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CONVERGENCE THEOREMS FOR NEWTON-LIKE METHODS FOR OPERATORS WITH GENERALIZED HÖLDER DERIVATIVE

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Abstract. In this paper, we present two semilocal convergence theorems for Newton-like approximations which improve and extend a fundamental result obtained by Yamamoto [15] in 1987.
In the first part of this paper, we obtain an improvement of Yamamoto's result. In fact we consider the case in which the involved operators satisfy Lipschitz hypotheses and we prove a convergence result for Newton-like approximations under weaker assumptions.
In the second part of this paper, we extend the previous result to the more general case studied recently by Argyros in [3] in which the involved operators satisfy generalized Hölder hypotheses. We conclude with a comparison between our result and the result obtained by Argyros in [3] and we show that, even if the two theorems are not in general comparable, in some interesting cases it is more convenient to apply our theorem.
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1. Introduction

Let X and Y be Banach spaces and let $f : D \subset X \to Y$ be an operator which is Fréchet differentiable in an open convex set $D_0 \subset D$. Many iterative methods for solving the equation

$$f(x) = 0 \tag{1}$$

can be written in the form

$$x_{n+1} = x_n - A(x_n)^{-1} f(x_n), \quad n \geq 0 \tag{2}$$

where $x_0 \in D_0$ of D and $A(x)$ denotes an invertible bounded linear operator which may be considered as an approximation to $f'(x)$.

In 1968, Rheinboldt [14] established a convergence theorem for Newton-like approximations (2) which includes the classical theorem of Kantorovich [10] for the Newton's method as a special case ($A(x) = f'(x)$).

In the years 1970-71, Dennis [6], [7] generalized Rheinboldt's result and later Miel [11], [12] improved their error bounds. Furthemore, in 1984 Moret [13] gave a sharper error bound than Miel's, but under a rather strong assumptions on $A(x)$.

In 1987, Yamamoto established a fundamental convergence theorem for (2) under Lipschitz conditions on the operators A and f' and improved the previous results of Rheinboldt, Dennis, Miel and Moret.

Later, Newton-like methods have been studied extensively under various hypotheses to generate a sequence converging to a solution of (1) (see, for example, [5], [4],[1], [2], [3]).

In this paper, we consider Newton-like approximations under Lipschtiz assumptions on the operators A and f' and we present an improvement of Yamamoto's result.

Moreover, we extend the previous result to the case recently studied by Argyros in [3] in which the derivative of the operator satisfies a generalized Hölder condition. Our theorem and the theorem obtained by Argyros in [3] are not in general comparable, but we show that in some interesting cases it is more convenient to apply our theorem.

The demonstrative technique applied in the following results is the same technique applied in [8], in which we improved a convergence theorem of Newton-Kantorovich approximations for nonlinear operators with derivative Hölder continuous.

2. Convergence of the Newton-like approximations under Yamamoto's hypotheses

In this section, we consider Newton-like approximations under Lipschitz hypotheses and we improve a result established by Yamamoto in [15].

Let X and Y be Banach spaces, $B(x_0, R)$ the closed ball in X centered in x_0 with radius R. Let $f : B(x_0, R) \to Y$ be a nonlinear operator Fréchet differentiable at interior points of $B(x_0, R)$ and let $A(x)$ be an element of $\mathcal{L}(X, Y)$ for any interior point x of $B(x_0, R)$.
In [15] Yamamoto prove the following

Theorem 2.1. *Suppose that $A(x_0)$ is invertible and that there exist some nonegative constants K, M, L, l, m with $l + m < 1$ such that the operators A and f' satisfy the following assumptions*

$$||A(x_0)^{-1}(f'(x) - f'(y))||_{\mathcal{L}(X,Y)} \leq K||x - y||, \quad \forall x, y \in \overset{\circ}{B} (x_0, R), \tag{3}$$

$$||A(x_0)^{-1}(A(x) - A(x_0))||_{\mathcal{L}(X,Y)} \leq L||x - x_0|| + l, \quad \forall x \in \overset{\circ}{B} (x_0, R), \tag{4}$$

$$||A(x_0)^{-1}(f'(x) - A(x))||_{\mathcal{L}(X,Y)} \leq M||x - x_0|| + m, \quad \forall x \in \overset{\circ}{B} (x_0, R)\,. \tag{5}$$

