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FIRST ORDER FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH BOTH ADVANCED AND RETARDED ARGUMENTS

IOAN A. RUS AND VERONICA DARZU-ILEA

Babeş-Bolyai University, Department of Applied Mathematics,
str. M. Kogălniceanu 1, 3400 Cluj-Napoca
e-mail: iarus@math.ubbcluj.ro, vdarzu@math.ubbcluj.ro

Abstract. The purpose of this paper is to study a boundary value problem with parameter for first order functional-differential equations with retarded and advanced arguments, by applying fixed point theory.
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1. Introduction

Comparing with functional differential equation with delay (which have had a great development in the last decades), functional differential equation of mixed type have had a slower contribution in mathematical researches. This is due the fact that this mixed type equations have two type of argument: with delay and with advance, which have different behavior. These equations have the important feature that the history and the future status of the system both affect its change rate at the present time. It is well known that it is not so complicate to study the case with odd derivative of the MFDE; but the case with even derivative could end really complicated.

This type of equations came from different fields of applications. For example, in optimal control problem with delay, the Euler equation determining the optimal solution often involves systems of functional differential equations with both advanced and delayed terms (L.S. Pontryagin, R.V. Gamkreledze and E.F. Mischenko[14]).

In 1989, A. Rustichini investigated a specific mixed functional differential equation arising in a special way from a competitive economy (A. Rustichini[19]), (A. Rustichini[20]).

A physical justification of mixed functional differential equation was also discussed by Schulman in 1973-1974 (L.S. Schulman[21]), (L.S. Schulman[22]) in the field of time symmetric electrodynamics and absorber theory of Wheeler and Feynman (J.A. Wheeler and R.P. Feynman[10]). The qualitative analysis of mixed functional differential equations(MFDE) is quite complicated and even the basic existence - uniqueness theory has not been establish.

In 1997, J. Wu and X. Zou [23] investigated the existence of periodic travelling waves and travelling wave fronts of a MFDE.

Some other fields of interest for this type of equations would be: population genetics (1975 - D.G. Aronson, H.F. Weinberger [23]), population growth (1993 - D. Hankerson, B. Zinner; 1992 - J. Wu, W. Krowcewicz; 1974 - S.A. Levin; 1996 - N. Madras, J. Wu, X. Zou; 1983 - M.A. Cohen, S. Grossberg; 1982-1984 - J.J. Hopfield; 1989 - F.J. Pineda [23]). Other examples include a model for propagation of nerve pulses in myelinated axons where the membrane is excitable only at spatially discreet site (1981 - J. Bell, N.F. Britton J.W. Cahn, S.N. Chow, E.S. Vleck [23]).

An important contribution in MFDE study are the papers of J. Mallet-Paret (J. Mallet-Paret[11]), (J. Mallet-Paret[13]), (J. Mallet-Paret[12]).

The purpose of this paper is to study a boundary value problem with parameter for first order functional-differential equations with advanced and retarded arguments, by applying fixed point theory.

2. Statement of the problem

Let $a,\ b \in R,\ a < b,\ h > 0,\ f \in C([a,b] \times R^3),\ \phi \in C([a-h,a])$ and $\psi \in C[b,b+h]$ be given.

The problem is to determine: $x \in C(a-h,b+h) \cap C^1[a,b]$ and $\lambda \in R$ such that

$$x'(t) = f(t, x(t), x(t-h), x(t+h)) + \lambda, \quad t \in (a,b) \tag{1}$$

$$x(t) = \phi(t), \quad t \in [a-h, a] \tag{2}$$

$$x(t) = \psi(t), \quad t \in [b, b+h] \tag{3}$$

There is an increasing need to investigate this type of MFDE since some biological and medical models involve such equations. In recent years, there have apparent a numbers of results of mixed-type equations reflecting a variety of research interests. For example, H. Chi, J. Bell, B. Hassard studied in 1986 a biological model [2]:

$$x'(t) = f(x(t)) + x(t-\sigma) - 2x(t) + x(t+\sigma), \quad -\infty < t < \infty \tag{4}$$

with $x(-\infty) = 0$ and $(+\infty) = 1$ by numerical methods.

Here we study the existence and uniqueness of the solution and the data dependence for the general case (1)+(2)+(3).

We can suppose the equation (1) as a model for a specific disease which depends on the physical condition of the subject (past argument) and the future treatment. The parameter can be an outside factor that can cause the death of the subject. The initial condition (2) is the past observation of the illness and condition (3) is, from a statistically point of view, the expectation of stabilization or recover.

