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NONNEGATIVE SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS IN ORDERED BANACH SPACES

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Abstract. The main purpose of this paper is to make Krasnoselskii's compression-expansion fixed point theorem applicable to nonlinear integral equations in ordered Banach spaces.
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1. Introduction

Let X be a real Banach space with norm $|.|$ and $K \subset X$ be a closed cone of X. The goal of this paper is to establish sufficient conditions for the existence of nonnegative solutions (i.e., with values in K) to the nonlinear integral equation

$$u(t) = \int_0^T k(t,s) F(u)(s)\, ds, \quad t \in [0,T] \tag{1.1}$$

where $k : [0,T] \times [0,T] \to \mathbf{R}$ and $F : C([0,T];X) \to C([0,T];X)$ is an operator. The main tool is Krasnoselskii's compression-expansion fixed point theorem. This technique has been extensively applied in the literature to scalar equations, when $X = \mathbf{R}$, see [1, 2, 3, 5, 6, 8, 9, 10, 12] and references therein. In this paper, for the first time by our knowledge, Krasnoselskii's Theorem is used to discuss nonlinear integral equations in Banach spaces. Our existence theorems extend to equations in ordered Banach spaces previous results established for scalar equations by Meehan and O'Regan [9] and Erbe and Wang [3].

Let us recall Krasnoselskii's compression-expansion fixed point theorem.

Theorem 1.1. [7] *Let $(E, |.|)$ be a Banach space, and let $C \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, and let $N : C \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to C$ be a completely continuous operator such that either*

(i) $|N(u)| \leq |u|\,,\ \ u \in C \cap \partial\Omega_1$ *and* $|N(u)| \geq |u|\,,\ u \in C \cap \partial\Omega_2$; *or*

(ii) $|N(u)| \geq |u|\,,\ \ u \in C \cap \partial\Omega_1$ *and* $|N(u)| \leq |u|\,,\ u \in C \cap \partial\Omega_2$.

Then N has a fixed point in $C \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Basic facts about ordered Banach spaces can be found in [4] and [11]. Here we just recall a few of them. Any cone C in E induces a partial order on E. Thus, $x \leq y$ if and only if $y - x \in C$. We say that the norm $|.|$ is increasing with respect to C if $|x| \leq |y|$ whenever $0 \leq x \leq y$.

We seek solutions of (1.1) in the space $C([0,T]; X)$ of all continuous functions from $[0,T]$ to X, which are nonnegative in the sense that their values belong to the cone K. Hence by a nonnegative solution of (1.1) we mean a function $u \in C([0,T]; K)$ satisfying (1.1). Here

$$C([0,T]; K) = \{u \in C([0,T]; X) : u(t) \in K,\ t \in [0,T]\}\,.$$

Notice $C([0,T]; K)$ is a cone of $C([0,T]; X)$ and if the norm of X is increasing with respect to K, then so is the norm $|.|_\infty$ with respect to $C([0,T]; K)$, where $|u|_\infty = \max_{t\in[0,T]} |u(t)|$.

2. Main result

To establish the existence of a solution u in $C([0,T]; K)$ of the integral equation (1.1) we introduce the following conditions:

(H1) For each $t \in [0,T]$, $k_t = k(t,.) \in L^1(0,T; \mathbf{R}_+)$ and the map $t \mapsto k_t$ is continuous from $[0,T]$ to $L^1(0,T)$;

(H2) There exists $\mu \in (0,1)$, $\kappa \in L^1(0,T)$ and an interval $[a,b] \subseteq [0,T]$, $a < b$, such that

$$\begin{aligned} k(t,s) &\leq \kappa(s)\,,\ t \in [0,T]\,,\ \text{a.e. } s \in [0,T]\,;\ \text{ and} \\ \mu\,\kappa(s) &\leq k(t,s)\,,\ t \in [a,b]\,,\ \text{a.e. } s \in [0,T]\,; \end{aligned}$$

(H3) $F : C([0,T];K) \to C([0,T];K)$ and there exists $\phi : K \to K$ such that

$$\phi(x) \le F(u)(t), \ \ t \in [a,b]$$

whenever $u \in C([0,T];K)$, $x \in K$ and $x \le u(t)$ for all $t \in [a,b]$;

