Fixed Point Theory, Volume 5, No. 1, 2004, 53-64
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

A FUNCTIONAL INTEGRAL INCLUSION INVOLVING DISCONTINUITIES

B. C. DHAGE

Kasubai, Gurukul Colony
Ahmedpur-413 515, Dist: Latur
Maharashtra, India
e-mail: bcd20012001@yahoo.co.in

Abstract. In this paper the existence of external solutions of a discontinuous functional integral inclusion is proved under certain monotonicity conditions. As applications, some existence results for initial and respectively boundary value problems of ordinary differential inclusions are given. Our results improve the results of Dhage [5] and Dhage and O'Regan under weaker conditions. [6].

Key Words and Phrases: Fixed point theorem, integral inclusion and existence theorem.
2000 Mathematics Subject Classification: 47H10.

1. Introduction

The topic of differential and integral inclusions is of much interest in the subject of set-valued analysis. The existence theorems for the problems involving the inclusions are generally obtained under the assumption that the set-function in question is either lower or upper semi-continuous on the domain of its definition. See Aubin and Cellina [2] and the reference therein. Therefore another approach of proving the existence theorems for the inclusion problem involving the discontinuous set-functions is interesting and in Dhage [5] and Dhage and O'Regan [6] some results in this direction have been proved. In this paper we study the following discontinuous functional integral inclusion. Let $\mathbb{R}$ denote the real line and $2^{\mathbb{R}}$ denote the class of all non-empty subsets of $\mathbb{R}$. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the integral inclusion

$$x(t) - q(t) \in \int_0^{\sigma(t)} k(t,s)F(s, x(\eta(s)))ds \tag{1}$$

for $t \in J$, where $\sigma, \eta : J \to J$, $q : J \to \mathbb{R}$, $k : J \times J \to \mathbb{R}$, and $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$.

The integral inclusion (1) has been studied recently by O'Regan [7] for the existence result under Carathéodory condition of F. In the present work we discuss the existence of extremal solutions of the integral inclusion (1) under certain monotonicity condition of the set-function F. We do not require any type of continuity condition of F in our discussion. The results of this paper are the improvement upon the results proved in Dhage [5] and Dhage and O'Regan [6]. In the following section we prove a lattice fixed point theorem for the set-maps which we need in the sequel.

2. Fixed Point Theorem for Set-maps

A partially ordered set $(L, \leq)$ is called a lattice if for any $x, y \in L$, $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist. Let A be any subset of L. By $\vee A$ we mean an element $a^* \in L$ such that $x \vee a^* = a^*$ for all $x \in A$. Similarly by $\wedge A$ we mean an element $a_* \in L$ such that $x \wedge a_* = a_*$ for all $x \in A$. The element a_* and a^* are respectively called the infimum and supremum of A. $(L, \leq)$ is called a complete lattice if every subset of L has a infimum and supremum in L. A mapping $f : L \to L$ is called an isotone increasing if for any $x, y \in L$, $x \leq y$ imply $fx \leq fy$. A lattice fixed point theorem for isotone mappings is

Theorem 2.1. *(Tarski [8]) Let f be a isotone increasing selfmap of a complete lattice L. Then f has a fixed point and the set of all fixed point is a complete lattice.*

A mapping $T : L \to 2^L$ is called a multi-valued or set-valued or simply set-map on L. A point $u \in L$ is called a fixed point of T if $u \in Tu$. By $\mathcal{F}$ we denote the set of all fixed point of T, i.e., $\mathcal{F} = \{u \in L \,|\, u \in Tu\}$.

For any $A, B \in 2^L$, we denote (Dhage [5])

$$A \leq_d B \quad \text{iff} \quad a \leq b \quad \text{for all} \quad a \in A \text{ and } b \in B \tag{2}$$

and

$$\begin{aligned} A \leq B \quad &\text{iff for every} \quad a \in A \quad \text{there exists} \quad b \in B \quad \text{such that} \quad a \leq b \\ &\text{and for every} \quad b' \in B \quad \text{there exists an } a' \in A \quad \text{such that} \quad a' \leq b'. \end{aligned} \tag{3}$$

Remark 2.1. *It is clear that (2) ⇒ (3), but the following simple example shows that the implication (3) ⇒ (2) may not hold.*

Example 2.1: Let $L = [0, 1]$ with the usual order relation $\leq$ in $\mathbb{R}$. Define a set-map $T : \mathbb{R} \to 2^{\mathbb{R}}$ by $Tx = [0, x]$. Then for any $x_1, x_2 \in L$ with $x_1 \leq x_2$, we have $Tx_1 \leq Tx_2$, but $Tx_1 \not\leq_d Tx_2$.