We could also look at this equation in the theory of epidemiology. This abstract model can be seen also as follows: ϕ - the population state , ψ - the way the population should be and λ - a control argument. Also $x\prime(t)$ is the speed of growth of the population with the law $x\prime = f + \lambda$. If a part of population is sacrificed then $\lambda < 0$, and if the population is numerically extending then $\lambda > 0$.

3. Existence and uniqueness

The purpose of this section is to find the conditions for the existence and uniqueness of the solution of problem (1)+(2)+(3).

Let (x, λ) be a solution of (1)+(2)+(3). We remark that it follows:

$$x(t) = \begin{cases} \phi(t), & t \in [a-h, a] \\ \phi(a) + \int_a^t f(s, x(s), x(s-h), x(s+h))ds + & \\ +\lambda(t-a), & t \in [a, b] \\ \psi(t), & t \in [b, b+h] \end{cases} \tag{5}$$

From the condition of continuity in $t = b$ we have:

$$\lambda = \frac{\psi(b) - \phi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))ds. \tag{6}$$

So the problem (1)+(2)+(3) is equivalent with

$$x = A(x) \;\; and \;\; \lambda \; = \; second \;\; part \;\; of \;\; (6),$$

where $A : C[a-h, b+h] \to C[a-h, b+h]$ and

$$A(x)(t) := \begin{cases} \phi(t), & t \in [a-h, a] \\ \phi(a) + \frac{t-a}{b-a}(\psi(b) - \phi(a)) - & \\ -\frac{t-a}{b-a}\int_a^b f(s, x(s), x(s-h), x(s+h))ds + & \\ +\int_a^t f(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\ \psi(t), & t \in [b, b+h] \end{cases} \tag{7}$$

Consider the Banach space $C[a-h, b+h]$ with Cebysev norm, $\| \cdot \|$.

We have for $t \in [a, b]$ $|A(x)(t) - A(y)(t)| = |\frac{t-a}{b-a}\int_a^b [f(s, x(s), x(s-h), x(s+h)) - f(s, y(s), y(s-h), y(s+h))]ds + \int_a^t [f(s, x(s), x(s-h), x(s+h)) - f(s, y(s), y(s-h), y(s+h))]ds| \leq 3L_f \int_a^b \| x - y \| + 3L_f \int_a^t \| x - y \| \leq 6L_f(b-a) \| x - y \|$.

Also we have for $t \in [a-h, a] \cup [b, b+h]$ $|A(x)(t) - A(y)(t)| = 0$.

Then A is Lipschitz with a Lipschitz constant $L_A = 6 \cdot L_f(b-a)$.

By the contraction principle we have:

Theorem 3.1. *We suppose that:*

(i) *there is* $L_f > 0$ *such that:*

$$\| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \| \leq L_f(\| u_1 - u_2 \| + \| v_1 - v_2 \| + \| w_1 - w_2 \|),$$

for all $t \in [a, b],\; u_i,\; v_i,\; w_i \in R,\; i = 1, 2;$

(ii) $6L_f(b-a) < 1$.

Then the problem (1)+(2)+(3) has a unique solution. Moreover if (x^*, λ^*) *is the unique solution of (1)+(2)+(3) , then*

$$x^* = \lim_{n\to\infty} A^n(x), \quad for \;\; all \;\; x \in C[a-h, b+h].$$

and

$$\lambda^* = \frac{\psi(b) - \phi(a)}{b-a} - \frac{1}{b-a}\int_a^b f(s, x^*(s), x^*(s-h), x^*(s+h))ds. \tag{8}$$

4. Data dependence

In this section we shall discus a theorem of data dependence for the solution of problem: (1)+(2)+(3). To prove data dependence relation we need the following lemma:

Lemma 4.1. *(I.A. Rus[16]) Let (X, d) be a complete metric space and $A, B : X \to X$ two operators. We suppose that:*

(i) A is an α-contraction;

(ii) there is $\eta > 0$ such that

$$d(A(x), B(x)) \leq \eta, \ \forall x \in X,$$

(iii) $x_B^* \in F_B$.

Then

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\alpha}.$$

when x_A^* is the unique fixed point of A.