If we denote with $a := ||A(x_0)^{-1}A(x_0)||\,, \quad$ *and* $\sigma := \max\left\{1, \frac{L+M}{K}\right\}$, *we assume that*

$$\sigma a K \leq \frac{(1 - l - m)^2}{2}.$$

Set

$$\phi(r) := \frac{Lr^2}{2} - (1 - l)r + a\,, \quad \phi_\sigma(r) := \frac{\sigma K r^2}{2} - (1 - l - m)r + a\,,$$

the scalar sequence t_n defined by recurrence formula

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{\phi_\sigma(t_n)}{\phi'(t_n)}, \quad n \geq 0$$

is monotonically converging to the smaller root t_* of the scalar equation $\phi_\sigma(t) = 0$.
The Newton-like approximations (2) are well defined for all n, belong to $B(x_0, t_*)$ and converge to a solution x_* of the equation (1) and the following error bounds hold

$$||x_n - x_{n-1}|| \leq t_n - t_{n-1}\,, \quad \forall\, n \in N\,, \tag{6}$$

$$||x_n - x_*|| \leq t_* - t_n\,, \quad \forall\, n \in N\,. \tag{7}$$

In our theorem, we replace the parameter σ with a more convenient parameter α.
To define the fundamental parameter α, we need to introduce the real functions

$$\phi_\beta(r) := \frac{\beta K r^2}{2} - (1-l-m)r + a \quad (\beta \geq 1)\,,$$
$$h(t) := \frac{Kt + 2(M+L)}{K(t+2)}\,.$$

Then the parameter α can be defined in the following way

$$\alpha := \min\left\{1 \leq \beta \leq \frac{(1-l-m)^2}{2aK} \;:\; \sup_{0\leq t\leq t_{max}} h(t) \;\leq \beta\right\}, \tag{8}$$

with

$$t_{max} := \max_{s\in[r_1, r_\beta]} -\frac{\phi_\beta(s)}{s\,\phi'(s)}$$

and

$$r_\beta = \frac{(1-l-m) - \sqrt{(1-l-m)^2 - 2a\beta K}}{\beta K}$$

is the smaller zero of ϕ_β.
We remark that $\phi_\beta(r)$ admits at least a positive zero if and only if

$$\beta \leq \frac{(1-l-m)^2}{2aK}\,.$$

We obtain the following improvement of Yamamoto's result [15]:

Theorem 2.2. *Suppose that $A(x_0)$ is invertible and that there exist some nonegative constants K, M, L, l, m with $l + m < 1$ such that the operators A and f' satisfy the assumptions* (3), (4) *and* (5).
If α is defined by (8)*, the scalar sequence $(r_n)_{n\in N}$ defined by recurrence formula*

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{\phi_\alpha(r_n)}{\phi'(r_n)}, \quad n \geq 0$$

is monotonically converging to r_α.
The Newton-like approximations (2) are well defined for all n, belong to $B(x_0, r_\alpha)$ and converge to a solution x_* of the equation (1) and the following error bounds hold

$$||x_n - x_{n-1}|| \leq r_n - r_{n-1}\,, \quad \forall\, n \in N\,, \tag{9}$$
$$||x_n - x_*|| \leq r_\alpha - r_n\,, \quad \forall\, n \in N\,, \tag{10}$$

Proof. We begin to remark that r_α satisfies the condition $Lr_\alpha + l < 1$. In fact, from the inequality

$$\alpha \geq h(0) = \frac{M+L}{K} > \frac{L}{K} \tag{11}$$

it follows

$$r_\alpha \leq r_{min} = \frac{1-l-m}{\alpha\, K} < \frac{1-l}{L}$$

where r_{min} denotes the minimum point of ϕ_α.

