FIRST ORDER FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH BOTH ADVANCED AND RETARDED ARGUMENTS

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Abstract. The purpose of this paper is to study a boundary value problem with parameter for first order functional-differential equations with retarded and advanced arguments, by applying fixed point theory.

Key Words and Phrases: functional-differential equations, boundary value problems, contraction principle.

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1. INTRODUCTION

Comparing with functional differential equation with delay (which have had a great development in the last decades), functional differential equation of mixed type have had a slower contribution in mathematical researches. This is due the fact that this mixed type equations have two type of argument: with delay and with advance, which have different behavior. These equations have the important feature that the history and the future status of the system both affect its change rate at the present time. It is well known that it is not so complicate to study the case with odd derivative of the MFDE; but the case with even derivative could end really complicated.

This type of equations came from different fields of applications. For example, in optimal control problem with delay, the Euler equation determining the optimal solution often involves systems of functional differential equations
with both advanced and delayed terms (L.S. Pontryagin, R.V. Gamkrelidze and E.F. Mischenko[14]).

In 1989, A. Rustichini investigated a specific mixed functional differential equation arising in a special way from a competitive economy (A. Rustichini[19]), (A. Rustichini[20]).

A physical justification of mixed functional differential equation was also discussed by Schulman in 1973-1974 (L.S. Schulman[21]), (L.S. Schulman[22]) in the field of time symmetric electrodynamics and absorber theory of Wheeler and Feynman (J.A. Wheeler and R.P. Feynman[10]). The qualitative analysis of mixed functional differential equations (MFDE) is quite complicated and even the basic existence - uniqueness theory has not been establish.

In 1997, J. Wu and X. Zou [23] investigated the existence of periodic travelling waves and travelling wave fronts of a MFDE.


An important contribution in MFDE study are the papers of J. Mallet-Paret (J. Mallet-Paret[11]), (J. Mallet-Paret[13]), (J. Mallet-Paret[12]).

The purpose of this paper is to study a boundary value problem with parameter for first order functional-differential equations with advanced and retarded arguments, by applying fixed point theory.

2. Statement of the problem

Let $a, b \in R$, $a < b$, $h > 0$, $f \in C([a, b] \times R^3)$, $\phi \in C([a - h, a])$ and $\psi \in C[b, b + h]$ be given.

The problem is to determine: $x \in C(a - h, b + h) \cap C^1[a, b]$ and $\lambda \in R$ such that

$$x'(t) = f(t, x(t), x(t - h), x(t + h)) + \lambda, \quad t \in (a, b)$$
\[ x(t) = \phi(t), \quad t \in [a-h, a] \quad (2) \]
\[ x(t) = \psi(t), \quad t \in [b, b+h] \quad (3) \]

There is an increasing need to investigate this type of MFDE since some biological and medical models involve such equations. In recent years, there have apparent a numbers of results of mixed-type equations reflecting a variety of research interests. For example, H. Chi, J. Bell, B. Hassard studied in 1986 a biological model [2]:
\[ x'(t) = f(x(t)) + x(t - \sigma) - 2x(t) + x(t + \sigma), \quad -\infty < t < \infty \quad (4) \]
with \( x(-\infty) = 0 \) and \( (+\infty) = 1 \) by numerical methods.

Here we study the existence and uniqueness of the solution and the data dependence for the general case (1)+(2)+(3).

We can suppose the equation (1) as a model for a specific disease which depends on the physical condition of the subject (past argument) and the future treatment. The parameter can be an outside factor that can cause the death of the subject. The initial condition (2) is the past observation of the illness and condition (3) is, from a statistically point of view, the expectation of stabilization or recover.

We could also look at this equation in the theory of epidemiology. This abstract model can be seen also as follows: \( \phi \) - the population state , \( \psi \) - the way the population should be and \( \lambda \) - a control argument. Also \( x'(t) \) is the speed of growth of the population with the law \( x' = f + \lambda \). If a part of population is sacrificed then \( \lambda < 0 \), and if the population is numerically extending then \( \lambda > 0 \).

3. Existence and uniqueness

The purpose of this section is to find the conditions for the existence and uniqueness of the solution of problem (1)+(2)+(3).