Definition 2.1. *A set-map $T : L \to 2^L$ is called isotone increasing if for any $x, y \in L$, $x \leq y$ implies $Tx \leq Ty$.*

A lattice fixed point theorem for set-maps is

Theorem 2.2. *Let $(L, \leq)$ be a complete lattice and let $T : L \to 2^L$. Suppose that*

(a) *T is isotone increasing, and*
(b) *$\wedge Tx \in Tx$ and $\vee Tx \in Tx$ for each $x \in L$.*

Then $\mathcal{F}$ is non-empty and has a minimal and a maximal element.

Proof. The proof is similar to that in Dhage [5] and Dhage and O'Regan [6] with appropriate modifications, but for the sake of completeness we give the details of it. Define two single-valued mappings $f, g : L \to L$ by

$$f(x) = \vee Tx$$

and

$$g(x) = \wedge Tx$$

for $x \in L$. Obviously f and g are well defined and isotone increasing on L. To see this, let $x, y \in L$ be such that $x \leq y$. Then by hypotheses (a) and (b),

$$f(x) = \vee Tx \leq Ty \leq \vee Ty = f(y).$$

Hence an application of Theorem 2.1 yields that f has a minimal fixed point x_* and a maximal fixed point x^*. Similarly the map g has a minimal fixed point y_* and a maximal fixed point y^*. Thus the set $\mathcal{F}$ of all fixed points of T is non-empty. We shall show that the fixed points y^* and x^* are respectively the minimal and maximal element of $\mathcal{F}$. Let $u \in L$ be any fixed point of T. Take

$p = \sup L$, which clearly does exist since L is complete lattice. Now consider the lattice interval $[u, p]$ which is obviously complete. Notice that the mappings f is isotone increasing on $[u, p]$. We only prove that $f : [u, p] \to [u, p]$. To do this, it is enough to prove that if $x \in L$ with $u \leq x$, then $u \leq fx$. By definition of f, $u \leq \vee Tu = fu$ and by isotonicity of f, $fu \leq fx$. Hence $u \leq fu \leq fx$. As a result f defines a mapping $f : [u, p] \to [u, p]$. Now an application of Theorem 2.1 yields that f has a fixed point in $[u, p]$. But x^* is the maximal fixed point of f is L. So we have $u \leq x^*$. Similarly it is proved that $y^* \leq u$. Thus for any fixed point u of T, $y^* \leq u \leq x^*$. Consequently $\mathcal{F}$ has a minimal and a maximal fixed point. This completes the proof. □

An interesting corollary to Theorem 2.2 in an applicable form is

Corollary 2.1. *Let X be a Banach space and let $(X, \leq)$ be a complete lattice. Suppose that $T : X \to 2^X$ be a set-map such that*

(a) *T is isotone increasing ,and*
(b) *Tx is closed for each $x \in X$.*

Then $\mathcal{F}$ is non-empty and has a minimal and a maximal element.

Proof. Since Tx is a closed subset of the complete lattice X, $(Tx, \leq)$ is complete lattice for each $x \in L$. As a result $\inf Tx \in Tx$ and $\sup Tx \in Tx$ for each $x \in X$. Now the desired conclusion follows by an application of Theorem 2.2. □

Remark 2.2. *We note that Theorem 2.2 is an improvement upon the fixed point theorems for the set-maps proved either in Dhage [5] and Dhage and O'Regan [6] in view of Remark 2.1.*

3. Existence Results

Let $M(J, \mathrm{I\!R})$ and $B(J, \mathrm{I\!R})$ denote respectively the space of all measurable and bounded real-valued functions on J. We shall obtain the existence of the extremal solutions of the functional integral inclusion (1) in the space $BM(J, \mathrm{I\!R})$ of all bounded and measurable real-valued functions on J. Define a norm $\|\cdot\|_{BM}$ and an order relation $\leq$ in $BM(J, \mathrm{I\!R})$ by

$$\|x\|_{BM} = \max_{t \in J} |x(t)|$$

and $x \leq y$ iff $x(t) \leq x(t)$ for all $t \in J$.