We have

Theorem 4.1. *Let f_i, ϕ_i, ψ_i, $i = 1, 2$ under the hypothesis of theorem (3.1). We suppose that there exist $\eta_i > 0$, $i = 1, 2, 3$, such that:*

$$|\phi_1(t) - \phi_2(t)| \leq \eta_1, \ \forall \ t \in [a-h, a],$$

$$|\psi_1(t) - \psi_2(t)| \leq \eta_2, \ \forall \ t \in [b, b+h]$$

and

$$|f_1(t, u, v, w) - f_2(t, u, v, w)| \leq \eta_3, \ \forall \ t \in [a, b], \ u, v, w \in R$$

Then:

$$|x_1^* - x_2^*| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3(b-a)}{1 - 6L_{f1}(b-a)}$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{\eta_1 + \eta_2}{b-a} + \eta_3.$$

where (x_i^*, λ_i^*), $i = 1, 2$ are solutions of the problems: (1)+(2)+(3) with data f_1, ϕ_1, ψ_1, respectively with data f_2,ϕ_2,ψ_2.

Proof. If (x_1^*, λ_1^*) solution of problem (1)+(2)+(3) with data f_1,ϕ_1,ψ_1, then we take

$$A(x)(t) := \begin{cases} \phi_1(t), & t \in [a-h, a] \\ \phi_1(a) + \frac{t-a}{b-a}(\psi_1(b) - \phi_1(a)) - \\ -\frac{t-a}{b-a}\int_a^b f_1(s, x(s), x(s-h), x(s+h))ds + \\ +\int_a^t f_1(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\ \psi_1(t), & t \in [b, b+h] \end{cases} \tag{9}$$

If (x_2^*, λ_2^*) solution of problem (1)+(2)+(3) with data f_2, ϕ_2, ψ_2, then

$$B(x)(t) := \begin{cases} \phi_2(t), & t \in [a-h, a] \\ \phi_2(a) + \frac{t-a}{b-a}(\psi_2(b) - \phi_2(a)) - \\ -\frac{t-a}{b-a}\int_a^b f_2(s, x(s), x(s-h), x(s+h))ds + \\ +\int_a^t f_2(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\ \psi_2(t), & t \in [b, b+h] \end{cases} \tag{10}$$

From the hypothesis of the theorem it follows that:
$|Ax - Bx| \leq |\phi_1(a) - \phi_2(a)| + |\frac{t-a}{b-a}(\phi_1(a) - \phi_2(a))| + |\frac{t-a}{b-a}(\psi_1(a) - \psi_2(a))|$
$+|\frac{t-a}{b-a}\int_a^b [f_1(s, x(s), x(s-h), x(s+h)) - f_2(s, x(s), x(s-h), x(s+h))]ds| +$
$+|\int_a^t [f_1(s, x(s), x(s-h), x(s+h)) - f_2(s, x(s), x(s-h), x(s+h))]ds| \leq$
$\leq 2\eta_1 + \eta_2 + 2\eta_3(b-a)$.
From lemma (4.1) we have:

$$|x_1^* - x_2^*| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3(b-a)}{1 - 6L_{f1}(b-a)}$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{\eta_1 + \eta_2}{b-a} + \eta_3.$$

So the proof is complete.

5. Examples

Let $\mu \in R$, $\phi \in C[-1, 0]$, $\psi \in C[2, 3]$, $\lambda \in R$ be given. We consider the problem

$$\begin{cases} x\prime(t) = \mu[x(t-1) + x(t+1)] + \lambda, & t \in [0, 2] \\ x(t) = \phi(t), & t \in [-1, 0] \\ x(t) = \psi(t), & t \in [2, 3] \end{cases} \tag{11}$$

Then

$$x(t) = \begin{cases} \phi(t), & t \in [-1, 0] \\ \phi(0) + \int_0^t \mu[x(s-1) + x(s+1)]ds + \lambda t, & t \in [0, 2] \\ \psi(t), & t \in [2, 3] \end{cases} \tag{12}$$

From the continuity in $t = b$ we have:

$$\lambda = \frac{\psi(2) - \phi(0)}{2} - \frac{1}{2}\int_0^2 \mu[x(s-1) + x(s+1)]ds. \tag{13}$$

We note that if we take $A : C[-1, 3] \to C[-1, 3]$ defined by

$$A(x)(t) := \begin{cases} \phi(t), & t \in [-1, 0] \\ \phi(0) + \frac{t}{2}(\psi(2) - \phi(0)) - & \\ -\frac{t}{2}\int_0^2 \mu[x(s-1) + x(s+1)]ds + & \\ + \int_0^t \mu[x(s-1) + x(s+1)]ds, & t \in [0, 2] \\ \psi(t), & t \in [2, 3] \end{cases} \tag{14}$$

then the problem (11) is equivalent with

$x = A(x)$ and $\lambda =$ second part of (13).