Let \((x, \lambda)\) be a solution of (1)+(2)+(3). We remark that it follows:
\[
x(t) = \begin{cases} 
\phi(t), & t \in [a-h, a] \\
\phi(a) + \int_{a}^{t} f(s, x(s), x(s-h), x(s+h))ds + \\
+\lambda(t - a), & t \in [a, b] \\
\psi(t), & t \in [b, b+h] 
\end{cases} \quad (5)
\]
We suppose that:
\[
\lambda = \frac{\psi(b) - \phi(a)}{b-a} - \frac{1}{b-a} \int_{a}^{b} f(s, x(s), x(s-h), x(s+h))ds.
\] (6)

So the problem (1)+(2)+(3) is equivalent with
\[
x = A(x) \text{ and } \lambda = \text{second part of (6)},
\]
where \(A : C[a-h, b+h] \rightarrow C[a-h, b+h]\) and
\[
A(x)(t) := \begin{cases} 
\phi(t), & t \in [a-h, a] \\
\phi(a) + \frac{t-a}{b-a}(\psi(b) - \phi(a)) - \frac{t-a}{b-a} \int_{a}^{b} f(s, x(s), x(s-h), x(s+h))ds + \\
+ \int_{a}^{t} f(s, x(s), x(s-h), x(s+h))ds, & t \in [a, b] \\
\psi(t), & t \in [b, b+h]
\end{cases}
\] (7)

Consider the Banach space \(C[a-h, b+h]\) with Cebysev norm, \(\| \cdot \|\).

We have for \(t \in [a, b]\) \(|A(x)(t) - A(y)(t)| = |\frac{t-a}{b-a} \int_{a}^{b} f(s, x(s), x(s-h), x(s+h)) - f(s, y(s), y(s-h), y(s+h))|ds + \int_{a}^{t} (f(s, x(s), x(s-h), x(s+h)) - f(s, y(s), y(s-h), y(s+h)))ds| \leq 3Lf \int_{a}^{b} \| x - y \| + 3Lf \int_{a}^{t} \| x - y \| \leq 6Lf(b-a) \| x - y \| .
\]

Also we have for \(t \in [a-h, a] \cup [b, b+h]\) \(|A(x)(t) - A(y)(t)| = 0\).

Then \(A\) is Lipschitz with a Lipschitz constant \(L_A = 6 \cdot L_f(b-a)\).

By the contraction principle we have:

**Theorem 3.1.** We suppose that:

(i) there is \(L_f > 0\) such that:

\[
\| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \| \leq L_f (\| u_1 - u_2 \| + \| v_1 - v_2 \| + \| w_1 - w_2 \| ),
\]

for all \(t \in [a, b], u_i, v_i, w_i \in R, i = 1, 2;\)

(ii) \(6L_f(b-a) < 1\).

Then the problem (1)+(2)+(3) has a unique solution. Moreover if \((x^*, \lambda^*)\) is the unique solution of (1)+(2)+(3), then

\[
x^* = \lim_{n \to \infty} A^n(x), \text{ for all } x \in C[a-h, b+h].
\]

and

\[
\lambda^* = \frac{\psi(b) - \phi(a)}{b-a} - \frac{1}{b-a} \int_{a}^{b} f(s, x^*(s), x^*(s-h), x^*(s+h))ds.
\] (8)
4. Data dependence

In this section we shall discuss a theorem of data dependence for the solution of problem: (1)+(2)+(3). To prove data dependence relation we need the following lemma:

Lemma 4.1. (I.A. Rus[16]) Let \((X, d)\) be a complete metric space and \(A, B : X \to X\) two operators. We suppose that:

(i) \(A\) is an \(\alpha\)-contraction;

(ii) there is \(\eta > 0\) such that

\[d(A(x), B(x)) \leq \eta, \forall x \in X,\]

(iii) \(x_B^* \in F_B\).

Then

\[d(x_A^*, x_B^*) \leq \frac{\eta}{1 - \alpha}.
\]

when \(x_A^*\) is the unique fixed point of \(A\).

We have

Theorem 4.1. Let \(f_i, \phi_i, \psi_i, i = 1, 2\) under the hypothesis of theorem (3.1). We suppose that there exist \(\eta_i > 0, i = 1, 2, 3,\) such that:

\[|\phi_1(t) - \phi_2(t)| \leq \eta_1, \forall t \in [a - h, a],\]

\[|\psi_1(t) - \psi_2(t)| \leq \eta_2, \forall t \in [b, b + h]\]

and

\[|f_1(t, u, v, w) - f_2(t, u, v, w)| \leq \eta_3, \forall t \in [a, b], u, v, w \in R\]

