Fixed Point Theory, Volume 5, No. 1, 2004, 37-52
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.htm

AN APPLICATION OF SCHAUDER'S FIXED POINT THEOREM IN STOCHASTIC MCSHANE'S MODELLING

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Abstract. The aim of this work is to establish an existence theorem for solutions of McShane's stochastic systems, applying Schauder's fixed point theorem. Also, we give new conditions relative to the coefficients for the continuity of solution with respect to the initial condition and respectively, the problem of parametric dependence of the solution process on the coefficients in McShane stochastic integral equations, generalizing the results of [1], [3], [4]. A short comment on continuous dependence of the solution on the disturbance and on modelling problems is given.

Key Words and Phrases: Stochastic integral equation of McShane's type, McShanes's stochastic belated integral, Schauder's fixed point theorem.
2000 Mathematics Subject Classification: 60H10, 47H10.

1. The mathematical modelling of several real-world problems leads to differential systems that involve radomness due to ignorance or uncertainties.

In the formulation of a mathematical model for a physical, biological or economical problems, we make errors in constructing the coefficients and errors in the initial conditions. For theoretical purposes it is sufficient to know that the change in the solution can be made arbitrary small by making the change in the coefficients and the initial values sufficiently small.

We consider families of stochastic integral equation systems of McShane type

$$X^i_\lambda(t,\omega) = \alpha^i_\lambda(t,\omega) + \sum_{j=1}^{r}\int_0^t g^i_{\lambda,j}(s, X_\lambda(s,\omega))dz_j(s,\omega) +$$

$$+\sum_{j,k=1}^{r}\int_0^t h^i_{\lambda,jk}(s,X_\lambda(s,\omega))dz_j(s,\omega)dz_k(s,\omega),\ \ i=1,2,...,n, \tag{1}$$

$t\in[0,a]$, $\omega\in\Omega$, $(\Omega,\mathcal{F},P)$ being some complete probability space, the coefficients depend on some parameter $\lambda\in\Lambda$, Λ being an open and bounded subset of R^n, and the integrals as McShane's stochastic belated integrals.

The continuous dependence of solutions on the parameter and respectively on the initial conditions (t_0,X_0) in McShane's stochastic integral equations were given by J.M.Angulo Ibáñez and R.Gutiérrez Jáimez [2] and A.Constantin [4].

Under the hypothesis of a weaker condition than the Lipschitz conditions on $g^i_{\lambda,j}$ and $h^i_{\lambda,jk}$ we prove the existence of solutions and the continuity with respect to the initial condition. Also, we consider the problem of parametric dependence of the solution process on the coefficients, generalizing the results of [2] and [4].

2. Let $\{\mathcal{F}_t\}_{0\le t\le a}$, $a\in\mathbb{R}_+$, be a family of complete $\sigma-$subalgebras of $\mathcal{F}$ such that if $0\le s\le t$, then $\mathcal{F}_s\subset\mathcal{F}_t$.

Let L_2 be the space of all random variables $y:\Omega\to\mathbb{R}$ with finite L_2-norm $||.||$ and let L_2^n be the space of all random vectors $x:(\Omega,\mathcal{F},P)\to\mathbb{R}^n$ with the norm $||.||_n$, $||x||_n^2=\sum\limits_{i=1}^{n}||x_i||^2$, $x=(x_1,...,x_n)\in L_2^n$. Let us denote also for each $\mathbb{R}^n$-valued process $X(t,\omega)$ the norm

$$|||X|||=\,=\sup_{t\in[0,a]}||X(t)||_n.$$

Let $\mathcal{C}([0,a])$ denote the space of all processes $x:[0,a]\to L_2^n$ which are continuous and adapted to $\{\mathcal{F}_t\}_{t\in[0,a]}$.

A solution to the equation (1) on $[0,a]$ is a process $x\in\mathcal{C}([0,a])$ which satisfies (1) on $[0,a]$.

Let us assume that:

$(\mathbf{H_1})$ the noise processes z_j, $j=1,...,r$, are defined on $[0,a]$ into $\mathbb{R}$, $\mathcal{F}_t$-measurable (i.e., adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,a]}$) process, satisfying for some positive constant K the inequalities:

$$|E\{[z(t,\omega)-z(s,\omega)]/\mathcal{F}_s\}|\le K(t-s),$$

$$E([z(t,\omega)-z(s,\omega)]^r/\mathcal{F}_s)\le K(t-s),\ \ r=2,4,$$

a.s., whenever $0\le s\le t\le a$;

($\mathbf{H_2}$) if φ_λ is any one of the functions $g^i_{\lambda,j},\ h^i_{\lambda,jk} : [0,a] \times L_2^n \to L_2,\ i = 1,2,...,n;\ j,k = 1,...,r,\ \lambda \in \Lambda$, fixed, then $\varphi_\lambda(s,x)$ is continuous in x on L_2^n for every $s \in [0,a]$, and for any $x_\lambda \in \mathcal{C}([0,a])$, the process $t \to \varphi_\lambda(t, x_\lambda(t))$ is measurable and $\mathcal{F}_t$- adapted with $t \to ||\varphi_\lambda(t, x_\lambda(t))||^2$ bounded on $[0,a]$;

($\mathbf{H_3}$) the initial condition α_λ belongs to $\mathcal{C}([0,a])$.