Since $\alpha \geq 1$, the sequence r_n is converging increasingly to r_α. In fact $\phi_\alpha(r_1) > 0$ and

$$\frac{a\alpha K}{(1-l)(1-l-m)} \leq \frac{a\alpha K}{(1-l-m)^2} < 1$$

implies

$$r_1 = \frac{a}{1-l} < \frac{1-l-m}{\alpha K} = r_{min}$$

Consequently $r_0 < r_1 \leq r_\alpha$.
Suppose now that $r_1 \leq r_2 \leq \cdots \leq r_n \leq r_\alpha$.
Then

$$r_{n+1} = r_n - \frac{\phi_\alpha(r_n)}{\phi'(r_n)} \geq r_n$$

since $r_n \leq r_\alpha$ implies $-\dfrac{\phi_\alpha(r_n)}{\phi'(r_n)} \geq 0$.

From (11) it follows

$$-\phi'(r_n) = 1 - l - Lr_n \geq 1 - l - m - Lr_n \geq 1 - l - m - \alpha K r_n = -\phi'_\alpha(r_n)$$

and consequently

$$r_{n+1} = r_n - \frac{\phi_\alpha(r_n)}{\phi'(r_n)} \leq r_n - \frac{\phi_\alpha(r_n)}{\phi'_\alpha(r_n)}\,.$$

Now we consider the Newton method for the equation $\psi_\alpha(r) = 0$ with initial point r_n.
Since $r_n \leq r_\alpha$ the first iteration $r_n - \dfrac{\phi_\alpha(r_n)}{\phi'_\alpha(r_n)}$ is less than or equal to r_α as well.
Now we prove (9) for induction on n; we have

$$||x_1 - x_0|| \leq a \leq \frac{a}{1-l} = r_1\,;$$

$$||x_2 - x_1|| = ||A(x_1)^{-1} f(x_1)|| \leq ||A(x_1)^{-1} A(x_0)||\, ||A(x_0)^{-1} f(x_1)||\,.$$

From the hypothesis (4), it follows

$$||A(x_0)^{-1}(A(x_1) - A(x_0))|| \leq L||x_1 - x_0|| + l \leq Lr_1 + l \leq Lr_\alpha + l < 1$$

and from the Banach's lemma, it follows that $A(x_1)$ is an invertible operator such that

$$||A(x_1)^{-1}\, A(x_0)||_{\mathcal{L}(Y,X)} \leq \frac{1}{1 - (Lr_1 + l)}\,.$$

Then we have

$$\begin{aligned}
||x_2 - x_1|| &\leq \frac{1}{1 - (Lr_1 + l)} ||A(x_0)^{-1}(f(x_1) - f(x_0) - A(x_0))(x_1 - x_0)|| \\
&\leq \frac{1}{1 - (Lr_1 + l)} \Big(||A(x_0)^{-1}(f(x_1) - f(x_0)) - f'(x_0))(x_1 - x_0))|| \\
&\qquad + ||A(x_0)^{-1}(f'(x_0) - A(x_0))(x_1 - x_0)|| \Big) \\
&\leq \frac{1}{1 - (Lr_1 + l)} \left(\int_0^1 ||A(x_0)^{-1}(f'(tx_1 + (1-t)x_0) - f'(x_0))||\, dt\, ||x_1 - x_0|| + m||x_1 - x_0|| \right) \\
&\leq \frac{1}{1 - (Lr_1 + l)} \left(K \int_0^1 t\, dt\, ||x_1 - x_0||^2 + m||x_1 - x_0|| \right) \\
&= \frac{1}{1 - (Lr_1 + l)} \left(\frac{K||x_1 - x_0||^2}{2} + m||x_1 - x_0|| \right) \leq \frac{1}{1 - (Lr_1 + l)} \left(\frac{Kr_1^2}{2} + mr_1 \right) \\
&\leq \frac{1}{1 - (Lr_1 + l)} \Big(\frac{\alpha K r_1^2}{2} + mr_1 \Big) = -\frac{\phi_\alpha(r_1)}{\phi'(r_1)} = r_2 - r_1\,.
\end{aligned}$$