If $0 < \mu < 1/8$ from Theorem 3.1 the problem (11) has a unique solution.

In what follows we discuss the data dependence of the solution.

Let ϕ_1, ϕ_2, ψ_1, ψ_2. We suppose that there are $\varepsilon_i > 0$, $i = 1, 2, 3$ such that:

$$|\phi_1(t) - \phi_2(t)| < \varepsilon_1$$

$$|\psi_1(t) - \psi_2(t)| < \varepsilon_2$$

$$|\mu_1 - \mu_2||x(t-1) + x(t+1)| < \varepsilon_3$$

Let us consider the problems:

$$\begin{cases} x\prime(t) = \mu_1[x(t-1) + x(t+1)] + \lambda, & t \in [0, 2] \\ x(t) = \phi_1(t), & t \in [-1, 0] \\ x(t) = \psi_1(t), & t \in [2, 3] \end{cases} \tag{15}$$

$$\begin{cases} x\prime(t) = \mu_2[x(t-1) + x(t+1)] + \lambda, & t \in [0, 2] \\ x(t) = \phi_2(t), & t \in [-1, 0] \\ x(t) = \psi_2(t), & t \in [2, 3] \end{cases} \tag{16}$$

If (x_1^*, λ_1^*) is a solution for the problem (15) and (x_2^*, λ_2^*) is a solution for the problem (16), we look for a estimation of $\| x_1^* - x_2^* \|$.

We have the operator

$$A(x)(t) := \begin{cases} \phi_1(t), & t \in [-1,0] \\ \phi_1(0) + \frac{t}{2}(\psi_1(2) - \phi_1(0)) - & \\ -\frac{t}{2}\int_0^2 \mu_1[x(s-1) + x(s+1)]ds + & \\ + \int_0^t \mu_1[x(s-1) + x(s+1)]ds, & t \in [0,2] \\ \psi_1(t), & t \in [2,3] \end{cases} \tag{17}$$

and

$$B(x)(t) := \begin{cases} \phi_2(t), & t \in [-1,0] \\ \phi_2(0) + \frac{t}{2}(\psi_2(2) - \phi_2(0)) - & \\ -\frac{t}{2}\int_0^2 \mu_2[x(s-1) + x(s+1)]ds + & \\ + \int_0^t \mu_2[x(s-1) + x(s+1)]ds, & t \in [0,2] \\ \psi_2(t), & t \in [2,3] \end{cases} \tag{18}$$

It follows that

$$|Ax - Bx| \leq 2\varepsilon_1 - \varepsilon_2 + 4\varepsilon_3$$

From Theorem 4.1 we have

$$|x_1^* - x_2^*| \leq \frac{2\varepsilon_1 + \varepsilon_2 + 4\varepsilon_3}{1 - 8\eta_1}$$

and

$$|\lambda_1^* - \lambda_2^*| \leq \frac{\varepsilon_1 + \varepsilon_2}{2} + \varepsilon_3.$$

6. Remarks

Remark 6.1. The Theorems (3.1) and (4.1) holds also if we make some changes on the arguments as follows: instead of $(x-h)$ we put $g_1(t)$ with $g_1 \in C([a,b],[a-h,b])$ and instead of $(x+h)$ we have $g_2(t)$, $g_2 \in C([a,b],[b,b+h])$.

Remark 6.2. Let $f \in C([a,b] \times X^3, X)$ with X a Banach space and $\phi \in C([a-h,a],X)$, $\psi \in C([b,b+h],X)$, $\lambda \in X$, $a,b \in R$, $a < b$. We take the problem (1)+(2)+(3).

In this case we consider the Cebysev norm on $C([a-h,b+h],X)$, then A is Lipschitz with the Lipschitz constant $L_A = 6L_f(b-a)$.