Clearly $BM(J,\mathbb{R})$ is a Banach space with respect to this maximum norm which is also again a complete lattice w.r.t. the above order relation $\leq$. See Birkhoff [4]. By $L^1(J,\mathbb{R})$ we denote the space of all Lebesgue integrable functions on J with the usual norm $\|\cdot\|_{L^1}$.

We use the following notations in the sequel.

For any $A,B \in 2^{BM(J,\mathbb{R})}$, denote

$$A \pm B = \{a \pm b | a \in A \quad \text{and} \quad b \in B\}$$

and

$$\lambda a = \{\lambda a | a \in A\} \text{ for } \lambda \in \mathbb{R}.$$

Again

$$|A| = \{|a| | a \in A\}$$

and

$$\|A\| = \sup\{|a| | a \in A\}.$$

Let us denote

$$S_F(x) = \{v \in M(J,\mathbb{R}) | v(t) \in F(t,x(t)), \quad \text{a.e. } t \in J\}$$

and

$$S_F^1(x) = \{v \in L^1(J,\mathbb{R}) | v(t) \in F(t,x(t)), \quad \text{a.e. } t \in J\},$$

where $x \in M(J,\mathbb{R})$.

Definition 3.1. *The set function $F(t,x)$ is called isotone increasing in x almost every-where for $t \in J$ if for any $x,y \in M(J,\mathbb{R})$, $x \leq y$ implies $S_F(x) \leq S_F(y)$.*

We consider the following set of hypotheses in the sequel.

(H_0) The functions $\sigma,\eta : J \to J$ are continuous.

(H_1) The function $q : J \to \mathbb{R}$ is bounded and measurable.

(H_2) The function $k : J \times J \to \mathbb{R}$ is continuous and nonnegative and let $c = \sup_{t,s\in J} k(t,s)$.

(H_3) $F(t,x)$ is closed for each $(t,x) \in J \times \mathbb{R}$.

(H_4) $S_F(x) \neq \emptyset$ for each $x \in BM(J,\mathbb{R})$.

(H_5) $F(t,x)$ is isotone increasing in x almost everywhere for $t \in J$.

(H_6) There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|F(t,x)| \leq h(t), \text{ a.e. } t \in J.$$

for all $x \in \mathbb{R}$.

Remark 3.1. *We note that if* $(H_4) - (H_6)$ *hold, then every* $v \in S_F(x)$ *is Lebesgue integrable for each* $x \in M(J, \mathbb{R})$*, i.e.,*$S_F(x) = S^1_F(x)\ \forall\ x \in M(J, \mathbb{R})$.

Theorem 3.1. *Assume that the hypotheses* (H_0)–(H_6) *hold. Then the functional integral inclusion (1) has a minimal and a maximal solution on J.*

Proof. Define a subset L of $BM(J, \mathbb{R})$ by

$$L = \{x \in BM(J, \mathbb{R}) | \|x\|_{BM} \leq M^*\} \tag{4}$$

where $M^* = \|q\|_{BM} + K\|h\|_{L^1}$.

Clearly L is a closed and bounded subset of the complete lattice $(BM(J, \mathbb{R}), \leq)$, and so $(L, \leq)$ is a complete lattice. See Birkhoff [4].

Define a set-map T on L by

$$\begin{aligned} Tx &= \{u \in BM(J, \mathbb{R}) | u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(\eta(s))ds, v \in S^1_F(x)(\eta(.))\} \\ &= (K \circ N)(x) \end{aligned} \tag{5}$$

where the operator $N : BM(J, \mathbb{R}) \to 2^L$ is defined by

$$N(x) = \{v \in L^1(J, \mathbb{R}) | v \in S^1_F(x)(\eta(.))\} \tag{6}$$

and the operator $K : L^1(J, \mathbb{R}) \to BM(J, \mathbb{R})$ is defined by

$$Ky(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(\eta(s))ds, \ \ t \in J. \tag{7}$$

First we show that T maps L into 2^L. Let $x \in L$. Then for each $u \in Tx$, there exists a $v \in S_F(x)(\eta(.))$ with

$$u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(\eta(s))ds.$$

So we have

$$\begin{aligned} |u(t)| &\leq |q(t)| + \int_0^{\sigma(t)} k(t,s)|v(\eta(s))|ds \\ &\leq \|q\|_{BM} + c\|h\|_{L^1} \\ &= M^* \end{aligned}$$

for all $t \in J$. As a result $T : L \to 2^L$.