(H4) There exists $\alpha > 0$ such that

$$|F(u)(t)| \le \frac{\alpha}{\sup\limits_{t \in [0,T]} \int_0^T k(t,s)\,ds}$$

for all $t \in [0,T]$ and $u \in C([0,T];K)$ with $|u|_\infty = \alpha$;

(H5) There exists $\beta > 0$, $\beta \ne \alpha$ and $t^* \in [0,T]$ such that

$$\inf\{|\phi(x)| : x \in K, \, |x| = \mu\beta\} \int_a^b k(t^*,s)\,ds \ge \beta;$$

(H6) The operator N defined by $N(u)(t) = \int_0^T k(t,s)F(u)(s)\,ds$ is completely continuous from $C([0,T];K)$ to $C([0,T];X)$.

Theorem 2.1. *If* (H1)-(H6) *are satisfied, then* (1.1) *has at least one solution* $u \in C([0,T];K)$ *such that*

$$\mu\, u(t) \le u(t') \ \ \text{for } t \in [0,T], \ \ t' \in [a,b]$$

and either $0 < \alpha \le |u|_\infty \le \beta$ *if* $\alpha < \beta$ *or* $0 < \beta \le |u|_\infty \le \alpha$ *if* $\beta < \alpha$.

Proof. Assume $\alpha < \beta$. To apply Krasnoselskii's Theorem let

$$E = C([0,T];X),$$

$$C = \left\{u \in C([0,T];K) : \mu\, u(t) \le u(t') \ \text{ for } t \in [0,T],\, t' \in [a,b]\right\}$$

and let Ω_1 and Ω_2 be given by

$$\Omega_1 = \{u \in C([0,T];X) : |u|_\infty < \alpha\}$$
$$\Omega_2 = \{u \in C([0,T];X) : |u|_\infty < \beta\}.$$

From (H1), (H3) and (H6) we have that N maps $C([0,T];K)$ into itself and is completely continuous.

Moreover, if $u \in C([0,T];K)$, $t \in [0,T]$ and $t' \in [a,b]$, then from (H3) we have

$$\mu N(u)(t) \le \mu \int_0^T \kappa(s)F(u)(s)\,ds \le \int_0^T k(t',s)F(u)(s)\,ds = N(u)(t').$$

Consequently, $N : C \to C$ and N is completely continuous.

Now we prove that condition (i) in Krasnoselskii's Theorem holds. Let $u \in C \cap \partial\Omega_1$. Then $u \in C([0,T];K)$ and $|u|_\infty = \alpha$. Using (H4) we deduce that

$$|N(u)(t)| \le \int_0^T k(t,s)\,|F(u)(s)|\,ds \le \alpha$$

for all $t \in [0,T]$. Hence $|N(u)|_\infty \le \alpha = |u|_\infty$, that is (i) holds.

Next we show that (ii) is satisfied. Let $u \in C \cap \partial\Omega_2$. Then $u \in C([0,T];K)$, $|u|_\infty = \beta$ and $\mu\, u(t) \le u(t')$ for all $t \in [0,T]$ and $t' \in [a,b]$. In particular, $\mu\, u(t_0) \le u(t')$ for all $t' \in [a,b]$ and $t_0 \in [0,T]$ with $|u(t_0)| = |u|_\infty$. Now (H3) implies that $\phi(\mu\, u(t_0)) \le F(u)(s)$ on $[a,b]$. Then

$$\begin{aligned} N(u)(t^*) &= \int_0^T k(t^*,s)\,F(u)(s)\,ds \\ &\ge \phi(\mu\, u(t_0)) \int_a^b k(t^*,s)\,ds \end{aligned}$$

and since $|.|$ is increasing with respect to K, we deduce that

$$|N(u)(t^*)| \ge |\phi(\mu\, u(t_0))| \int_a^b k(t^*,s)\,ds.$$

This together with (H5) guarantees that $|N(u)|_\infty \ge \beta = |u|_\infty$. Thus (ii) also holds.

Therefore, Krasnoselskii's Theorem applies. □

Remark 2.1. Multiple solutions to equation (1.1) are guaranteed by Theorem 2.1 if the nonlinearity F satisfies assumptions (H3)-(H5) for several disjoint intervals $[\alpha,\beta]$ (or $[\beta,\alpha]$).

