem we obtain the existence of a point $x_* \in \overline{B}_r$, such that $F(x_*) = x_*$, $x_* \in S_r$. Then we have

$$0 > G(f(x_*), x_*) = G\left(\frac{\|f(x_*)\|}{r} x_*, x_*\right) \geq \frac{\|f(x_*)\|}{r} G(x_*, x_*) \geq 0$$

which is impossible. Hence, we must have that $0 \in \overline{f(\overline{B}_r)}$ and the proof is complete. □

We note that, if G satisfies the condition $G(x,x) \neq 0$ if $x \neq 0$ and only (g_2') then we can replace (2.5) by

$$G(x,x)G(f(x),x) < 0 \text{ for all } x \in S_r$$

Indeed, from $x_* = F(x_*)$ it follows

$$\|f(x_*)\| G(x_*, x_*) = rG(f(x_*), x_*), \quad x_* \in S_r$$

and consequently

$$G(x_*, x_*)G(f(x_*), x_*) \geq 0$$

which contradicts the precedent inequality.

2.4. **Some solvability results.** It is clear that if the hypothesis of the Theorem 2.1 are satisfied and

$$0 \notin (\overline{B}_r) \text{ implies } 0 \notin \overline{f(\overline{B}_r)} \tag{2.6}$$

then (2.1) is solvable. Indeed, if (2.6) is valid then F has a fixed point and a contradiction is then obtained. Using this remark we can obtain the following corollary.

Corollary 2.1. *If the hypothesis of the Theorem 2.1 are satisfies and if there exists $a > 0$ such that*

$$\|f(x) - f(y)\| \geq a\|x - y\| \textit{ for all } x, y \in \overline{B}_r \tag{2.7}$$

then the equation (2.1) is solvable and has a unique solution.

Proof. Suppose that the implication (2.6) is not valid, i.e. $0 \notin f(\overline{B}_r)$ but $0 \in \overline{f(\overline{B}_r)}$. If $x_m \in \overline{B}_r$ such that $f(x_m) \to 0$, then we have

$$\|x_m - x_p\| \leq \frac{1}{a}\|f(x_m) - f(x_p)\|,$$

which implies that $\{x_m\}$ is a Cauchy sequence and consequently $x_m \to \overline{x} \in \overline{B}_r$. Because f is continuous we have that $f(\overline{x}) = 0$ which contradicts the hypothesis. Therefore (2.6) holds. Because f is injective the solution of (2.1) is unique. □

Now, we consider the case where (2.7) is replaced by

$$\|f(x) - f(y)\| \leq a\|x - y\| \text{ for all } x, y \in \overline{B}_r \tag{2.8}$$

Set

$$M = \sup_{x \in \overline{B}_r} \|f(x)\|$$

Note that M is finite, because f is completely continuous.

Theorem 2.2. *Suppose that:*
i) $f : \overline{B}_r \to E$ *is a completely continuous mapping,*
ii) inequality (2.8) is satisfied,
iii) G *is an inner-product subordinated to the norm* $\|\cdot\|$, *i.e.*

$$|G(x, y)| \leq k\|x\|\|y\|, \quad x, y \in E, \ k > 0.$$

iv) the inequality

$$G(f(x), x) \leq -c \tag{2.9}$$

is satisfied for all $x \in S_r$, *where*

$$c = rk[ar + M]. \tag{2.10}$$

Then the equation (2.1) is solvable on $\overline{B}_r$.

Proof. Define the function $g : \overline{B}_r \to \mathbb{R}$ by $g(x) = G(f(x), x)$. By (2.9) we have

$$g(x) \leq -c < 0 \text{ for all } x \in S_r. \tag{2.11}$$

On the other hand

$$|g(x) - g(y)| = |G(f(x) - f(y), x) + G(f(y), x - y)| \leq$$

$$\leq k[\|f(x) - f(y)\|\|x\| + \|f(y)\|\|x - y\|] \leq k[ar + M]\|x - y\|$$

therefore

$$|g(x) - g(y)| \leq \alpha \|x - y\|, \quad x, y \in \overline{B}_r \tag{2.12}$$