It is known that if $f : [0,a] \to L_2$ is a measurable process $\mathcal{F}_t$- adapted and if $t \to ||f(t)||^2$ is Lebesgue integrable on $[0,a]$ then ([18], [19]) if z_1 and z_2 satisfy the hypothesis ($\mathbf{H_1}$) the McShane integrals

$$\int_0^a f(s)dz_1(s), \quad \int_0^a f(s)dz_1(s)dz_2(s)$$

exist and the following estimates are true:

$$||\int_0^a f(s)dz_1(s)|| \le C\{\int_0^a ||f(s)||^2 ds\}^{\frac{1}{2}},$$

$$||\int_0^a f(s)dz_1(s)dz_2(s)|| \le C\{\int_0^a ||f(s)||^2 ds\}^{\frac{1}{2}},$$

where $C = 2Ka^{\frac{1}{2}} + K^{\frac{1}{2}}$.

We give the following result:

Theorem 1. *Let u be a continuous function on $[0,a]$ with $u(0) = 0$, $u(t) > 0$ for $t > 0$ and having non-negative derivative $u' \in L_1((0,a])$ such that if φ_λ is any one of the function $g^i_{\lambda,j}$ and $h^i_{\lambda,jk}$, then*

$$||\varphi_\lambda(t,x) - \varphi_\lambda(t,y)||^2 \le \frac{u'(t)}{\alpha u(t)} w(||x-y||_n^2),\ t \in [0,a],\ x,y \in L_2^n, \qquad (2)$$

with $\lambda \in \Lambda$, fixed, $\alpha = 2nAC^2(r + r^2)(1 + r + r^2)$, $w \in \mathcal{C}(\mathbb{R}+, \mathbb{R}+)$ monotone nodecreasing function, $w(0) = 0$, $w(t) > 0$ for $t > 0$ and having bounded derivative on $\mathbb{R}_+$, with $\sup_{t \in \mathbb{R}_+} |w'(t)| = A$.

Further, let

$$||g^i_{\lambda,j}(t,x)||^2 + ||h^i_{\lambda,jk}(t,x)||^2 \le \beta(t)(||x||_n^2 + 1),\ t \in [0,a],\ ||x||_n \le M,\ M > 0 \qquad (3)$$

for $x \in L_2^n$, $\lambda \in \Lambda$, fixed, $\beta > 0$ a continuous function on $(0,a]$ with $\lim_{t \to 0^+} \frac{\beta(t)}{u'(t)} = 0$.

If the hypotheses ($\mathbf{H_1}$) − ($\mathbf{H_3}$) are satisfied for $\lambda \in \Lambda$, fixed, the functions $g_{\lambda,j}$, $h_{\lambda,j,k}$ and the integrators z_j are common, then the family (1) of systems has a unique solution $X_\lambda(t, \alpha_\lambda(t)) \in \mathcal{C}([0,b])$, with $b \in (0,a]$.

If $\lim_{m\to\infty} |||\alpha_{\lambda,m} - \alpha_\lambda||| = 0$, *for* α_λ, $\alpha_{\lambda,m} \in \mathcal{C}([0,a])$, *we have that*

$$\lim_{m\to\infty} |||X_{\lambda,m} - X_\lambda||| = 0 \tag{4}$$

for every λ *fixed in* Λ, $X_\lambda \in \mathcal{C}([0,a])$ *is the solution of equation (1) and* $X_{\lambda,m}$ *is the solution of (1) with the initial condition* $\alpha_{\lambda,m} \in \mathcal{C}([0,a])$, $m \in I\!N$.

Proof. Let us first prove the existence of a solution on some interval $(0,b]$ with $b \in (0,a]$. We will made a reasoning similar to Ladde and Seikkala [16] and A. Constantin [7].

We consider the operator T defined on $\mathcal{C}([0,a])$ by

$$TX_\lambda(t) = \alpha_\lambda(t) = \sum_{j=1}^{r} \int_0^t g_{\lambda,j}(s, X_\lambda(s))dz_j(s) +$$

$$+ \sum_{j,k=1}^{r} h_{\lambda,jk}(s, X_\lambda(s))dz_j(s)dz_k(s),\ 0 \le t \le a.$$

By the hypotheses, these integrals exist and T maps $\mathcal{C}([0,a])$ into itself.