In particular, Theorem 2.1 yields the following existence result for the abstract integral equation

$$u(t) = \int_0^T k(t,s)\,f(u(s))\,ds, \quad t \in [0,T]. \tag{2.1}$$

Theorem 2.2. *Assume* (H1), (H2) *hold and there exists* $t^* \in [0,T]$ *with* $\varepsilon^* := \int_a^b k(t^*,s)\,ds > 0$*. In addition assume that* $f : K \to K$ *is completely continuous and increasing with respect to* K*, there exists* $\alpha > 0$ *such that*

$$|f(x)| \le \frac{\alpha}{\sup\limits_{t\in[0,T]} \int_0^T k(t,s)\,ds} \tag{2.2}$$

for all $x \in K$ *with* $|x| \leq \alpha$, *and*

$$\lim_{|x| \to 0} \frac{|f(x)|}{|x|} > \frac{1}{\mu\,\varepsilon^*} \quad or \quad \lim_{|x| \to \infty} \frac{|f(x)|}{|x|} > \frac{1}{\mu\,\varepsilon^*}. \tag{2.3}$$

Then (2.1) *has at least one non-zero solution in* $C([0,T];K)$.

Proof. Obviously (H3) is true with $\phi(x) = f(x)$ and (H4) is implied by (2.2). If the first inequality in (2.3) holds, then there is $\delta > 0$ such that $|f(x)| \geq \frac{1}{\mu\,\varepsilon^*}|x|$ for $|x| \leq \delta$. Now (H5) is true if we choose any $0 < \beta \leq \frac{\delta}{\mu}$ with $\beta < \alpha$. If the second inequality in (2.3) is true, then we may find a $\delta > 0$ with $|f(x)| \geq \frac{1}{\mu\,\varepsilon^*}|x|$ for $|x| \geq \delta$. So (H5) holds for any $\beta \geq \frac{\delta}{\mu}$ with $\beta > \alpha$. Thus Theorem 2.1 applies. □

Remark 2.2. In case that both inequalities in (2.3) hold, the existence of two non-zero solutions u_1 and u_2 is guaranteed, with $0 < |u_1|_\infty \leq \alpha \leq |u_2|_\infty$.

Theorem 2.2 immediately yields existence results for two point boundary value problems in Banach spaces.

As an example, we can extend to abstract equations the results from [3] concerning the problem

$$\begin{cases} u''(t) + q(t)\, f(u(t)) = 0, \quad 0 < t < 1 \\ A\,u(0) - B\,u'(0) = 0 \\ C\,u(1) + D\,u'(1) = 0. \end{cases} \tag{2.4}$$

Theorem 2.3. *Assume* $q \in C([0,1];\mathbf{R}_+)$, $q(t) > 0$ *on* $(0,1)$ *and* A, B, C, $D \in \mathbf{R}_+$ *and* $CB + AC + AD > 0$. *In addition assume that* $f : K \to K$ *is completely continuous, increasing with respect to* K *and*

$$\lim_{|x| \to 0} \frac{|f(x)|}{|x|} = 0 \quad and \quad \lim_{|x| \to \infty} \frac{|f(x)|}{|x|} = \infty. \tag{2.5}$$

Then (2.4) *has at least one non-zero solution in* $C([0,1];K)$.

Proof. In this case $k = G$, the Green's function for the boundary value problem

$$\begin{cases} -u'' = h, \quad 0 < t < 1 \\ A\,u(0) - B\,u'(0) = 0 \\ C\,u(1) + D\,u'(1) = 0 \end{cases}$$

and $[a,b] = \left[\frac{1}{4}, \frac{3}{4}\right]$ (see [3]). Also, the first equality in (2.5) implies that for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|f(x)| \leq \varepsilon\,|x|$ for $|x| \leq \delta_\varepsilon$. Now take

$\varepsilon = \varepsilon_0 = \frac{1}{\sup\limits_{t\in[0,1]} \int_0^1 G(t,s)ds}$ and $\alpha = \delta_{\varepsilon_0}$ to obtain (2.2). Furthermore, the second equality in (2.5) guarantees the second inequality in (2.3). Thus Theorem 2.2 applies. $\square$

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