Next we show that Tx is closed subset of L for each $x \in L$. To finish, it is enough to show that the values of the operator N are closed in $L^1(J, \mathbb{R})$. Let $\{\omega_n\}$ be a sequence in $L^1(J, \mathbb{R})$ such that $\omega_n \to \omega$. Then $\omega_n \to \omega$ in measure. So there exists a subsequence S of the positive integers such that $\omega_n \to \omega$ a.e. $n \to \infty$ through S. Since the hypothesis (H_3) holds, the values of N are closed in $L^1(J, \mathbb{R})$. Thus for each $x \in L$, Tx is a non-empty, closed and bounded subset of L.

Finally we show that T is isotone increasing on L. Let $x, y \in L$ be such that $x \leq y$. Let $a_1 \in Tx$. Then there exists $u_1 \in S_F^m(x)(\eta(.))$ such that

$$a_1(t) = q(t) + \int_0^{\sigma(t)} k(t,s)u_1(\eta(s))ds, \ \ t \in J.$$

By hypothesis (H_5), there exists a $v_1 \in S_F^1(y)(\eta(.))$ such that $u_1(t) \leq v_1(t), \forall t \in J$. As a result we have

$$\begin{aligned} a_1(t) &= q(t) + \int_0^{\sigma(t)} k(t,s)u_1(\eta(s))ds \\ &\leq q(t) + \int_0^{\sigma(t)} k(t,s)v_1(\eta(s))ds \\ &= b_1(t) \end{aligned}$$

for all $t \in J$; here $b_1 \in Ty$. Similarly let $b_2 \in Ty$. Then there exists a $v_2 \in S_F^1(y)(\eta(.))$ such that

$$b_2(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v_2(\eta(s))ds, \ \ t \in J.$$

Now by (H$_5$), there exists a $u_2 \in S_F^m(y)(\eta(.))$ such that $u_2(t) \leq v_2(t)$ for $t \in J$. Hence we have

$$\begin{aligned} b_2(t) &= q(t) + \int_0^{\sigma(t)} k(t,s)v_2(\eta(s))ds \\ &\geq q(t) + \int_0^{\sigma(t)} k(t,s)u_2(\eta(s))ds \\ &= a_2(t) \end{aligned}$$

for all $t \in J$; here $a_2 \in Tx$. Hence $Tx \leq Ty$ i.e. T is isotone increasing on L.

Thus all the conditions of Corollary 2.1 are satisfied and hence an application of it yields that the fixed point set of T is non-empty and that it has minimal and maximal elements. This further implies that the integral inclusion (1) has a maximal and a minimal solution on J. This completes the proof. □

We note that the hypothesis (H$_6$) in Theorem 3.1 may be replaced with the following condition.

(H$_7$) There exists a function $\phi \in L^1(J, \mathbb{R})$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$|F(t,x)| \leq \phi(t)\psi(|x|), \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

Theorem 3.2. *Assume that the hypothesis (H$_0$)–(H$_5$) and (H$_7$) hold. Further if $\sigma(t) \leq t$, $\eta(t) \leq t, \forall t \in J$ and*

$$\int_{\|q\|_{BM}}^{\infty} \frac{ds}{\psi(s)} > c\|\phi\|_{L^1}, \tag{8}$$

then the integral inclusion (1) has a minimal and a maximal solution on J.

Proof. Define a subset L of $BM(J, \mathbb{R})$ by

$$L = \{x \in BM(J,\mathbb{R}) | x(t) \leq a(t), \ \forall t \in J\}$$

where $\alpha(t) = J^{-1}\left(c\int_0^t \phi(s)ds\right)$ and $J(z) = \int_{\|q\|_{BM}}^z \frac{ds}{\psi(s)}$.