Let $q \in I\!N$ be such that $\sqrt{u(a)} \le (q-1)\frac{\|\alpha(t)\|_n+1}{\sqrt{n}C(r+r^2)}$ and denote $Q_1 = q(\|\alpha(t)\|_n+1)$. Let $b \in (0,a]$ be such that $\|x\|_n \le Q_1$ implies $\|\varphi_\lambda(t,x)\|^2 \le u'(t)$ for $t \in [0,b]$ and for every φ_λ of the form $g^i_{\lambda,j}$, $h^i_{\lambda,jk}$.

The set

$$B = \{X_\lambda \in \mathcal{C}([0,b]);\ \|X_\lambda(t)\|_n \le Q_1 \text{ for } t \in [0,b]\}$$

is a closed bounded and convex subset of the Banach space $(\mathcal{C}([0,b]), |||.|||)$.

If $X_\lambda \in B$ we have that

$$\| \int_0^t g^i_{\lambda,j}(s, X_\lambda(s))dz_j(s)\| \le C\{\int_0^t \|g^i_{\lambda,j}(s, X_\lambda(s))\|^2 ds\}^{\frac{1}{2}} \le$$

$$\le C\{\int_0^t u'(s)ds\}^{\frac{1}{2}} = C\sqrt{u(t)},\ \ j = \overline{1,r},\ \ 0 \le t \le b,$$

and

$$\| \int_0^t h^i_{\lambda,jk}(s, X_\lambda(s))dz_j(s)dz_k(s)\| \le C\sqrt{u(t)},\ \ j,k = \overline{1,r},\ \ 0 \le t \le b.$$

Then we obtain that

$$\|TX_\lambda\|_n \le \|\alpha_\lambda(t)\|_n + \sqrt{n}C(r+r^2)\sqrt{u(t)},\ \ 0 \le t \le b,$$

and

$$\|TX_\lambda\|_n \le \|\alpha_\lambda(t)\|_n + (q-1)(\|\alpha_\lambda(t)\|_n + 1) \le Q_1, \ \ 0 \le t \le b,$$

which implies

$$|||TX_\lambda|||_n \le Q_1, \ \ X_\lambda \in B,$$

i.e. $T(B) \subseteq B$.

Also if $X_\lambda \in B$ we have that

$$\|TX_\lambda(t) - TX_\lambda(s)\|_n \le \sqrt{n}\sqrt{u(t)-u(s)}, \ \ 0 \le s \le t \le b,$$

thus the set $T(B)$ is equicontinuous.

Moreover, we have for $X_\lambda, Y_\lambda \in B$ that

$$\|(TX_\lambda)^i(t) - (TY_\lambda)^i(t)\| \le C\sum_{j=1}^{r}\{\int_0^t \|g^i_{\lambda,j}(s, X_\lambda(s)) - g^i_{\lambda,j}(s, Y_\lambda(s))\|^2 ds\}^{\frac{1}{2}} +$$

$$+C\sum_{j,k=1}^{r}\{\int_0^t \|h^i_{\lambda,jk}(s, X_\lambda(s)) - h^i_{\lambda,jk}(s, Y_\lambda(s))\|^2 ds\}^{\frac{1}{2}}, \quad 0 \le t \le b.$$

It is easy to see that for every φ_λ of the form $g^i_{\lambda,j}$, $h^i_{\lambda,jk}$ we have

$$\|\varphi_\lambda(s, X_\lambda(s)) - \varphi_\lambda(s, Y_\lambda(s))\|^2 \le 4u'(t), \ \ 0 \le t \le b$$

and by the continuity of $g^i_{\lambda,j}(s, X_\lambda)$, $h^i_{\lambda,jk}(s, X_\lambda(s))$ in X_λ and by the Lebesgue convergence theorem we deduce that T is continuous.

Applying the Schauder fixed point theorem we obtain that T has a fixed point in B, thus equation (1) has a solution on $[0, b]$.

Let us now prove the uniqueness of solutions of the problem (1).

If X_λ and Y_λ are solutions defined on the same probability space $(\Omega, \mathcal{F}, P)$ with the same reference family $\{\mathcal{F}_t\}_{t\ge0}$ on some interval $[0, c]$ with $0 < c \le a$, and with the same initial condition $\alpha_\lambda(t)$, we obtain

$$\|X^i_\lambda(t) - Y^i_\lambda(t)\|^2 \le \{\sum_{j=1}^{r} C^2 \int_0^t \|g^i_{\lambda,j}(s, X_\lambda(s)) - g^i_{\lambda,j}(s, Y_\lambda(s))\|^2 ds + \tag{5}$$

$$+ \sum_{j,k=1}^{r} C^2 \int_0^t \|h^i_{\lambda,jk}(s, X_\lambda(s)) - h^i_{\lambda,jk}(s, Y_\lambda(s))\|^2 ds\}(r + r^2) \le$$

$$\le C^2(r + r^2)^2 \int_0^t \frac{u'(t)}{\alpha u(t)} w(\|X_\lambda(s) - Y_\lambda(s)\|_n^2) ds, \ 0 \le t \le c.$$