We suppose that the x_k are well defined and that (9) holds for all $k \leq n$. Then we have

$$||x_{n+1} - x_n|| \leq ||A(x_n)^{-1}A(x_0)||\, ||A(x_0)^{-1}f(x_n)||$$

As above, from the Banach's Lemma it follows

$$||A(x_n)^{-1}A(x_0)|| \leq \frac{1}{1 - (Lr_n + l)}$$

then we have

$$\begin{aligned}
||x_{n+1} - x_n|| &\leq \frac{1}{1 - (Lr_n + l)} ||A(x_0)^{-1}(f(x_n) - f(x_{n-1}) - A(x_{n-1}))(x_n - x_{n-1})|| \\
&\leq \frac{1}{1 - (Lr_n + l)} \Big(||A(x_0)^{-1}(f(x_n) - f(x_{n-1}) - f'(x_{n-1}))(x_n - x_{n-1}))|| \\
&\qquad + ||A(x_0)^{-1}(f'(x_{n-1}) - A(x_{n-1}))(x_n - x_{n-1}))|| \Big)
\end{aligned}$$

$$\le \frac{1}{1-(Lr_n+l)}\Bigg(\int_0^1 ||A(x_0)^{-1}(f'(tx_n+(1-t)x_{n-1})-f'(x_{n-1}))||\,dt\,||x_n-x_{n-1}||$$

$$+||A(x_0)^{-1}(f'(x_{n-1})-A(x_{n-1}))||\,||x_n-x_{n-1}||\Bigg)$$

$$\le \frac{1}{1-(Lr_n+l)}\left(K\int_0^1 t\,dt\,||x_n-x_{n-1}||^2+(M||x_{n-1}-x_0||+m)||x_n-x_{n-1}||\right)$$

$$= \frac{1}{1-(Lr_n+l)}\left(\frac{K||x_n-x_{n-1}||^2}{2}+(M||x_{n-1}-x_0||+m)||x_n-x_{n-1}||\right)$$

$$\le \frac{1}{1-(Lr_n+l)}\left(\frac{K(r_n-r_{n-1})^2}{2}+(Mr_{n-1}+m)(r_n-r_{n-1})\right)$$

$$= \frac{1}{1-(Lr_n+l)}\Bigg(\frac{K(r_n-r_{n-1})^2}{2}+(M+L)r_{n-1}(r_n-r_{n-1})$$

$$+(m+l)(r_n-r_{n-1})-(Lr_{n-1}+l)(r_n-r_{n-1})\Bigg)$$

$$= \frac{1}{1-(Lr_n+l)}\Bigg[r_{n-1}(r_n-r_{n-1})\left(\frac{K}{2}\Big(\frac{r_n}{r_{n-1}}-1\Big)+(M+L)\right)$$

$$+(m+l)(r_n-r_{n-1})-(Lr_{n-1}+l)(r_n-r_{n-1})\Bigg]$$

By the definition of the parameter α, we have

$$\frac{Kt}{2}+M+L\le\frac{\alpha K}{2}(t+2)\quad\forall\,t\in[0,t_{max}]$$

and for $t:=\frac{r_n}{r_{n-1}}-1$, we obtain

$$||x_{n+1}-x_n||\le\frac{1}{1-(Lr_n+l)}\Bigg[\frac{\alpha K}{2}\,r_{n-1}(r_n-r_{n-1})\left(\frac{r_n}{r_{n-1}}+1\right)$$

$$+(m+l)(r_n-r_{n-1})-(Lr_{n-1}+l)(r_n-r_{n-1})\Bigg]$$

$$= \frac{1}{1-(Lr_n+l)}\left(\frac{\alpha K}{2}(r_n^2-r_{n-1}^2)+(m+l)(r_n-r_{n-1})-(Lr_{n-1}+l)(r_n-r_{n-1})\right)$$

$$= -\frac{1}{\phi'(r_n)}\Big(\phi_\alpha(r_n)-\phi_\alpha(r_{n-1})-\phi'(r_{n-1})(r_n-r_{n-1})\Big)=-\frac{\phi_\alpha(r_n)}{\phi'(r_n)}=r_{n+1}-r_n\,.$$

Consequently the sequence x_n is a Cauchy sequence converging to a solution x_* of the equation (1). Moreover, $x_* \in B(x_0, r_\alpha)$ and for $n \to +\infty$ we obtain the estimate (10).