Clearly the set L is well defined since α is a real-valued bounded function on J in view of condition (8). Obviously L is closed and bounded subset of $BM(J, \mathbb{R})$ and hence is a complete lattice. Define a set-map T on L by (5).

We first show that $T : L \to 2^L$. Let $x \in L$. Then for any $u \in Tx$, there exists a $v \in S^1_F(x)(\eta(.))$ such that

$$u(t) = q(t) + \int_0^{\sigma(t)} k(t,s)v(\eta(s))ds.$$

Therefore for any $t \in J$,

$$\begin{aligned} |u(t)| &\leq |q(t)| + \int_0^{\sigma(t)} k(t,s)|v(\eta(s))|ds \\ &\leq |q(t)| + \int_0^{\sigma(t)} k(t,s)|v(s)|ds \\ &\leq \|q\|_{BM} + c\int_0^t \phi(s)\psi(|x(\eta(s)))ds \\ &= \|q\|_{BM} + c\int_0^t \alpha'(s)ds \\ &= \alpha(t) \end{aligned}$$

since

$$\int_{\|q\|_{BM}}^{\alpha(s)} du/\psi(u) = c\int_0^s \phi(\tau)d\tau.$$

Hence we have $T : L \to 2^L$. It is further shown as in the proof of Theorem 3.1 that T is isotone increasing on L and Tx is closed for each $x \in L$. Now the desired conclusion follows by an application of Corollary 2.1. The proof is complete. □

4. Applications

In this section we obtain the existence theorems for extremal solutions to initial and boundary value problems of ordinary differential inclusions by the applications of the main existence result of the previous section.

4.1. Initial Value Problem: Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the initial value problem (in short IVP) of ordinary functional differential inclusion,

$$\begin{aligned} &x' \in F(t, x(\eta(t)) \quad \text{a.e. } t \in J \\ &x(0) = x_0 \in \mathbb{R}, \end{aligned} \tag{9}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ and $\eta : J \to J$ is continuous.

By the solution of the IVP (9) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies the relations in (9) on J, that is, there exists a $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, x(\eta(t))$ for all $t \in J$ such that $x'(t) = v(t)$ a.e. $t \in J$ and $x(0) = x_0$, where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

Clearly $AC(J, \mathbb{R})$ is a Banach space with respect to the norm $\|\cdot\|_C$ given by $\|x\|_C = \sup\{|x(t)| \ t \in J\}$ which is also a complete lattice with respect to the order relation $\leq$ defined by $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in J$.

Theorem 4.1. *Assume that hypotheses (H_0)–(H_6) hold. Then the functional differential inclusion (9) has a maximal and minimal solution on J.*

Proof. A function $x : J \to \mathbb{R}$ is a solution of the IVP (9) if and only if it is a solution of the integral inclusion

$$x(t) - x_0 \in \int_0^t F(s, x(\eta(s)))ds, \ \ t \in J. \tag{10}$$

Now the desired conclusion follows by an application of Theorem 3.1 with $q(t) = x_0$, $\sigma(t) = t$ for all $t \in J$ and $k(t, s) = 1 \ \forall \ t, s \in J$, since $AC(J, R) \subset BM(J, R)$. $\square$

Theorem 4.2. *Assume that hypotheses (H_3)–(H_5) and (H_7) hold. Further if $\eta(t) \leq t \ \forall \ t \in J$ and if condition (8) holds, then IVP (9) has a maximal and a minimal solution on J.*

Proof. The proof is similar to Theorem 4.1 and now the conclusion follows by an application of Theorem 3.2. $\square$

4.2. Boundary Value Problems: Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the first and second boundary value problems (in short BVPs) of ordinary functional differential inclusion

$$\begin{aligned} &x''(t) \in F(t, x(\eta(t)), a.e. t \in J \\ &x(0) = 0 = x(1) \end{aligned} \tag{11}$$

and

$$\begin{aligned} &x''(t) \in F(t, x(\eta(t)), a.e. t \in J \\ &x(0) = 0 = x'(1) \end{aligned} \tag{12}$$

where $F : J \times \mathbb{R} \to 2^{\mathbb{R}}$ and $\eta : J \to J$ is continuous. By the solution of the BVP (11) or (12) we mean a function $x \in AC^1(J, \mathbb{R})$ that satisfies the relations in (11) or (12), where $AC^1(J, \mathbb{R})$ is the space of all continuous real-valued functions whose first derivative exists and is absolutely continuous on J. A solution x_M of BVP (11) or (12) is called maximal if for any solution x of such BVP, $x(t) \le x_M(t)$ for all $t \in J$. Similarly a minimal solution of BVP (11) or (12) is defined.