Then:

\[|x_1^* - x_2^*| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3(b - a)}{1 - 6L_f(b - a)}
\]

and

\[|\lambda_1^* - \lambda_2^*| \leq \frac{\eta_1 + \eta_2}{b - a} + \eta_3.
\]

where \((x_1^*, \lambda_1^*), i = 1, 2\) are solutions of the problems: (1)+(2)+(3) with data \(f_1, \phi_1, \psi_1\), respectively with data \(f_2, \phi_2, \psi_2\).
Proof. If \((x_1^*, \lambda_1^*)\) solution of problem (1)+(2)+(3) with data \(f_1, \phi_1, \psi_1\), then we take

\[
A(x)(t) := \begin{cases}
\phi_1(t), & t \in [a-h, a] \\
\phi_1(a) + \frac{t-a}{b-a} (\psi_1(b) - \phi_1(a)) - \\
- \frac{t-a}{b-a} \int_a^b f_1(s, x(s), x(s-h), x(s+h)) ds + \\
+ \int_a^t f_1(s, x(s), x(s-h), x(s+h)) ds, & t \in [a, b] \\
\psi_1(t), & t \in [b, b+h]
\end{cases}
\]

(9)

If \((x_2^*, \lambda_2^*)\) solution of problem (1)+(2)+(3) with data \(f_2, \phi_2, \psi_2\), then

\[
B(x)(t) := \begin{cases}
\phi_2(t), & t \in [a-h, a] \\
\phi_2(a) + \frac{t-a}{b-a} (\psi_2(b) - \phi_2(a)) - \\
- \frac{t-a}{b-a} \int_a^b f_2(s, x(s), x(s-h), x(s+h)) ds + \\
+ \int_a^t f_2(s, x(s), x(s-h), x(s+h)) ds, & t \in [a, b] \\
\psi_2(t), & t \in [b, b+h]
\end{cases}
\]

(10)

From the hypothesis of the theorem it follows that:

\[
|Ax - Bx| \leq |\phi_1(a) - \phi_2(a)| + \left| \frac{t-a}{b-a} (\phi_1(a) - \phi_2(a)) \right| + \left| \frac{t-a}{b-a} (\psi_1(a) - \psi_2(a)) \right| + \left| \frac{t-a}{b-a} \int_a^b f_1(s, x(s), x(s-h), x(s+h)) ds \right| + \\
+ \left| \frac{t-a}{b-a} \int_a^b f_2(s, x(s), x(s-h), x(s+h)) ds \right| + \\
+ \left| \int_a^t [f_1(s, x(s), x(s-h), x(s+h)) - f_2(s, x(s), x(s-h), x(s+h))] ds \right| \leq \\
\leq 2\eta_1 + \eta_2 + 2\eta_3 (b - a).
\]

From lemma (4.1) we have:

\[
|x_1^* - x_2^*| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3 (b - a)}{1 - 6L f_1(b - a)}
\]

and

\[
|\lambda_1^* - \lambda_2^*| \leq \frac{\eta_1 + \eta_2}{b - a} + \eta_3.
\]

So the proof is complete.

5. Examples

Let \(\mu \in R, \phi \in C[-1, 0], \psi \in C[2, 3], \lambda \in R\) be given. We consider the problem

\[
\begin{cases}
x(t) = \mu [x(t-1) + x(t+1)] + \lambda, & t \in [0, 2] \\
x(t) = \phi(t), & t \in [-1, 0] \\
x(t) = \psi(t), & t \in [2, 3]
\end{cases}
\]

(11)
Then
\[
x(t) = \begin{cases} 
\phi(t), & t \in [-1, 0] \\
\phi(0) + \int_0^t \mu[x(s - 1) + x(s + 1)]ds + \lambda t, & t \in [0, 2] \\
\psi(t), & t \in [2, 3]
\end{cases}
\] (12)

From the continuity in \( t = b \) we have:
\[
\lambda = \frac{\psi(2) - \phi(0)}{2} - \frac{1}{2} \int_0^2 \mu[x(s - 1) + x(s + 1)]ds.
\] (13)

We note that if we take \( A : C[-1, 3] \rightarrow C[-1, 3] \) defined by
\[
A(x)(t) := \begin{cases} 
\phi(t), & t \in [-1, 0] \\
\phi(0) + \frac{t}{2}(\psi(2) - \phi(0)) - \\
-\frac{1}{2} \int_0^2 \mu[x(s - 1) + x(s + 1)]ds + \\
+ \int_0^t \mu[x(s - 1) + x(s + 1)]ds, & t \in [0, 2] \\
\psi(t), & t \in [2, 3]
\end{cases}
\] (14)

then the problem (11) is equivalent with
\[
x = A(x) \text{ and } \lambda = \text{second part of (13)}.
\]

If \( 0 < \mu < 1/8 \) from Theorem 3.1 the problem (11) has a unique solution.