$\square$

Remark. In the case $\dfrac{M+L}{K} < 1$, we obtain an improvement of the Yamamoto's result.
In fact since $\alpha = \max\limits_{0 \le t \le t_{max}} \dfrac{Kt + 2(M+L)}{K(t+2)} < 1 = \sigma$ our hypothesis $\alpha K \le \dfrac{(1-l-m)^2}{2}$ is weaker of Yamamoto's hypothesis $K \le \dfrac{(1-l-m)^2}{2}$.

3. Convergence of the Newton-like approximations under generalized Hölder conditions

In this section, we consider the more general case in which the operators A and f' satisfy generalized Hölder assumptions and therefore we extend the theorem proved in the previous section.
In [3] Argyros consider this case but Yamamoto's result is not a particular case of his theorem.
We recall the result proved by Argyros.

Theorem 3.1. [3] *Suppose that $A(x_0)$ is invertible and that the operators A and f' satisfy the following conditions*

$$||A(x_0)^{-1}(f'(x) - f'(y))||_{\mathcal{L}(X,Y)} \le \omega_f(||x-y||), \quad \forall x, y \in \overset{\circ}{B}(x_0, R), \tag{12}$$

$$||A(x_0)^{-1}(A(x) - A(x_0))||_{\mathcal{L}(X,Y)} \le \omega_A(||x-x_0||), \quad \forall x \in \overset{\circ}{B}(x_0, R), \tag{13}$$

$$||A(x_0)^{-1}(f'(x) - A(x))||_{\mathcal{L}(X,Y)} \le \omega(||x-x_0||), \quad \forall x \in \overset{\circ}{B}(x_0, R), \tag{14}$$

with $\omega_f, \omega_A, \omega : [0, +\infty[\to [0, +\infty[$ increasing functions and $\omega_f(0) = 0$.
If, as in Section 2, we denote with $a = ||A(x_0)^{-1}A((x_0)||$, we suppose that the function

$$\psi(r) := \int_0^r \big(\sup_{0 \le s \le t} \omega_f(s) + \omega_A(t-s)\big)dt - (1-\omega(r))r + a\,,$$

admits a zero r_* in $[0, R]$ and that $\psi(R) \le 0$.
Set

$$\varphi(r) := \int_0^r \omega_A(t)dt - r + a\,,$$

the scalar sequence (s_n) *defined by*

$$s_0 = 0,\ s_{n+1} = s_n - \frac{\psi(s_n)}{\varphi'(s_n)}\,.$$

is monotonically converging to r_*.
The Newton-like approximations (2) are well defined for all n*, belong to* $B(x_0, r_*)$ *and converge to a solution* x_* *of the equation (1) and the following error bounds hold for all* $n \in N$*:*

$$||x_n - x_{n-1}|| \le s_n - s_{n-1}\,, \tag{15}$$

$$||x_n - x_*|| \le r_* - s_n\,, \tag{16}$$

We remark that the theorem established by Yamamoto in [15] is not a particular case of the theorem proved by Argyros.
In fact, if

$$\omega_f(t) := Kt\,,\ \omega_A(t) := Lt + l\,,\ \omega(t) := Mt + m\,, \tag{17}$$

set $\tau := \max\{K, L\}$, we have

$$\sup_{0\le s\le t} \omega_f(s) + \omega_A(t-s) = \sup_{0\le s\le t} (K-L)s + Lt + l = \tau t + l$$

and the function ψ defined by

$$\psi(r) = \frac{\tau + 2M}{2} r^2 - (1 - l - m)r + a$$

is not in general comparable with the function ϕ_σ defined in Section 2.
In our theorem, we define a function φ_α which coincides with ϕ_α in the particular case in which the conditions (17) hold and is more convenient of the function ψ in some interesting applications.
To define φ_α, we need to introduce the scalar functions φ_β

$$\varphi_\beta(r) := \beta \int_0^r \omega_f(s)ds - (1 - \omega_A(0) - \omega(0))r + a\,, \quad \beta \ge 1\,.$$