Taking into account the hypotheses, we have

$$\|X_\lambda(t) - Y_\lambda(t)\|_n^2 = \sum_{i=1}^{n} \|X_\lambda^i(t) - Y_\lambda^i(t)\|^2 \leq \tag{6}$$

$$\leq nC^2(r+r^2)^2 \int_0^t \frac{u'(t)}{\alpha u(t)} w(\|X_\lambda(s) - Y_\lambda(s)\|_n^2) ds, \ \ 0 \leq t \leq c.$$

If we set $v(t) = \|X_\lambda(t) - Y_\lambda(t)\|_n^2$ we deduce that

$$v(t) \leq \int_0^t \frac{u'(s)}{Au(s)} w(v(s)) ds, \ \ 0 \leq t \leq c. \tag{7}$$

In view of (3) and the condition on β, there exists $b \in (0, a]$ such that

$$\|\varphi_\lambda(t, X_\lambda)\|^2 \leq \frac{\epsilon A}{4\alpha} u'(t), \ 0 < t \leq b, \ \ \|X_\lambda\|_n \leq M, \ M > 0,$$

for any φ_λ of the form $g_{\lambda,j}^i$, $h_{\lambda,jk}^i$.

We deduce that

$$\|X_\lambda^i(t) - Y_\lambda^i(t)\|^2 \leq \frac{\epsilon}{n} u(t), \ \ 0 < t \leq b,$$

and

$$v(t) = \|X_\lambda(t) - Y_\lambda(t)\|_n^2 \leq \epsilon u(t), \ \ 0 < t \leq b, \tag{8}$$

thus

$$\lim_{t\to 0^+} \frac{w(v(t))}{u(t)} \leq \lim_{t\to 0^+} \frac{w(\epsilon u(t))}{u(t)} = \lim_{t\to 0_+} \frac{w'(\epsilon u(t))\epsilon u'(t)}{u'(t)} \leq A\epsilon \tag{9}$$

so that

$$\lim_{t\to 0_+} \frac{w(v(t))}{u(t)} = 0 \tag{10}$$

Using these relations we can deduce that if we denote

$$V(t) = \int_0^t \frac{u'(s)}{Au(s)} w(v(s)) ds, \ \ 0 < t \leq c, \tag{11}$$

then V is differentiable on $(0, c]$ and for $t \in (0, c] \setminus \{t : w'(V(t)) = 0\}$ we have

$$V'(t) = \frac{u'(t)}{Au(t)} w(v(t)) \leq \frac{u'(t)}{u(t)} \frac{w(V(t))}{w'(V(t))}. \tag{12}$$

Hence we deduce that

$$u(t)w'(V(t))V'(t) - u'(t)w(V(t)) \leq 0, \ \ 0 < t \leq c$$

since on the set $\{t : w'(V(t)) = 0\}$ we can write

$$0 \cdot V'(t)u(t) - u'(t)w(V(t)) \leq 0.$$

Then $(\frac{w(V(t))}{u(t)})' \leq 0$ and thus $\frac{w(V(t))}{u(t)}$ is a nonincreasing function of t on $(0, c]$. Moreover, since u is a continuous nonnegative function on $(0, c]$ we deduce that

$$\lim_{t \to 0^+} \frac{w(V(t))}{u(t)} = 0 \tag{13}$$

since, as in (8), for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $v(s) \leq \epsilon u(s)$ on $(0, \delta(\epsilon)]$ and thus we have

$$\lim_{t \to 0^+} \frac{w(V(t))}{u(t)} = \lim_{t \to 0^+} \frac{w'(V(t))V'(t)}{u'(t)} = \lim_{t \to 0^+} \frac{w'(V(t))\frac{u'(t)w(v(t))}{Au(t)}}{u'(t)} \leq$$

$$\leq \lim_{t \to 0^+} \frac{w(v(t))}{u(t)} = 0.$$

Then for every $\epsilon > 0$ there exists a $\gamma(\epsilon) > 0$ such that $w(v(s)) \leq u(s)$ on $[0, \gamma(\epsilon)]$ and thus

$$V(t) \leq \int_0^t \frac{\epsilon}{A} u'(s) ds = \frac{\epsilon}{A} u(t) \leq \frac{\epsilon}{A} u(\gamma(\epsilon)) \text{ on } [0, \gamma(\epsilon)]$$

i.e. $V(0^+) = 0$. So we must have $V(t) = 0$ and thus $v(t) = 0$ on $[0, c]$, proving that

$$\|X_\lambda(t) - Y_\lambda(t)\|_n^2 = 0, \quad 0 \leq t \leq c,$$

and the uniqueness of solutions for the equations (1) is proved.

Remark. For various choices of $u(t)$, $w(t)$ and $\beta(t)$ we get some known criteria for the existence and uniqueness of solutions for the problem (1), as those from [2], [4], [5], [6], [7], [12], [16], [18].