In what follows we discuss the data dependence of the solution.

Let \( \phi_1, \phi_2, \psi_1, \psi_2 \). We suppose that there are \( \varepsilon_i > 0, i = 1, 2, 3 \) such that:
\[
|\phi_1(t) - \phi_2(t)| < \varepsilon_1 \\
|\psi_1(t) - \psi_2(t)| < \varepsilon_2 \\
|\mu_1 - \mu_2||x(t - 1) + x(t + 1)| < \varepsilon_3
\]

Let us consider the problems:
\[
\begin{cases} 
x(t) = \mu_1[x(t - 1) + x(t + 1)] + \lambda, & t \in [0, 2] \\
x(t) = \phi_1(t), & t \in [-1, 0] \\
x(t) = \psi_1(t), & t \in [2, 3]
\end{cases}
\] (15)

\[
\begin{cases} 
x(t) = \mu_2[x(t - 1) + x(t + 1)] + \lambda, & t \in [0, 2] \\
x(t) = \phi_2(t), & t \in [-1, 0] \\
x(t) = \psi_2(t), & t \in [2, 3]
\end{cases}
\] (16)

If \( (x_1^*, \lambda_1^*) \) is a solution for the problem (15) and \( (x_2^*, \lambda_2^*) \) is a solution for the problem (16), we look for an estimation of \( \| x_1^* - x_2^* \| \).
We have the operator

\[
A(x)(t) := \begin{cases} 
\phi_1(t), & t \in [-1, 0] \\
\phi_1(0) + \frac{t}{2}(\psi_1(2) - \phi_1(0)) - \\
- \frac{t}{2} \int_0^t \mu_1 [x(s - 1) + x(s + 1)] ds + \\
+ \int_0^t \mu_1 [x(s - 1) + x(s + 1)] ds, & t \in [0, 2] \\
\psi_1(t), & t \in [2, 3] 
\end{cases}
\]  
(17)

and

\[
B(x)(t) := \begin{cases} 
\phi_2(t), & t \in [-1, 0] \\
\phi_2(0) + \frac{t}{2}(\psi_2(2) - \phi_2(0)) - \\
- \frac{t}{2} \int_0^t \mu_2 [x(s - 1) + x(s + 1)] ds + \\
+ \int_0^t \mu_2 [x(s - 1) + x(s + 1)] ds, & t \in [0, 2] \\
\psi_2(t), & t \in [2, 3] 
\end{cases}
\]  
(18)

It follows that

\[|Ax - Bx| \leq 2\varepsilon_1 - \varepsilon_2 + 4\varepsilon_3\]

From Theorem 4.1 we have

\[|x_1^* - x_2^*| \leq \frac{2\varepsilon_1 + \varepsilon_2 + 4\varepsilon_3}{1 - 8\eta_1}\]

and

\[|\lambda_1^* - \lambda_2^*| \leq \frac{\varepsilon_1 + \varepsilon_2}{2} + \varepsilon_3.\]

6. Remarks

**Remark 6.1.** The Theorems (3.1) and (4.1) holds also if we make some changes on the arguments as follows: instead of \((x - h)\) we put \(g_1(t)\) with \(g_1 \in C([a, b], [a - h, b])\) and instead of \((x + h)\) we have \(g_2(t)\), \(g_2 \in C([a, b], [b, b + h])\).

**Remark 6.2.** Let \(f \in C([a, b] \times X^3, X)\) with \(X\) a Banach space and \(\phi \in C([a - h, a], X)\), \(\psi \in C([b, b + h], X)\), \(\lambda \in X, a, b \in R, a < b\). We take the problem (1)+(2)+(3).

In this case we consider the Cebyshev norm on \(C([a - h, b + h], X)\), then \(A\) is Lipschitz with the Lipschitz constant \(L_A = 6L_f(b - a)\).

By the contraction principle we have
Theorem 6.1. We suppose that:

(i) there is $L_f > 0$ such that:
\[ \| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) \| \leq L_f (\| u_1 - u_2 \| + \| v_1 - v_2 \| + \| w_1 - w_2 \|), \]
for all $t \in [a, b]$, $u_i$, $v_i$, $w_i \in X$, $i = 1, 2$;
(ii) $6L_f(b - a) < 1$.

