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STABILITY BY FIXED POINT METHODS FOR HIGHLY NONLINEAR DELAY EQUATIONS

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Abstract. Using contraction mappings we study $x' = -a(t)g(x(t-r))$ and obtain a stability result allowing $a(t)$ to change sign, but requiring that it be positive on average. Next, we study stability properties of delay equations of the form $x'(t) = -a(t)g_1(x(t-r)) - b(t)g_2(x(t-r))$ where the g_i are odd increasing functions and $g_2(x)/g_1(x) \to 0$ as $|x| \to 0$. We show that stability properties can be proved by studying $x'(t) = -[a(t)+b(t)]g_1(x(t-r))$. In effect, we can borrow the coefficient of the higher order term and add it to the coefficient of the lower order term to stabilize the equation. Finally, we return to the first equation and obtain a stability result when $a(t)$ can change sign, but is negative on average. Some of these stability results are of a very different type than previously discussed in the literature; and this is a result of using fixed point techniques.
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1. Introduction

Perron's theorem [14] tells us that if the linear approximation of a differential equation is uniformly asymptotically stable then the higher order terms can be ignored. Here, we take the reverse view: Higher order terms can be used with lower order terms to stabilize an equation.

In a series of papers [2-9] Furumochi and the author have investigated stability by means of fixed point theory. The general project was started some years ago in an attempt to overcome a list of difficulties encountered in stability investigations using Liapunov's direct method. The present work continues from [3].

In this paper we use fixed point theory to study stability properties of scalar equations of the form

$$x'(t) = -a(t)g(x(t-r))$$

where r is a positive constant, $a(t)$ can change sign, while g is nonlinear, $xg(x) > 0$ for $x \neq 0$, and g satisfies monotonicity and growth conditions.

Three things of interest are accomplished. First, we prove a fixed point theorem of contraction type which is suited to such stability problems; it is essentially a known result, but is suited to problems addressed here. Next, we obtain a stability criterion for $x' = -a(t)g(x(t-r))$ when $a(t)$ can change sign, but on average is positive. Then we obtain results which allow us to "average" the coefficients of terms of different orders in order to get a stability result. We show that stability of the zero solution of

$$x'(t) = -a(t)x(t-r) - b(t)x^{2n+1}(t-r)$$

can be established from the stability of

$$x'(t) = -(a(t) + b(t))x(t-r);$$

the coefficient of the higher order term is simply borrowed and added to the coefficient of the lower order term. There is, of course, a "cost" for doing so. In the same way, stability of the zero solution of

$$x'(t) = -a(t)x^{2n+1}(t-r) - b(t)x^{2n+3}$$

can be established from examination of

$$x'(t) = -(a(t) + b(t))x^{2n+1}(t-r).$$

The aforementioned "cost" goes down as n increases. We believe that this is a type of result not previously seen in the theory of differential equations. In one of the later results we show that our first equation can still be stable when $a(t)$ changes sign and is, on average, negative.

We focus some of our attention on polynomial problems in order to give clear insight into what can be proved. The same techniques work on far more general problems.

These averaging results are in marked contrast to classical linear dominant theory, as portrayed in Perron [14]; in fact, nonlinear terms combine with linear terms to facilitate proof of stability. They are also in sharp contrast to Liapunov results begun in Krasovskii [12] where strongly stable ordinary differential equations are perturbed with a higher order delay term, typified by

$$x'(t) = -a(t)g(x(t)) + b(t)h(x(t-r))$$

with the pointwise relations

$$a(t) - |b(t+r)| \geq 0, \qquad xg(x) \geq 0, \quad \text{and} \quad |g(x)| \geq |h(x)|$$

so that with Liapunov functional

$$V(t, x_t) = |x(t)| + \int_{t-r}^{t} |b(s+r)h(x(s))|ds$$

we get the relation

$$V'(t, x_t) \leq -a(t)|g(x(t))| + |b(t)h(x(t-r))|$$

$$+|b(t+r)h(x(t))| - |b(t)h(x(t-r))| \leq 0.$$

All of the stability comes from the part without a delay, $-a(t)g(x(t))$, and all the relations are pointwise. See Hatvani [11; pp. 3565-3570] and Zhang [15; p. 1381] for recent extensions of the Krasovskii idea. In [10], for example, is found Liapunov theory with the stability being derived from the delay term. In the results presented here all the relations are various kinds of averages and all the stability will come from terms containing a delay. Recently, Zhang [16] has presented a systematic way of constructing Liapunov functionals for linear delay equations when information about the quasi-characteristic roots is known, even when all the stability comes from the delay terms. We will have more to say on the history in Section 3.