For the second part we observe that we can write

$$||X^i_{\lambda,m}(t) - X^i_\lambda(t)||^2 \leq (1 + r + r^2)\{||\alpha^i_{\lambda,m}(t) - \alpha^i_\lambda(t)||^2 +$$

$$+\sum_{j=1}^{r} || \int_0^t [g^i_{\lambda,j}(s, X_{\lambda,m}(s)) - g^i_{\lambda,j}(s, X_\lambda(s))] dz_j(s)||^2 +$$

$$+\sum_{j,k=1}^{r} || \int_0^t [h^i_{\lambda,jk}(s, X_{\lambda,m}(s)) - h^i_{\lambda,jk}(s, X_\lambda(s))] dz_j(s) dz_k(s)||^2\} \leq$$

$$\leq (1 + r + r^2)\{||\alpha^i_{\lambda,m}(t) - \alpha^i_\lambda(t)||^2 +$$

$$+(r + r^2)C^2 \int_0^t \frac{u'(s)}{\alpha u(s)} w(||X_{\lambda,m}(s) - X_\lambda(s)||^2_n) ds\}, \ t \in (0, a]$$

where $\lambda \in \Lambda$, fixed.

If we denote by

$$v_{\lambda,m}(t) = ||X_{\lambda,m}(t) - X_\lambda(t)||_n^2,$$

$$V_{\lambda,m}(t) = \int_0^t \frac{u'(s)}{2Au(s)} w(v_{\lambda,m}(s))ds,$$

$$D = 1 + r + r^2,$$

adding for $i = 1, ..., n$ and taking supreme on $[0, a]$ to the initial condition term, we obtain

$$v_{\lambda,m}(t) \leq D|||\alpha_{\lambda,m} - \alpha_\lambda|||^2 + V_{\lambda,m}(t) \text{ for } t \in (0, a],\ \lambda \in \Lambda,\ \text{fixed .} \tag{14}$$

In view of (3) and the condition on β, for every $\epsilon > 0$ there exists $b \in (0, a]$ such that

$$||\varphi_\lambda(t, x)||^2 \leq \frac{\epsilon A u'(t)}{4\alpha} \text{ for } 0 \leq t \leq b,\ ||x||_n \leq M,\ M > 0, \tag{15}$$

for any φ_λ of the form $g^i_{\lambda,j},\ h^i_{\lambda,jk},\ \lambda \in \Lambda$ fixed.

From (15) and (8) we deduce that

$$v_{\lambda,m}(t) \leq D|||\alpha_{\lambda,m} - \alpha_\lambda|||^2 + \epsilon u(t) \text{ for } 0 \leq t \leq b. \tag{16}$$

If we denote by $v_\lambda(t) = \lim_{m\to\infty} v_{\lambda,m}(t)$, then, from Lebesgue convergence theorem, we obtain that

$$v_\lambda(t) \leq \int_0^t \frac{u'(s)}{2Au(s)} w(v_\lambda(s))ds = V_\lambda(t) \text{ for } t \in (0, a]$$

From (16) we deduce that

$$\lim_{t\to 0^+} \frac{w(v_\lambda(t))}{2Au(t)} \leq \lim_{t\to 0^+} \frac{w(\epsilon u(t))}{2Au(t)} = \lim_{t\to 0^+} \frac{w'(\epsilon u(t))\epsilon u'(t)}{2Au'(t)} \leq \frac{\epsilon}{2}\ ,$$

so we have that

$$\lim_{t\to 0^+} \frac{w(v_\lambda(t))}{u(t)} = 0$$

On the other hand

$$V'_\lambda(t) = \frac{u'(t)}{2Au(t)} w(v_\lambda(t)) \leq \frac{u'(t)w(V_\lambda(t))}{u(t)w'(V_\lambda(t))} \text{ on } (0, a].$$

Then the function $(\frac{w(V_\lambda(t))}{u(t)})' \leq 0$ and thus $\frac{w(V_\lambda(t))}{u(t)}$ is a positive nonincreasing function on $(0, a]$. But u is a positive continuous function of t on $(0, a]$ and

$$\lim_{t\to 0^+} \frac{w(V_\lambda(t))}{u(t)} = 0$$

so that $V_\lambda(0^+) = 0$ and thus $v_\lambda(t) = 0$ on $(0, a]$ and the theorem is proved.

3. Now we consider the problem of the convergence of the solution processes in McShane's stochastic integral equation systems (1) with coefficients $\{\alpha_\lambda, g_{\lambda,j}, h_{\lambda,jk};\ \lambda \in \Lambda\}$ depending on a parameter $\lambda \in \Lambda$ (being the integrator z_j common to the all elements of the family). We will suppose that the mentioned family satisfies the hypotheses $(\mathbf{H_1}) - (\mathbf{H_3})$ for every $\lambda \in \Lambda$. By the above theorem we have that for every $\lambda \in \Lambda$, equation (1) has a unique solution $X_\lambda \in \mathcal{C}([0,a])$. Let λ_0 be a fixed point in Λ.