Theorem 4.3. *Assume that hypotheses (H_3)–(H_6) hold. Then BVP (11) has a minimal and a maximal solution on J.*

Proof. A function $x : J \to R$ is a solution of BVP (11) if and only if it is a solution of the integral inclusion

$$x(t) \in \int_0^t G(t,s)F(s, x(\eta(s)))ds, \quad t \in J \tag{13}$$

where $G(t,s)$ is a Green's function associated with the homogeneous linear BVP

$$\begin{gathered} x''(t) \in F(t, x(\eta(t))), \quad \text{a. e. } t \in J \\ x(0) = 0 = x'(1). \end{gathered}$$

It is known that $G(t,s)$ is a continuous and nonnegative real-valued function on $J \times J$. Now an application of Theorem 3.1 with $q(t) = 0$, $s(t) = 1$ for all $t \in J$ and $k(t,s) = G(t,s), \forall\ t,s \in J$ yields that BVP (11) has a minimal and a maximal solution on J, $AC1(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. $\square$

Theorem 4.4. *Assume that hypotheses (H_3)–(H_5) and (H_7) hold. Further if $\eta(t) \le t$, $\forall\ t \in J$ and if the condition (8) holds, then BVP (11) has a minimal and a maximal solution on J.*

Proof. The proof is similar to Theorem 4.3 and now the conclusion follows by an application of Theorem 3.2. $\square$

Theorem 4.5. *Assume that hypotheses (H_3)–(H_6) hold. Then BVP (12) has a minimal and a maximal solution on J.*

Proof. A function $x : J \to \mathbb{R}$ is a solution of BVP (12) if and only if it is a solution of the integral inclusion

$$x(t) \in \int_0^1 H(t,s)F(s, x(\eta(s)))ds, \quad t \in J$$

where $H(t,s)$ is a Green's function for the BVP

$$x''(t) \in F(t, x(\eta(t))), \quad \text{a. e. } t \in J,$$

$$x(0) = 0 = x'(1).$$

It is known that $H(t,s)$ is a continuous and nonnegative real-valued function on $J \times J$. Now an application of Theorem 3.1 with $q(t) = 0$, $\sigma(t) = 1$ for all $t \in J$ and $k(t,s) = H(t,s), \forall\ t, s \in J$ yields that BVP (12) has a minimal and a maximal solution on J, $AC^1(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. $\square$

Theorem 4.6. *Assume that hypotheses (H_3)–(H_5) and (H_7) hold. Further if $\eta(t) \in J$ and if the condition (8) holds, then BVP (12) has a minimal and a maximal solution on J.*

Proof. The proof is similar to Theorem 4.5 and now the conclusion follows by an application of Theorem 3.2. $\square$

References

[1] J. Appel, H. T. Nguven and P. Zabreiko, *Multi-valued superposition operators in ideal spaces of vector functions*, Indag. Math., **3**(1992), 1-8.

[2] J. Aubin and A. Cellina , *Differential Inclusions,* Springer Verlag, 1984.

[3] P. B. Bailey, L. F. Shampine and P. E. Waltman, *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York, 1968.

[4] G. Birkhoff, *Lattice Theory,* Amer. Math. Soc. Coll. Publ. Vol 25, New York, 1967.

[5] B. C. Dhage, *A lattice fixed point theorem for multi-valued mappings with applications*, Chinese J. Math., **19**(1991), 11-22.

[6] B. C. Dhage and D. O' Regan, *A lattice fixed point theorem and multi-valued differential equations*, Functional Diff. Equations, **9**(2002), 109-115.

[7] D. O' Regan, *Integral inclusions of upper semi-continuous or lower semi-continuous type*, Proc. Amer. Math. Soc., **124**(1996), 2391-2399.

[8] A. Tarski, *A lattice theoretical fixed point theorem and its applications*, Pacific J. Math., **5**(1955), 285-310.