where

$$\alpha = k[ar + M] \tag{2.13}$$

By (2.10) and (2.13) it follows that

$$c = r\alpha \tag{2.14}$$

Set

$$\delta_k = \frac{c}{2^k \alpha}. \tag{2.15}$$

It is easy to see that

$$\sum_{k=1}^{\infty} \delta_k = r.$$

Because

$$g(x) = g(x) - g(y) + g(y) \leq \alpha \|x - y\| + g(y),$$

it follows that for all $y \in S_r$ and $x \in \overline{B}_r$ such that

$$\|x - y\| \leq \delta_1 \tag{2.16}$$

we have

$$g(x) \leq \alpha\delta_1 - c \leq -\frac{c}{2}. \tag{2.17}$$

Set

$$r_1 = r - \delta_1, \quad C_1 = \{x \in \overline{B}_r |\ r_1 \leq \|x\| \leq r\}.$$

By (2.15), it follows that $r_1 > 0$. On the other hand for every $x \in C_1$ there exists $y \in C_1$ such that (2.16) hold. Consequently

$$g(x) \leq -\frac{c}{2} \text{ for all } x \in C_1.$$

Continuing this process we get

$$g(x) \leq -\frac{c}{2^k} \text{ for all } x \in C_k$$

where

$$C_k = \{x |\ r_k \leq \|x\| \leq r_{k-1}\}, \quad k \geq 1,\ r_0 = r.$$

Now we suppose that (2.6) is not true, then $0 \notin f(\overline{B}_r)$ but there is a sequence $(x_m) \subset \overline{B}_r$ such that $f(x_m) \to 0$. Therefore

$$g(x_m) \to 0, \quad m \to \infty. \tag{2.18}$$

We want to show that

$$x_m \to 0, \quad m \to \infty. \tag{2.19}$$

If (2.19) is not valid, there exists a subsequence, again denoted by (x_m), such that

$$\|x_m\| \geq \beta > 0, \quad \beta \leq r.$$

Because $r_k \to 0$, there exists $r_k \leq \beta$. Therefore $(x_m) \subset \bigcup_{j=1}^{k} C_j$ and consequently

$$g(x_m) \leq -\frac{c}{2^k}, \text{ for all } m \geq 1 \tag{2.20}$$

which contradicts (2.18).

But, from (2.19) we have that $f(0) = 0$ which contradicts the assumption $0 \notin f(\overline{B}_r)$. □

3. A solvability result in locally convex spaces

3.1. **Preliminaries.** Now we give an extension of Theorem 2.1 to locally convex spaces. To do this, we need to recall the following fixed point theorem.

Theorem (Arino-Gautier-Penot) *Let (E, τ) be a metrizable locally convex space and let $C \subset E$ be a weakly compact convex set.*

Then any weakly sequentially continuous mapping from C into C has a fixed point.

The reader can find a proof of this result in [1]. □

3.2. **Functions B and G.** Let $E(\tau)$ be a metrizable locally convex space. Suppose given a continuous mapping $B : E \to \mathbb{R}$ satisfying the following properties:

(b_1) $B(x) > 0$ for any $x \in E$, $x \neq 0$,

(b_2) $B(\lambda x) = \lambda B(x)$ for any $x \in E$ and $\lambda > 0$, $\lambda \in \mathbb{R}$.

Given $r > 0$, $(r \in \mathbb{R})$ we denote by $S_r^B = \{x \in E |\ B(x) = r\}$.

Suppose also given a mapping $G : E \times E \to \mathbb{R}$ satisfying conditions $(g_1), (g_2)$ where the set S_r is replaced by the set S_r^B.

We recall that a mapping $f : E \to E$ is called strongly continuous if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in E, weakly convergent to an element $x_* \in E$, we have that $\{f(x_n)\}_{n\in\mathbb{N}}$ is τ-convergent to $f(x_*)$.

3.3. **Main result.** We have the following result.

Theorem 3.1. *Let $E(\tau)$ be a metrizable locally convex space and suppose given a mapping $B : E \to \mathbb{R}$ satisfying conditions (b_1) and (b_2) and a mapping $G : E \times E \to \mathbb{R}$ satisfying conditions $(g_1), (g_2)$ with respect to S_r^B, for a particular r. Suppose that the set $D_r^B = \overline{conv}(S_r^B)$, is weakly compact.*

If $f : E \to E$ is a strongly continuous mapping such that $G(f(x), x) < 0$, for all $x \in S_r^B$, then either there exists an element $x_* \in D_r^B$ such that $f(x_*) = 0$ or $0 \in \overline{f(D_r^B)}$.