We assume that for every process $X \in \mathcal{C}([0,a])$ we have

$$(\mathbf{H_4})\ \ \varphi_\lambda(t, X(t)) \overset{\text{P}}{\to} \varphi_{\lambda_0}(t, X(t)) \text{ as } \lambda \to \lambda_0,$$

if φ_λ is any one of the functions $g^i_{\lambda,j},\ h^i_{\lambda,jk}$, where P expresses the convergence in probability.

From [3, Lemma 1 and the Remark] we know that the condition $(\mathbf{H_4})$ implies that for every $X \in \mathcal{C}([0,a])$

$$\lim_{\lambda\to\lambda_0} \int_0^a ||\varphi_\lambda(s, X(s)) - \varphi_{\lambda_0}(s, X(s))||^2 ds = 0$$

Theorem 2. *If the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_4})$ *and (2)-(3) are satisfied, then* $\lim\limits_{\lambda\to\lambda_0} |||\alpha_\lambda - \alpha_{\lambda_0}||| = 0$ *implies* $\lim\limits_{\lambda\to\lambda_0} ||X_\lambda - X_{\lambda_0}||^2_n = 0$.

Proof. We can write that

$$||X^i_\lambda(t) - X^i_{\lambda_0}(t)||^2 \leq (1 + r + r^2)\{||\alpha^i_\lambda(t) - \alpha^i_{\lambda_0}(t)||^2 +$$

$$+ C^2 \sum_{j=1}^r \int_0^t ||g^i_{\lambda,j}(s, X_\lambda(s)) - g^i_{\lambda_0,j}(s, X_{\lambda_0}(s))||^2 ds +$$

$$+ C^2 \sum_{j,k=1}^r \int_0^t ||h^i_{\lambda,jk}(s, X_\lambda(s)) - h^i_{\lambda_0,jk}(s, X_{\lambda_0}(s))||^2 ds\} \leq$$

$$\leq (1 + r + r^2)\{||\alpha^i_\lambda(t) - \alpha^i_{\lambda_0}(t)||^2 +$$

$$+ 2C^2 \sum_{j=1}^r \int_0^t ||g^i_{\lambda,j}(s, X_\lambda(s)) - g^i_{\lambda,j}(s, X_{\lambda_0}(s))||^2 ds +$$

$$+ 2C^2 \sum_{j=1}^r \int_0^t ||g^i_{\lambda,j}(s, X_{\lambda_0}(s)) - g^i_{\lambda_0,j}(s, X_{\lambda_0}(s))||^2 ds +$$

$$+ 2C^2 \sum_{j,k=1}^r \int_0^t ||h^i_{\lambda,jk}(s, X_\lambda(s)) - h^i_{\lambda,jk}(s, X_{\lambda_0}(s))||^2 ds +$$

$$+2C^2\sum_{j,k=1}^{r}\int_0^t||h^i_{\lambda,jk}(s,X_{\lambda_0}(s)-h^i_{\lambda_0,jk}(s,X_{\lambda_0}(s))||^2ds\},\ 0\leq t\leq a,$$

since if φ_λ is any one of the functions $g^i_{\lambda,j}$, $h^i_{\lambda,jk}$ we have

$$\int_0^t||\varphi_\lambda(s,X_\lambda(s))-\varphi_{\lambda_0}(s,X_{\lambda_0}(s))||^2ds\leq\int_0^t(||\varphi_\lambda(s,X_\lambda(s))-\varphi_\lambda(s,X_{\lambda_0}(s))+$$

$$+\varphi_\lambda(s,X_{\lambda_0}(s))-\varphi_{\lambda_0}(s,X_{\lambda_0}(s))||)^2ds\leq$$

$$\leq 2\{\int_0^t(||\varphi_\lambda(s,X_\lambda(s))-\varphi_\lambda(s,X_{\lambda_0}(s)||^2+||\varphi_\lambda(s,X_{\lambda_0}(s))-\varphi_{\lambda_0}(s,X_{\lambda_0}(s))||^2)ds\}$$

for $0\leq t\leq a$.

We denote by $I^i_{1,\lambda}(t),\ I^i_{2,\lambda}(t),\ I^i_{3,\lambda}(t)$ and $I^i_{4,\lambda}(t),\ 0\leq t\leq a$, the last terms in the last member of the previous inequalities.