By the contraction principle we have

Theorem 6.1. *We suppose that:*

(i) *there is* $L_f > 0$ *such that:*

$$\| f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2) \| \leq L_f(\| u_1-u_2 \| + \| v_1-v_2 \| + \| w_1-w_2 \|),$$

for all $t \in [a,b],\ u_i,\ v_i,\ w_i \in X,\ i = 1,2;$

(ii) $6L_f(b-a) < 1.$

Then the problem (1)+(2)+(3) has a unique solution. Moreover if (x^*,λ^*) *is the unique solution of (1)+(2)+(3) , then*

$$x^* = \lim_{n\to\infty} A^n(x), \quad for \quad all \quad x \in C([a-h,b+h],X)$$

and

$$\lambda^* = \frac{1}{b-a}(\psi(b) - \phi(a)) - \frac{1}{b-a}\int_a^b f(s,x^*(s),x^*(s-h),x^*(s+h))ds. \quad (19)$$

By lemma (4.1) we have

Theorem 6.2. *Let* $f_i,\ \phi_i,\ \psi_i,\ i = 1,2$ *under the hypothesis of theorem (6.1). We suppose there exist* $\eta_i > 0,\ i = 1,2,3,$ *such that:*

$$\| \phi_1(t) - \phi_2(t) \| \leq \eta_1,\ \forall\ t \in [a-h,a],$$

$$\| \psi_1(t) - \psi_2(t) \| \leq \eta_2,\ \forall\ t \in [b,b+h]$$

and

$$\| f_1(t,u,v,w) - f_2(t,u,v,w) \| \leq \eta_3,\ \forall\ t \in [a,b],\ u,v,w \in X$$

Then:

$$\| x_1^* - x_2^* \| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3(b-a)}{1 - 6L_{f1}(b-a)}$$

and

$$\| \lambda_1^* - \lambda_2^* \| \leq \frac{\eta_1 + \eta_2}{b-a} + \eta_3.$$

where $(x_i^*,\lambda_i^*),\ i = 1,2$ *are solutions of the problems: (1)+(2)+(3) with data* $f_1,\ \phi_1,\ \psi_1,$ *respectively with data* $f_2,\phi_2,\psi_2.$

Remark 6.3. Let $f \in C([a,b] \times R^n \times R^n \times R^n, R^n)$, $\lambda \in R^n$, $\phi \in C([a-h,a],R^n)$, $\psi \in C([b,b+h],R^n)$, $a,b \in R$, $a < b$.

We extend the same discussion to n populations, with the specification that the populations are - in the same environment - prade or predator.

Let $x_1(t)$, $x_2(t)$, $\cdots$, $x_n(t)$ lows of growing, being continues and derivabless.

Then we have the system:

$$\begin{aligned} x_i'(t) = \quad & f_i(t, x_1(t), x_2(t), \cdots, x_n(t), x_1(t-h), x_2(t-h), \cdots, x_n(t-h), \\ & x_1(t+h), x_2(t+h), \cdots, x_n(t+h)) + \lambda_i, t \in [a,b], i = 1, 2, \cdots, n. \end{aligned} \tag{20}$$

where $f_i \in C([a,b] \times R^n \times R^n \times R^n, R)$, $i = 1, 2, \cdots, n$ and the initial conditions:

$$\begin{aligned} x_i(t) = \phi_i(t), \quad & t \in [a-h, a], i = 1, 2, \cdots, n \\ x_i(t) = \psi_i(t), \quad & t \in [b, b+h], i = 1, 2, \cdots, n. \end{aligned} \tag{21}$$

The problem is to determine $x_i \in C(a-h, b+h) \cap C^1[a,b]$ and $\lambda_i \in R$, $i = 1, 2, \cdots, n$ that suits for problem (20)+(21).

Let (x_i, λ_i) a solution of (20)+(21). We remark that it follows:

$$x_i(t) = \begin{cases} \phi_i(t), & t \in [a-h, a] \\ \phi_i(a) + \int_a^t f_i(s, x_1(s), x_2(t), \cdots, x_n(t), & \\ x_1(s-h), x_2(s-h), \cdots, x_n(s-h), & \\ x_1(s+h), x_2(s+h), \cdots, x_n(s+h))ds + \lambda_i(t-a), & t \in [a,b] \\ \psi_i(t), & t \in [b, b+h] \end{cases} \tag{22}$$

and

$$\lambda_i = \frac{\psi_i(b) - \phi_i(a)}{b-a} -$$

$$-\frac{1}{b-a} \int_a^b f_i(s, x_1(s), x_2(t), \cdots, x_n(t), x_1(s-h), x_2(t-h), \cdots, x_n(t-h),$$

$$x_1(s+h), x_2(t+h), \cdots, x_n(t+h))ds, \ i = 1, 2, \cdots, n. \tag{23}$$

So the problem (20)+(21) is equivalent with

$$x_i = A_i(x) \ \ \textit{and} \ \ \lambda_i \ = \ \textit{second} \ \ \textit{part} \ \ \textit{of} \ \ (23), \ \ i = 1, 2, \cdots, n$$