Proof. If there exists an element $x_* \in D_r^B$, such that $f(x_*) = 0$, then in this case the proof is complete.

Suppose that $f(x) \neq 0$ for any $x \in D_r^B$. We show that in this case $0 \in \overline{f(D_r^B)}$.

Indeed, suppose that $0 \notin \overline{f(D_r^B)}$. In this case we consider the mapping

$$F(x) = \frac{r}{B(f(x))} f(x), \text{ for any } x \in D_r^B.$$

We have that F is well defined and $F(D_r^B) \subseteq S_r^B \subseteq D_r^B$. Because $0 \notin \overline{f(D_r^B)}$, we can show that $F : D_r^B \to D_r^B$ is (w)-sequentially continuous. By Arino-Gautier-Penot Fixed Point Theorem there exists an element $x_* \in D_r^B$, such that $F(x_*) = x_* \in S_r^B$. We have that

$$f(x_*) = \frac{B(f(x_*))}{r} x_*,$$

which implies that

$$0 > G(f(x_*), x_*) \geq \frac{B(f(x_*))}{r} G(x_*, x_*) \geq 0$$

which is a contradiction. Therefore, we must have $0 \in \overline{f(D_r^B)}$, and the proof is complete. □

Remark 1. If the mapping B is also sequentially weakly continuous, then in this case, we can suppose in Theorem 3.1 that f is sequentially continuous from the weak to the weak topology.

Remark 2. The reader can find several continuity tests for a nonlinear mapping $f : E \to E$ with respect to the given topology and the weak topology in [7].

Remark 3. In Theorems 2.1 and 3.1 we can replace the inequality $G(f(x), x) < 0$ by $G(f(x), x) > 0$, if G is homogeneous with respect to the first variable, since the zeros of f and of $-f$ are the same.

3.4. **Second existence result.** In 1970, F. E. Browder introduced in [2] the notion of nonlinear operator of class $(S)_+$ and for this kind of nonlinear mapping he defined a topological degree.

I. V. Skrypnik in [10] presented a generalization of this topological degree and its applications to partial differential equations.

Let $(E, \|\cdot\|)$ be a Banach space and E^* the topological dual of E. We recall that a mapping $f : E \to E^*$ is demicontinuous on a set $D \subset E$ if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ convergent in norm to $x_0 \in D$, we have the equality

$$\lim_{n\to\infty} \langle f(x_n), u\rangle = \langle f(x_0), u\rangle \text{ for all } u \in E,$$

where $\langle\cdot,\cdot\rangle$ is the duality between E and E^*.

Also, we say that f is of class $(S)_+$ on D, if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D$ with $\{x_n\}$ weakly convergent to $x_0 \in E$ and such that $\limsup_{n\to\infty}\langle f(x_n), x_n - x_0\rangle \leq 0$, we have that $\{x_n\}_{n\in\mathbb{N}}$ is convergent in norm to x_0.

Finally, we say that $f : E \to E^*$ is ρ-strongly monotone if there exists a continuous strictly increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho(0) = 0$ and

$$\langle f(x) - f(y), x - y\rangle \geq \rho(\|x - y\|) \text{ for any } x, y \in E.$$

We have the following result.

Theorem 3.2. *Let $(E, \|\cdot\|)$ be a separable reflexive Banach space. Let $f : E \to E$ be a demicontinuous ρ-strongly monotone mapping.*

If there exists $r > 0$ such that $\langle f(x), x\rangle > 0$ for any $x \in S_r$, then the equation $f(x) = 0$ has a solution in $\overline{B}_r$.

Proof. This result is a consequence of Theorem 4.4 proved in [10] (p. 47), because we take $\overline{D} = \overline{B}_r$. We have $0 \in \overline{B}_r \setminus S_r$ and the assumption $\langle f(x), x\rangle > 0$ implies also that $f(x) \neq 0$ for any $x \in S_r$.

Moreover, in [6] was proved that any ρ strongly monotone mapping is of class $(S)_+$. Therefore, all the assumption of Theorem 4.4 ([10], p. 47) are satisfied and the theorem follows. We note that, the proof of this Theorem 4.4 is based on the Skrypnik-topological degree. □

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