2. A natural metric for a contraction mapping

It is well known that if $f(t, x)$ satisfies a uniform Lipschitz condition in x in the supremum norm, then a new norm can be defined (a weighted norm) so that $\int_0^t f(s, x(s))ds$ is a contraction. But if $f(t, s, x)$ does not satisfy a uniform Lipschitz condition in x then it can be challenging to define a norm so that $\int_0^t f(t, s, x(s))ds$ is a contraction.

We will have a mapping with several terms which needs to be a contraction. The terms will be considered individually with each term being a contraction and having a contraction constant as small as we please. The following theorem shows how this is done and it will be used repeatedly in the rest of the paper. The general idea of the theorem is very well known.

Theorem 2.1. *Let $L > 0$, $\psi(0)$ be a fixed number,*

$$M = \{\phi : [0, \infty) \to R | \phi \in C, \phi(0) = \psi(0), |\phi(t)| \leq L\},$$

and $f : [-L, L] \to R$ satisfy a Lipschitz condition with constant $K > 0$. Suppose also that $a : [0, \infty) \to R$ is continuous, $h : [0, \infty) \to R$ is continuous, and for $\phi \in M$ define

$$(P\phi)(t) = h(t) + \int_0^t e^{-\int_s^t a(u+r)du} a(s + r) f(\phi(s))ds.$$

If $P : M \to M$ then for each $d > 1$ there is a metric ρ on M such that P is a contraction with constant $1/d$ and (M, ρ) is a complete metric space.

Proof. Let $(X, |\cdot|_K)$ be the Banach space of continuous $\phi : [0, \infty) \to R$ for which

$$|\phi|_K := \sup_{t\geq 0} e^{-(dK+2)\int_0^t |a(s+r)|ds} |\phi(t)|$$

exists. If $\phi, \eta \in M$ then

$$\begin{aligned}
&|P\phi - P\eta|_K \\
&\leq \sup_{t\geq 0} e^{-(dK+2)\int_0^t |a(s+r)|ds} \int_0^t e^{-\int_s^t a(u+r)du} |a(s + r)||f(\phi(s)) - f(\eta(s))|ds \\
&\leq \sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u+r)du} |a(s+r)|K|\phi(s)-\eta(s)| e^{-(dK+2)(\int_0^s |a(u+r)|du + \int_s^t |a(u+r)|du)} ds \\
&\leq |\phi - \eta|_K \sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u+r)du} |a(s + r)|K e^{-(dK+2)\int_s^t |a(u+r)|du} ds \\
&\leq |\phi - \eta|_K \sup_{t\geq 0} \int_0^t e^{-dK\int_s^t |a(u+r)|du} K|a(s + r)|ds \\
&\leq |\phi - \eta|_K \frac{K}{dK} = (1/d)|\phi - \eta|_K.
\end{aligned}$$

Now M is a subset of the Banach space X and M is closed so M is complete. Thus, $P : M \to M$ has a unique fixed point.

The proof is complete.

3. A beginning stability theorem

Let r be a positive constant, $a : [0,\infty) \to R$ be continuous, and consider the scalar equation

$$x'(t) = -a(t)g(x(t-r)) \tag{3.1}$$

with continuous initial function $\psi : [-r,0] \to R$, where g is continuous, locally Lipschitz, and odd, while $x - g(x)$ is nondecreasing and $g(x)$ is increasing on an interval $[0,L]$ for some $L > 0$. This requirement may be reduced, in part, when g has a derivative by writing

$$a(t)g(x(t-r)) = (Da(t))(g(x(t-r))/D),$$

where $|\frac{d}{dx}g(x)| \le D$ on $[0,L]$, and then renaming $a(t)$ as $Da(t)$.