If we suppose that the function φ_β admits at least a zero and we denote with r_β the smaller zero of φ_β, we define the parameter α

$$\alpha := \min\left\{\beta \ge 1\ :\ \varphi_\beta(u) \ge 0;\ r_1 \le u \le r_\beta \text{ and}\right.$$

$$\sup\left\{\frac{\int_0^{v-u} \omega_f(s)ds + (\omega(u) - \omega(0) + \omega_A(u) - \omega_A(0))(v-u)}{\int_u^v \omega_f(s)ds}\right. \le \beta,$$

$$\forall\, v, u :\; v - u = -\frac{\varphi_\beta(u)}{\varphi'(u)}, r_1 \le u \le r_\beta\}\Big\}\,,$$

and the fundamental function φ_α

$$\varphi_\alpha(r) := \alpha \int_0^r \omega_f(s)ds - (1 - \omega_A(0) - \omega(0))r + a\,.$$

We obtain the following

Theorem 3.2. *Suppose that $A(x_0)$ is invertible and that the operators A and f' satisfy the conditions* $(12), (13)$ *and* (14)*.*
Suppose that the function φ_α admits a zero r_α in $[0, R]$ such that $\varphi_\alpha(R) \le 0$ and $\omega_A(r_\alpha) < 1$.
Then the scalar sequence (r_n) defined by

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{\varphi_\alpha(r_n)}{\varphi'(r_n)}, \quad n \ge 0\,,$$

is monotonically converging to r_α.
The Newton-like approximations (2) are well defined for all n, belong to $B(x_0, r_\alpha)$ and converge to a solution x_* of the equation (1) and the following error bounds hold

$$||x_n - x_{n-1}|| \le r_n - r_{n-1}\,, \quad \forall\, n \in N\,, \tag{18}$$

$$||x_n - x_*|| \le r_\alpha - r_n\,, \quad \forall\, n \in N\,, \tag{19}$$

Proof. The monotone convergence of the sequence r_n follows as in Section 2.
From the Banach's lemma, it follows that $A(x)$ is an invertible operator for all point $x \in B(x_0, R)$ such that $\omega_A(||x - x_0||) < 1$ and that

$$||A(x)^{-1}\, A(x_0)||_{\mathcal{L}(Y,X)} \le \frac{1}{1 - \omega_A(||x - x_0||)}\,.$$

We prove (18) for induction on n; we have

$$||x_1 - x_0|| \le a \le \frac{a}{1 - \omega_A(0)} = r_1\,;$$

$$\begin{aligned} ||x_2 - x_1|| &= ||A(x_1)^{-1} f(x_1)|| \le ||A(x_1)^{-1}\, A(x_0)||\; ||A(x_0)^{-1}\, f(x_1)|| \\ &\le \frac{1}{1 - \omega_A(||x_1 - x_0||)}\, ||A(x_0)^{-1}\,(f(x_1) - f(x_0) - A(x_0))(x_1 - x_0)|| \\ &\le \frac{1}{1 - \omega_A(||x_1 - x_0||)} \Big(||A(x_0)^{-1}\,(f(x_1) - f(x_0) - f'(x_0))(x_1 - x_0)|| \end{aligned}$$

$$+||A(x_0)^{-1}\,(f'(x_0)-A(x_0))(x_1-x_0)||\Big)$$

$$\le \frac{1}{1-\omega_A(||x_1-x_0||)}\left(\int_0^1 ||A(x_0)^{-1}\,(f'(tx_1+(1-t)x_0)-f'(x_0))||\,dt\,||x_1-x_0||\right.$$

$$\left.+||A(x_0)^{-1}\,(f'(x_0)-A(x_0))||\,||x_1-x_0||\right)$$

$$\le \frac{1}{1-\omega_A(||x_1-x_0||)}\left(\int_0^1 \omega_f(t||x_1-x_0||)\,dt\,||x_1-x_0||+\omega(0)||x_1-x_0||\right)$$