Then the problem $(1)+(2)+(3)$ has a unique solution. Moreover if $(x^*, \lambda^*)$ is the unique solution of $(1)+(2)+(3)$, then
\[ x^* = \lim_{n \to \infty} A^n(x), \text{ for all } x \in C([a - h, b + h], X) \]
and
\[ \lambda^* = \frac{1}{b - a} (\psi(b) - \phi(a)) - \frac{1}{b - a} \int_a^b f(s, x^*(s), x^*(s - h), x^*(s + h))ds. \quad (19) \]

By lemma (4.1) we have

Theorem 6.2. Let $f_i, \phi_i, \psi_i, i = 1, 2$ under the hypothesis of theorem (6.1). We suppose there exist $\eta_i > 0$, $i = 1, 2, 3$, such that:
\[ \| \phi_1(t) - \phi_2(t) \| \leq \eta_1, \forall t \in [a - h, a], \]
\[ \| \psi_1(t) - \psi_2(t) \| \leq \eta_2, \forall t \in [b, b + h] \]
and
\[ \| f_1(t, u, v, w) - f_2(t, u, v, w) \| \leq \eta_3, \forall t \in [a, b], u, v, w \in X \]

Then:
\[ \| x^*_1 - x^*_2 \| \leq \frac{2\eta_1 + \eta_2 + 2\eta_3(b - a)}{1 - 6L f_1(b - a)} \]
and
\[ \| \lambda^*_1 - \lambda^*_2 \| \leq \frac{\eta_1 + \eta_2 + \eta_3}{b - a} + \eta_3. \]

where $(x^*_i, \lambda^*_i), i = 1, 2$ are solutions of the problems: $(1)+(2)+(3)$ with data $f_1, \phi_1, \psi_1$, respectively with data $f_2, \phi_2, \psi_2$.

Remark 6.3. Let $f \in C([a, b] \times R^n \times R^n \times R^n), \lambda \in R^n, \phi \in C([a - h, a], R^n), \psi \in C([b, b + h], R^n), a, b \in R, a < b$.

We extend the same discussion to $n$ populations, with the specification that the populations are - in the same environment - prade or predator.

Let $x_1(t), x_2(t), \cdots, x_n(t)$ lows of growing, being continues and derivables.
Then we have the system:

\[
x'_i(t) = f_i(t, x_1(t), x_2(t), \cdots, x_n(t), x_1(t-h), x_2(t-h), \cdots, x_n(t-h),
x_1(t+h), x_2(t+h), \cdots, x_n(t+h)) + \lambda_i, t \in [a, b], i = 1, 2, \cdots, n.
\]  

(20)

where \( f_i \in C([a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}) \), \( i = 1, 2, \cdots, n \) and the initial conditions:

\[
x_i(t) = \phi_i(t), \quad t \in [a-h, a], i = 1, 2, \cdots, n
\]  

\[
x_i(t) = \psi_i(t), \quad t \in [b, b+h], i = 1, 2, \cdots, n.
\]  

(21)

The problem is to determine \( x_i \in C(a - h, b + h) \cap C^1[a, b] \) and \( \lambda_i \in \mathbb{R}, \quad i = 1, 2, \cdots, n \) that suits for problem (20)+(21).

Let \((x_i, \lambda_i)\) a solution of (20)+(21). We remark that it follows:

\[
x_i(t) = \begin{cases} 
\phi_i(t), & t \in [a - h, a] \\
\phi_i(a) + \int_a^t f_i(s, x_1(s), x_2(t), \cdots, x_n(t), \\
x_1(s-h), x_2(s-h), \cdots, x_n(s-h), \\
x_1(s+h), x_2(s+h), \cdots, x_n(s+h))ds + \lambda_i(t-a), & t \in [a, b] \\
\psi_i(t), & t \in [b, b+h] 
\end{cases}
\]  

(22)

and

\[
\lambda_i = \frac{\psi_i(b) - \phi_i(a)}{b - a} - \frac{1}{b - a} \int_a^b f_i(s, x_1(s), x_2(t), \cdots, x_n(t), x_1(s-h), x_2(t-h), \cdots, x_n(t-h), \\
x_1(s+h), x_2(t+h), \cdots, x_n(t+h))ds, \quad i = 1, 2, \cdots, n.
\]  

(23)