In [3] we studied this equation for $n = 0$ (the linear case). We proved that if there is an $\alpha < 1$ such that

$$\int_{t-r}^{t} |a(u+r)|du + \int_0^t |a(s+r)|e^{-\int_s^t a(u+r)du}\Big|\int_{s-r}^{s} a(u+r)du\Big|ds \le \alpha$$

and if $\int_0^t a(s)ds \to \infty$ as $t \to \infty$, then the zero solution of (3.1) is asymptotically stable.

To prepare for our next theorem write (3.1) as

$$x'(t) = -a(t+r)g(x(t)) + \frac{d}{dt}\int_{t-r}^{t} a(s+r)g(x(s))ds$$

and then as

$$x'(t) = -a(t+r)x(t) + a(t+r)[x(t) - g(x(t))] + \frac{d}{dt}\int_{t-r}^{t} a(s+r)g(x(s))ds.$$

Existence theory is found in [1; pp. 186-191], for example. For each $t_0 \ge 0$, Equation (3.1) requires a continuous initial function $\psi : [t_0 - r, t_0] \to R$ to specify a solution $x(t,t_0,\psi)$. In this problem the computations are the same for any $t_0 \ge 0$ so we take $t_0 = 0$. By the variation of parameters formula we have

$$\begin{aligned} x(t) =& \psi(0)e^{-\int_0^t a(s+r)ds} + \int_0^t e^{-\int_s^t a(u+r)du}a(s+r)[x(s) - g(x(s))]ds \\ &+ \int_0^t e^{-\int_s^t a(u+r)du}\frac{d}{ds}\int_{s-r}^{s} a(u+r)g(x(u))duds. \end{aligned}$$

Integration by parts of the last term yields

$$x(t) = \psi(0)e^{-\int_0^t a(s+r)ds} - e^{-\int_0^t a(u+r)du}\int_{-r}^{0} a(u+r)g(\psi(u))du$$

$$+\int_0^t e^{-\int_s^t a(u+r)du}a(s+r)[x(s)-g(x(s))]ds + \int_{t-r}^t a(u+r)g(x(u))du$$

$$-\int_0^t e^{-\int_s^t a(u+r)du}a(s+r)\int_{s-r}^{s} a(u+r)g(x(u))duds. \tag{3.2}$$

Note that if $0 < L_1 < L$, then the conditions on g given with (3.1) hold on $[-L_1, L_1]$. Also, note that if $\phi : [-r,\infty) \to R$ with $\phi_0 = \psi$, if ϕ is continuous and $|\phi(t)| \le L$, then for $t \ge 0$ we have

$$|\phi(t) - g(\phi(t))| \le L - g(L)$$

since $x - g(x)$ is odd and nondecreasing on $(0, L)$. The symbol ϕ_0 denotes the segment of ϕ on $[-r, 0]$.

For any continuous ψ on $[-r, 0]$ with $|\psi(t)| < L$ we take

$$M = \{\phi : [-r,\infty) \to R | \phi_0 = \psi, \phi \in C, |\phi(t)| \le L\}.$$

The size of ψ will be further restricted later.

Theorem 3.1. *Let g be odd, increasing on $[0, L]$, satisfy a Lipschitz condition, and let $x - g(x)$ be nondecreasing on $[0, L]$. Suppose also that for each $L_1 \in (0, L]$ we have*

$$|L_1 - g(L_1)| \sup_{t\ge 0}\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|ds + g(L_1)\sup_{t\ge 0}\int_{t-r}^t |a(u+r)|du$$

$$+g(L_1)\sup_{t\ge 0}\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^{s}|a(u+r)|duds < L_1 \tag{3.3}$$

and there exists $J > 0$ such that

$$-\int_0^t a(s+r)ds \le J \qquad \textit{for} \qquad t \ge 0. \tag{3.4}$$

Then the zero solution of (3.1) is stable.