$$=\frac{1}{1-\omega_A(||x_1-x_0||)}\left(\int_0^{||x_1-x_0||} \omega_f(t)\,dt\,+\omega(0)||x_1-x_0||\right)$$

$$\le \frac{1}{1-\omega_A(r_1)}\left(\int_0^{r_1}\omega_f(t)\,dt\,+\omega(0)r_1\right)$$

$$\le \frac{1}{1-\omega_A(r_1)}\left(\alpha\int_0^{r_1}\omega_f(t)\,dt\,+\omega(0)r_1\right)=-\frac{\varphi_\alpha(r_1)}{\varphi'(r_1)}=r_2-r_1\,.$$

We suppose that the x_k are well defined and that (18) holds for all $k\le n$. Then we have

$$||x_{n+1}-x_n||=||A(x_n)^{-1}f(x_n)||$$

$$\le ||A(x_n)^{-1}A(x_0)||\,||A(x_0)^{-1}\,(f(x_n)-f(x_{n-1})-A(x_{n-1}))(x_n-x_{n-1})||$$

$$\le\frac{1}{1-\omega_A(||x_n-x_0||)}\left(\int_0^1 ||A(x_0)^{-1}\,(f'(tx_n+(1-t)x_{n-1})-f'(x_{n-1}))t||\,dt\,||x_n-x_{n-1})||\right.$$

$$\left.+||A(x_0)^{-1}\,(f'(x_{n-1})-A(x_{n-1}))||\,||x_n-x_{n-1}||\right)$$

$$\le\frac{1}{1-\omega_A(||x_n-x_0||)}\left(\int_0^1\omega_f(t||x_n-x_{n-1}||)\,dt\,||x_n-x_{n-1}||+\omega(||x_{n-1}-x_0||)||x_n-x_{n-1}||\right)$$

$$=\frac{1}{1-\omega_A(||x_n-x_0||)}\left(\int_0^{||x_n-x_{n-1}||}\omega_f(t)\,dt\,+\omega(||x_{n-1}-x_0||)||x_n-x_{n-1}||\right)$$

$$\le \frac{1}{1-\omega_A(r_n)}\left(\int_0^{r_n-r_{n-1}}\omega_f(t)\,dt\,+\omega(r_{n-1})(r_n-r_{n-1})\right)$$

$$\le \frac{1}{1-\omega_A(r_n)}\left(\int_0^{r_n-r_{n-1}}\omega_f(t)\,dt\,+(\omega(r_{n-1})-\omega(0)+\omega_A(r_{n-1})\right.$$

$$-\omega_A(0))(r_n - r_{n-1}) + (\omega(0) + \omega_A(0))(r_n - r_{n-1}) - \omega_A(r_{n-1})(r_n - r_{n-1})\Big)$$

By the definition of the parameter α, we have

$$\int_0^{v-u} \omega_f(s)ds + (\omega(u) - \omega(0) + \omega_A(u) - \omega_A(0))(v-u) \le \alpha \int_u^v \omega_f(s)ds$$

for all $r_1 \le u \le v \le r_\alpha,\ v - u = -\dfrac{\varphi_\alpha(u)}{\varphi'(u)}$.

Then for $u = r_{n-1},\ v = r_n$, we obtain

$$||x_{n+1} - x_n|| \le \frac{1}{1-\omega_A(r_n)}\Bigg(\alpha \int_0^{r_n} \omega_f(t)\,dt\ - \alpha \int_0^{r_{n-1}} \omega_f(t)\,dt$$

$$+(\omega(0) + \omega_A(0))(r_n - r_{n-1}) - \omega_A(r_{n-1})(r_n - r_{n-1})\Bigg)$$

$$= \frac{1}{1-\omega_A(r_n)}\big(\varphi_\alpha(r_n) - \varphi_\alpha(r_{n-1}) + (1 - \omega_A(r_{n-1}))(r_n - r_{n-1})\big)$$

$$= -\frac{1}{\phi'(r_n)}\big(\varphi_\alpha(r_n) - \varphi_\alpha(r_{n-1}) - \varphi'(r_{n-1})(r_n - r_{n-1})\big)$$

$$= -\frac{\varphi_\alpha(r_n)}{\varphi'(r_n)} = r_{n+1} - r_n\,.$$

Consequently the sequence x_n is a Cauchy sequence converging to a solution x_* of the equation (1). Moreover, $x_* \in B(x_0, r_\alpha)$ and the estimate (19) holds. □