So the problem (20)+(21) is equivalent with

\[
x_i = A_i(x) \quad \text{and} \quad \lambda_i = \text{second part of (23)}, \quad i = 1, 2, \cdots, n
\]
where \( A_i : C([a-h,b+h],\mathbb{R}^n) \to C([a-h,b+h],\mathbb{R}^n) \) and
\[
A_i(x)(t) := \begin{cases}
\phi_i(t), & t \in [a-h,a] \\
\phi_i(a) + \frac{t-a}{b-a}(\psi_i(b) - \phi_i(a)) - \frac{t-a}{b-a} \int_a^b f_i(s, x_1(s), x_2(t), \ldots, x_n(t)), \\
x_1(s-h), x_2(t-h), \ldots, x_n(t-h), \\
x_1(s+h), x_2(t+h), \ldots, x_n(t+h) \\
+ \int_a^t f_i(s, x_1(s), x_2(t), \ldots, x_n(t)), \\
x_1(s-h), x_2(t-h), \ldots, x_n(t-h), \\
x_1(s+h), x_2(t+h), \ldots, x_n(t+h)) ds, & t \in [a,b] \\
\psi_i(t), & t \in [b,b+h]
\end{cases}
\] (24)

By the contraction principle we have

**Theorem 6.3.** We suppose that:

(i) there is \( L_f > 0 \) such that:
\[
|f(t, u_{i1}, \ldots, u_{1n}, v_{j1}, \ldots, u_{1n}, w_{i1}, \ldots, w_{1n}) - \\
- f(t, u_{21}, \ldots, u_{2n}, v_{j1}, \ldots, u_{2n}, w_{21}, \ldots, w_{2n})| \leq \\
L_f(|u_{i1} - u_{21}| \cdot \cdot \cdot |u_{1n} - u_{2n}| + |v_{j1} - v_{21}| \cdot \cdot \cdot + |v_{1n} - v_{2n}| + \\
+ |w_{i1} - w_{21}| \cdot \cdot \cdot + |w_{1n} - w_{2n}|) \] (25)
for all \( t \in [a,b], u_{ji}, \ v_{ji}, \ w_{ji} \in \mathbb{R}, j = 1, 2, \ i = 1, 2, \ldots, n; \)

(ii) \( 6nL_f(b - a) < 1. \)

Then the problem (20)+(21) has a unique solution. Moreover if \((x^*, \lambda^*)\) is the unique solution of (20)+(21) , then
\[
x_i^* = \lim_{n \to \infty} A_i^n(x), \quad \text{for all } x_i \in C[a-h,b+h]
\]
and
\[
\lambda_i^* = \frac{\psi_i(b) - \phi_i(a)}{b-a} - \frac{1}{b-a} \int_a^b f_i(s, x_1^*(s), x_2^*(t), \ldots, x_n^*(t)), \\
x_1^*(s-h), x_2^*(t-h), \ldots, x_n^*(t-h), x_1^*(s+h), x_2^*(t+h), \ldots, x_n^*(t+h)) \] (26)

Applying lemma 4.1 we have:
Theorem 6.4. Let $f_{ik}, \phi_{ik}, \psi_{ik}, k = 1, 2, i = 1, 2, \cdots, n$ under the hypothesis of theorem (6.3). We suppose that there exist $\eta_{ij} > 0, j = 1, 2, 3, i = 1, 2, \cdots, n$, such that:

$$
|\phi_{i1}(t) - \phi_{i2}(t)| \leq \eta_{i1}, \forall t \in [a - h, a],
$$

$$
|\psi_{i1}(t) - \psi_{i2}(t)| \leq \eta_{i2}, \forall t \in [b, b + h]
$$

and

$$
|f_{i1}(t, u_1, \cdots, u_n, v_1, \cdots, w_1, \cdots, w_n) - f_{i2}(t, u_1, \cdots, u_n, v_1, \cdots, v_1, \cdots, w_n)| \leq \eta_{i3}, \quad (27)
$$

$t \in [a, b], u_i, v_i, w_i \in R$

Then:

$$
|x_{i1}^* - x_{i2}^*| \leq \frac{2\eta_{i1} + \eta_{i2} + 2\eta_{i3}(b - a)}{1 - 6nL_f(b - a)}
$$

and

$$
|\lambda_{i1}^* - \lambda_{i2}^*| \leq \frac{\eta_{i1} + \eta_{i2}}{b - a} + \eta_{i3}.
$$

where $(x_{ik}^*, \lambda_{ik}^*), i = 1, 2, \cdots, n, k = 1, 2$ are solutions of the problems: (20)+(21) with data $f_{i1}, \phi_{i1}, \psi_{i1}$, respectively with data $f_{i2}, \phi_{i2}, \psi_{i2}$.

References