Proof. We will first define a mapping $P : M \to M$ using (3.2) so that for $\phi \in M$ we have

$$(P\phi)(t) = \psi(t), \qquad -r \le t \le 0,$$

and for $t \geq 0$

$$(P\phi)(t) = \psi(0)e^{-\int_0^t a(s+r)ds} - e^{-\int_0^t a(u+r)du} \int_{-r}^0 a(u+r)g(\psi(u))du$$
$$+ \int_0^t e^{-\int_s^t a(u+r)du} a(s+r)[\phi(s) - g(\phi(s))]ds + \int_{t-r}^t a(u+r)g(\phi(u))du$$
$$- \int_0^t e^{-\int_s^t a(u+r)du} a(s+r) \int_{s-r}^s a(u+r)g(\phi(u))duds. \tag{3.5}$$

By (3.3) there is an $\alpha < 1$ such that if $\phi \in M$ then

$$|(P\phi)(t)| \leq \|\psi\|e^J + e^J\|g(\psi)\| \int_{-r}^0 |a(s+r)|ds$$
$$+|L - g(L)| \sup_{t\geq 0} \int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|ds$$
$$+g(L) \int_{t-r}^t |a(u+r)|du + g(L) \int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)| \int_{s-r}^s |a(u+r)|duds$$
$$\leq e^J[\|\psi\| + \|g(\psi)\| \int_{-r}^0 |a(s+r)|ds] + \alpha L.$$

Choose $\delta > 0$ so that $\|\psi\| < \delta$ and K the Lipschitz constant for g on $[0, L]$ implies that

$$e^J[\delta + K\delta \int_{-r}^0 |a(s+r)|ds] < (1 - \alpha)L.$$

Then $|(P\phi)(t)| \leq L$ so we can show that $P : M \to M$. Since the mapping is given by integrals and the functions are Lipschitz we will be able to show that P is a contraction. (At this point, if (3.3) holds only for L, itself, then we have a boundedness result.) For a given $\epsilon > 0$, $\epsilon < L$, substitute ϵ for L and obtain the usual stability proof since a fixed point of P will be a unique solution and lie in M.

We now want to change the metric so that we will have a contraction. For $\phi, \eta \in M$ we have

$$|(P\phi)(t) - (P\eta)(t)| \leq \int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)||\phi(s) - g(\phi(s)) - \eta(s) + g(\eta(s))|ds$$
$$+ \int_{t-r}^t |a(u+r)||g(\phi(u)) - g(\eta(u))|du$$
$$+ \int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)| \int_{s-r}^s |a(u+r)||g(\phi(u)) - g(\eta(u))|duds. \tag{3.6}$$

Now $g(x)$ and $x - g(x)$ both satisfy a Lipschitz condition with the same constant K, so we proceed as in the proof of Theorem 2.1 and take the metric on M as that induced by the norm

$$|\phi|_K := \sup_{t\geq 0} e^{-(dK+2)\int_0^t |a(u+r)|du}|\phi(t)|$$

where we will find that $d > 3$ will suffice. As shown in Theorem 2.1, the first term on the right-hand-side of (3.6) has a contraction constant $1/d$. The second term satisfies

$$\begin{aligned}
&\sup_{t\geq 0} e^{-(dK+2)\int_0^t |a(u+r)|du} \int_{t-r}^t |a(u+r)||g(\phi(u)) - g(\eta(u))|du \\
&\leq \sup_{t\geq 0} \int_{t-r}^t |a(u+r)|K|\phi(u) - \eta(u)|e^{-(dK+2)\int_0^u |a(s+r)|ds}e^{-(dK+2)\int_u^t |a(s+r)|ds}du \\
&\leq \sup_{t\geq 0} \int_{t-r}^t |a(s+r)|Ke^{-(dK+2)\int_s^t |a(u+r)|du}ds|\phi - \eta|_K \\
&\leq (K/dK)|\phi - \eta|_K \\
&= (1/d)|\phi - \eta|_K.
\end{aligned}$$