where $A_i : C([a-h,b+h],R^n) \to C([a-h,b+h],R^n)$ and

$$A_i(x)(t) := \begin{cases} \phi_i(t), & t \in [a-h,a] \\ \phi_i(a) + \frac{t-a}{b-a}(\psi_i(b) - \phi_i(a)) - & \\ -\frac{t-a}{b-a}\int_a^b f_i(s, x_1(s), x_2(t), \cdots, x_n(t), & \\ x_1(s-h), x_2(t-h), \cdots, x_n(t-h), & \\ x_1(s+h), x_2(t+h), \cdots, x_n(t+h)) & \\ +\int_a^t f_i(s, x_1(s), x_2(t), \cdots, x_n(t), & \\ x_1(s-h), x_2(t-h), \cdots, x_n(t-h), & \\ x_1(s+h), x_2(t+h), \cdots, x_n(t+h))ds, & t \in [a,b] \\ \psi_i(t), & t \in [b, b+h] \end{cases} \tag{24}$$

By the contraction principle we have

Theorem 6.3. *We suppose that:*

(i) there is $L_f > 0$ such that:

$$|f(t, u_{11}, \cdots, u_{1n}, v_{11}, \cdots, u_{1n}, w_{11}, \cdots, w_{1n}) -$$

$$-f(t, u_{21}, \cdots, u_{2n}, v_{21}, \cdots, u_{2n}, w_{21}, \cdots, w_{2n})| \leq$$

$$L_f(|u_{11} - u_{21}| \cdots + |u_{1n} - u_{2n}| + |v_{11} - v_{21}| \cdots + |v_{1n} - v_{2n}| +$$

$$+|w_{11} - w_{21}| \cdots + |w_{1n} - w_{2n}|) \tag{25}$$

for all $t \in [a,b]$, u_{ji}, v_{ji}, $w_{ji} \in R$, $j = 1,2$, $i = 1,2,\cdots,n$;

(ii) $6nL_f(b-a) < 1$.

Then the problem (20)+(21) has a unique solution. Moreover if (x^*, λ^*) is the unique solution of (20)+(21) , then

$$x_i^* = \lim_{n\to\infty} A_i^n(x), \quad for \quad all \quad x_i \in C[a-h, b+h]$$

and

$$\lambda_i^* = \frac{\psi_i(b) - \phi_i(a)}{b-a} - \frac{1}{b-a}\int_a^b f_i(s, x_1^*(s), x_2^*(t), \cdots, x_n^*(t),$$

$$x_1^*(s-h), x_2^*(t-h), \cdots, x_n^*(t-h), x_1^*(s+h), x_2^*(t+h), \cdots, x_n^*(t+h)). \tag{26}$$

Applying lemma 4.1 we have:

Theorem 6.4. *Let f_{ik}, ϕ_{ik}, ψ_{ik}, $k = 1,2$, $i = 1,2,\cdots,n$ under the hypothesis of theorem (6.3). We suppose that there exist $\eta_{ij} > 0$, $j = 1,2,3$, $i = 1,2,\cdots,n$, such that:*

$$|\phi_{i1}(t) - \phi_{i2}(t)| \leq \eta_{i1},\ \forall\ t \in [a-h, a],$$

$$|\psi_{i1}(t) - \psi_{i2}(t)| \leq \eta_{i2},\ \forall\ t \in [b, b+h]$$

and

$$\begin{gathered} |f_{i1}(t, u_1, \cdots, u_n, v_1, \cdots, v_n, w_1, \cdots, w_n) - \\ f_{i2}(t, u_1, \cdots, u_n, v_1, \cdots, v_n, w_1, \cdots, w_n)| \leq \eta_{i3}, \\ t \in [a,b],\ u_i, v_i, w_i \in R \end{gathered} \tag{27}$$

Then:

$$|x^*_{i1} - x^*_{i2}| \leq \frac{2\eta_{i1} + \eta_{i2} + 2\eta_{i3}(b-a)}{1 - 6nL_f(b-a)}$$

and

$$|\lambda^*_{i1} - \lambda^*_{i2}| \leq \frac{\eta_{i1} + \eta_{i2}}{b-a} + \eta_{i3}.$$

where (x^*_{ik}, λ^*_k), $i = 1,2,\cdots,n$, $k = 1,2$ are solutions of the problems: (20)+(21) with data f_{i1}, ϕ_{i1}, ψ_{i1}, respectively with data f_{i2},ϕ_{i2},ψ_{i2}.

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