The hypothesis (2) enables us to write that

$$||X^i_\lambda(t)-X^i_{\lambda_0}(t)||^2\leq(1+r+r^2)\{||\alpha^i_\lambda(t)-\alpha^i_{\lambda_0}(t)||^2+I^i_{2,\lambda}(t)+$$

$$+I^i_{4,\lambda}(t)+2(r+r^2)C^2\int_0^t\frac{u'(s)}{\alpha u(s)}w(||X_\lambda(s)-X_{\lambda_0}(s)||^2_n)ds\},$$

$0\leq t\leq a,\ \lambda\in\Lambda$. Adding for $i=1,2,...,n$ and noting

$$M_1(\lambda)=(1+r+r^2)\{|||\alpha_\lambda-\alpha_{\lambda_0}|||^2+\sum_{i=1}^n I^i_{2,\lambda}(a)+\sum_{i=1}^n I^i_{4,\lambda}(a)\},$$

$$M_2=2n(1+r+r^2)(r+r^2)C^2,$$

we see that by the hypothesis $(\mathbf{H_4})$ we have

$$\lim_{\lambda\to\lambda_0}\sum_{i=1}^n(I^i_{2,\lambda}(a)+I^i_{4,\lambda}(a))=0$$

and so, since $\lim_{\lambda\to\lambda_0}|||\alpha_\lambda-\alpha_{\lambda_0}|||^2=0$, we deduce that $\lim_{\lambda\to\lambda_0}M_1(\lambda)=0$.

We obtain, for every $t\in[0,a]$, that

$$||X_\lambda(t)-X_{\lambda_0}(t)||^2_n\leq M_1(\lambda)+M_2\int_0^t\frac{u'(s)}{\alpha u(s)}w(||X_\lambda(s)-X_{\lambda_0}(s)||^2_n)ds,$$

$0\leq t\leq a$, and so that by the Lebesgue convergence theorem we have

$$\lim_{\lambda\to\lambda_0}||X_\lambda(t)-X_{\lambda_0}(t)||^2_n\leq\int_0^t\frac{u'(s)}{Au(s)}w(\lim_{\lambda\to\lambda_0}||X_\lambda(t)-X_{\lambda_0}(t)||^2_n)ds=$$

$$=\int_0^t\frac{u'(s)}{Au(s)}w(v(s))ds,\ 0\leq t\leq a,$$

where we denoted by

$$v(t) = \lim_{\lambda \to \lambda_0} ||X_\lambda(t) - X_{\lambda_0}(t)||_n^2$$

$$\leq \lim_{\lambda \to \lambda_0} [D|||\alpha_\lambda - \alpha_{\lambda_0}||| + \epsilon u(t)] = \epsilon u(t), \ 0 \leq t \leq b.$$

Noting by $V(t) = \int_0^t \frac{u'(s)}{Au(s)} w(v(s))ds$, in a similar way as in the first part, we deduce that $V(t) = 0$ on $[0, a]$ and $v(t) = 0$ on $[0, a]$, hence the theorem is proved.

4. In modelling, analyzing and predicting aspects of economic reality, physical, biological and social phenomena researches are placing greater and greater emphasis upon stochastic methods.

Such methods are expected to capture the various complexities measurement errors and uncertainties that are associated with the world reality. The question that arises naturally is: How can combinations of complexity, uncertainty and ignorance which are present in the process of theorizing be incorporated into dynamic analysis ?

Stochastic calculus appears to be one of the natural tools for the study of models of these phenomena having some non-deterministic elements. For example, in the description of brownian motion the stochastic nature is adequately described by a linear differential equation with a random forcing term which is identified as a white noise process or has a formal derivative of the Wiener process. These developments culminated in the establishment of the stochastic calculus (Ito (1951), Doob (1953), Ghihman (1950), Stratonovich (1964), Kunita and Watanabe (1967), the French probabilists notably Meyer, Dellacherie and Jacod (1967-1980)).

However, when the results of the stochastic calculus were applied to other types phenomena, certain difficulties arose in the process of interpretation of stochastic differentials and approximation process. In many models, white noise process is explicitly introduced and the basic physical process in question is visualised as an approximation. Hence it is reasonable to expect some kind of a stability in the sense that the solutions that are obtained by approximating the white noise process should themselves approximate the process in question.

Ito stochastic calculus failed to satisfy this requirement of stability (see Wong and Zakai [25]).

The symmetric integral defined by Stratonovich [23] and the calculus based on this definition were gainfully employed in modelling and analysis of noise driven processes, since the Stratonovich interpretation in some situations may be the most appropriate. On the other hand, the specific feature of the Ito model of not looking into the feature seems to be a reason for choosing the Ito interpretation in many cases, for example in bilogy.

In any case, because of the explicit connection between the two models, it will for many purposes suffice to do the general mathematical treatment for one of the two type integrals. In general, one can say that Stratonovich integral has the advantage of leading to ordinary chain rule formula under a transformation (change of variable), i.e., there are no 2^{nd} order terms in the Stratonovich analogue of the Ito transformation formula. This property makes the Stratonovich integral natural to use for example in connection with stochastic differential on manifolds (see Elworthy [12]). However, Stratonovich integrals are not martingales as the Ito integrals are and which gives the Ito integrals an important computational advantage, even though it does not behave so nicely under transformation.

All these aspects were studied in great depth by McShane (1969, 1970, 1974) whose attempts culminated in the establishment of a more general calculus encompassing both ordinary and Ito Calculus.