Remark

As we shown above, the functions ψ and ϕ_σ are not comparable. In the particular case in which the conditions (17) hold, the function φ_α coincides with the function ϕ_α which is less or equal to the function ϕ_σ.

In general also the functions ψ and φ_α are not comparable except for the some particular cases.

A very interesting case is the one in which $A(x) = f'(x)$ with f' Hölder continuous.

We have $\omega_f(t) = Kt^\theta\,,\ (0 < \theta \le 1)$,

$$\sup_{0\le s\le t} s^\theta + (t-s)^\theta = 2^{1-\theta}\, t^\theta,$$

$$\psi(r) = 2^{1-\theta}\frac{r^{1+\theta}}{1+\theta} - r + a$$

and

$$\varphi_\alpha(r) = \alpha(\theta)\frac{r^{1+\theta}}{1+\theta} - r + a$$

where the parameter $\alpha(\theta)$ introduced in [8] is defined by

$$\alpha(\theta) := \min\{\beta \geq 1 : \max_{0\leq t\leq t(\beta)} g(t) \leq \beta\}\,, \tag{20}$$

where

$$t(\beta) := \frac{\beta\,\theta^\theta}{(1+\theta)(\beta(1+\theta)^\theta - \theta^\theta)}\,, \quad g(t) := \frac{t^{1+\theta} + (1+\theta)t}{(1+t)^{1+\theta} - 1}\,.$$

In fact

$$\alpha(\theta) = \min\left\{\beta \geq 1 :\ \sup\left\{\frac{(v-u)^{1+\theta} + (1+\theta)(v-u)}{v^{1+\theta} - u^{1+\theta}} \leq \beta \right.\right.$$

$$\left.\left.\forall\, v, u :\ v - u = -\frac{\phi_\beta(u)}{\phi'(u)} r_1 \leq u \leq r_\beta\right\}\right\}$$

$$= \min\left\{\beta \geq 1 :\ \sup\left\{\frac{\left(\frac{v}{u} - 1\right)^{1+\theta} + (1+\theta)\left(\frac{v}{u} - 1\right)}{\left(\frac{v}{u}\right)^{1+\theta} - 1} \leq \beta \right.\right.$$

$$\left.\left.\forall\, v, u :\ v - u = -\frac{\phi_\beta(u)}{\phi'(u)} r_1 \leq u \leq r_\beta\right\}\right\}.$$

Set

$$t := \frac{v-u}{u}\,,$$

we proved in [8] that

$$0 \leq t \leq t(\beta)$$

from which we obtain (20).
Finally, from the inequality

$$\alpha(\theta) < 2^{1-\theta} \text{ for all } 0 < \theta < 1$$

(see [8] and [9] for the proof), it follows that $\varphi_\alpha < \psi$.

We conclude with an example of a particular case in which we can apply Theorem 3.2 while Theorem 3.1 is not applicable.
Set $\omega_f(t) := \omega_A(t) := \omega(t) := Kt$, we have

$$\psi(r) = \frac{3}{2}Kr^2 - r + a\,,$$

$$\alpha \leq \max_{r_1 \leq u \leq v \leq r_\alpha} \frac{v+3u}{v+u} = 2$$

and consequently we obtain

$$\varphi_\alpha(r) \leq Kr^2 - r + a < \psi(r)\,.$$

Then, if $\frac{1}{6} < aK \leq \frac{1}{4}$, the function φ_α admits at least a positive zero while $\psi(r) > 0$ for all $r > 0$.

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