In McShane's Calculus, the standard equation

$$X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s, X(s,\omega))ds + \sum_{j=1}^{r} \int_0^t g^i_j(s, X(s,\omega))dz_j(s,\omega) \tag{17}$$

is replaced by what he calls a **canonical extension** (or canonical form or canonical system) of equation (17):

$$X^i(t,\omega) = X^i(0,\omega) + \int_0^t f^i(s, X(s,\omega))ds + \tag{18}$$

$$\sum_{j=1}^{r} \int_0^t g^i_j(s, X(s,\omega))dz_j(s,\omega) + \frac{1}{2} \sum_{j,k=1}^{r} \int_0^t g^i_{j,k}(s, X(s,\omega))dz_j(s,\omega)dz_k(s,\omega)$$

in which

$$g^i_{j,k}(t,x,\omega) = \sum_{m=1}^{n} [\partial g^i_j(t,x,\omega)/\partial x^m] g^m_k(t,x,\omega)$$

$i = 1, 2, ..., n;\ j, k = 1, 2, ..., r;\ t \in [0, a];\ x \in \mathbb{R}^n.$

We are now able to describe the method by which we shall construct stochastic models of physical systems which in the physically realizable case of lipschitzian noises are known to satisfy the integral equation (17).

If $g^i_{j,k}(t,x,\omega)$ are functions defined for $t \in [0,a]$ and $x \in \mathbb{R}^n$ and bounded on bounded sets of (t,x), then the solution $X^i(t,\omega)$ of (17) is also a solution of (18) since the last integral vanishes for lipschitzian noises.

The McShane Calculus is better suited modelling dynamical phenomena described typically by McShane systems where $z_j(t,\omega)$ are noises processes.

McShane stochastic integral systems enjoy the following three important properties:

(i) The property of inclusiveness: the model must apply to systems in which the permitted noises are processes belonging to some family large enough to include processes with sample paths having lipschitzian property, all brownian motion processes, and such modifications as have proved convenient in applications;

(ii) The property of consistency: for lipschitzian noises, the solutions of the equations should coincide with the solutions of the equations that are normally believed to be applicable to physical systems;

(iii) The property of stability: the model must be such that if the noise process $z_j(t,\omega)$ is replaced by another permissible process $z^0_j(t,\omega)$ close to it, then the corresponding solutions $X^i(t,\omega)$, $X^i_0(t,\omega)$ are also close to each other (in the sense that an extreme degree of closeness corresponds to practical imposibility of distinguishing the process by means of available experimental procedures).

However, in this analysis of stochastic models, McShane assumed the sample continuity of the noise processes $z_j(t,\omega)$. In fact there are many situations in communication and control theory where the sample path of noise processes are superposition of step functions. Stochastic point processes do fall under this category; purely discontinuous Markov processes (PDMP) of Feller-Kolmogorov type also do come under this class. It is known that higher order integrals involving z (of order ≥ 3) do not vanish if the processes z admit finite jumps. An extension of the modelling procedure of McShane so as to include systems driven by discontinuous Markov processes is given in [22] where the canonical form is defined by

$$dx^i(t) = g^i(x(t))dt + \sum_{m=1}^{\infty}\sum_{j=1}^{r}\frac{1}{m!}D_j^{m-1}(g_j^i(x(t)))(dz_j(t))^m$$

where

$$D_j^0 = 1, \quad D_j^{m+1} = D_j(D_j^m), \quad D_j = \sum_{i=1}^{n} g_j^i(x)\partial/\partial x^i$$

and the integrals are McShane belated integrals. The existence of such canonical forms has been established for PDMP as well as for weighted pointed processes (see [21], [22]).

The canonical choice is advantageous not only from simple computational convenience, but besides this it is known [18] that the difference between the solution of a stochastic differential equation and the Cauchy-Maruyama approximations can be disappointingly slow in its convergence to zero, and this is true for Runge-Kutta approximation also. But when the equations have the canonical form and some additional hypotheses are satisfied, a set of approximations closely related to the Cauchy-Maruyama approximations converges considerably faster.

Moreover, for the Runge-Kutta method applied to stochastic differential equation (17), without second-order terms, the method gives estimates and converge as mesh Π tends to zero. But the surprising feature is that the estimates converge, not to a solution of (17) itself, but to a solution of its canonical extension. This establishes another distinctive property of the canonical form, and also enables us to compute solutions of canonical extension without ever having to compute the functions $g_{j,k}^i$.

The solution of the stochastic differential equation in canonical form has a stability under modification of the z_j that is not possessed by the equations with other choices for $g_{j,k}^i$ (including the traditional choice zero). We can be sure that any other extension that gives different solutions certainly lacks stability.

Moreover, the solutions of the canonical extension do not depend on the coordinate system in which we choose to expres them, the property that is not in general possessed by other equations. Also, it must be mentioned an especial suitability for retaining adequate agrement with experiment when the noises are idealized to "white noise" (see [18, pp 228-234], [19], [21], [22]).

In conclusion, McShane's Calculus had proved to be very valuable in modelling and in finding application in physics under canonical form.

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