Multiply the third term by $e^{-(dk+2)\int_s^t |a(u+r)|du}$ obtaining

$$\begin{aligned}
&\int_0^t e^{-\int_s^t a(u+r)du}|a(s+r)|\int_{s-r}^s |a(u+r)|K|\phi(u) - \eta(u))|\times \\
&e^{-(dK+2)(\int_0^u |a(v+r)|dv)-(dK)\int_u^t |a(v+r)|dv)-2\int_s^t |a(v+r)|dv}duds \\
&\leq |\phi - \eta|_K \sup_{t\geq 0} \int_{t-r}^t |a(u+r)|Ke^{-dK\int_u^t |a(v+r)|dv}du \\
&\leq (1/d)|\phi - \eta|_K.
\end{aligned}$$

We take $d > 3$ for a contraction. As in the proof of Theorem 2.1 there is a unique fixed point. This completes the proof.

Our result is an interesting complement to the well-known 3/2-Theorem which stresses the pointwise extremes of functions, while ours stresses averages and allows $a(t)$ to be negative. There is a large theory centered around the scalar equation

$$x'(t) = -a(t)x(t - r(t))$$

where $a(t), r(t)$ are continuous nonnegative functions. General results and a survey is found in Krisztin [13] containing nonlinear extensions.

The $\frac{3}{2}$-theorem. *If there are nonnegative constants α and q with $a(t) \leq \alpha, r(t) \leq q$ such that $\alpha q \leq 3/2$, then the zero solution of $x'(t) = -a(t)x(t-r(t))$ is uniformly stable and 3/2 is the best possible constant.*

That result allows r to be a function of t, but our (3.3) allows $a(t)$ to be negative and also as large as we please. We have been unable to find any results on a 3/2-theorem in which $a(t)$ is allowed to change sign. Condition (3.3) does imply that $a(t)$ has essentially a positive average. By contrast, Example 3 in Section 5 will show stability when $a(t)$ has essentially a negative average. Thus, fixed point theory seems to add something entirely new to the solution of that type of problem.

4. Borrowing from a higher order term

It is a fairly simple matter to use exactly the same proof as that of Theorem 3.1 to show that if (3.1) satisfies the conditions of Theorem 3.1 and if (3.1) is perturbed with a higher order term then the conclusion of Theorem 3.1 still holds for the perturbed system. But if the linear part does not satisfy Theorem 3.1 then we note an interesting fact: a higher order term can stabilize the equation in the sense that by borrowing from the coefficient of a higher order term we can fulfill conditions parallel to those of Theorem 3.1.

Remark 1. We could continue here with

$$x'(t) = -a(t)x(t-r) - b(t)g(x(t-r))$$

where $x - g(x)$ is odd, positive and increasing on $(0, L)$. But the impact of the next two results will depend crucially on the reader being able to see the exact value of two constants. See Remark 6 for a comparison of results.

Thus, we consider

$$x'(t) = -a(t)x(t-r) - b(t)x^3(t-r) \tag{4.1}$$

where we assume that

$$c(t) := b(t) + a(t) \geq 0, \tag{4.2}$$

and that there is a constant $\alpha < 1$ with

$$\sup_{t \geq 0} 2\int_{t-r}^{t} c(u+r)du + \frac{2}{3}\int_{0}^{t} e^{-\int_s^t c(u+r)du}|b(s)|ds \leq \alpha. \tag{4.3}$$

Example 1. If $1 < k < 3/2$ then

$$x'(t) = -(1 - k\sin t)x(t-r)$$

will not satisfy (3.3), but when

$$2r + (2/3)k < 1$$

then

$$x'(t) = -(1 - k\sin t)x(t-r) - (k\sin t)x^3(t-r)$$

will satisfy (4.3) and the conditions of our next theorem will hold.

Problem. It would be very interesting to see a Liapunov functional constructed for this last equation showing stability. Such construction might well lead Liapunov theory into a fruitful new phase. Indeed, it seems challenging even for $r = 0$ since the condition then includes $k < 3/2$.

Remark 2. Theorem 3.1 averaged the values of $a(t)$ to produce stability. Our next result will produce stability by averaging the coefficients of terms having different powers.

Theorem 4.1. *If (4.2) and (4.3) hold then there is a $\delta > 0$ such that if ψ is a continuous initial function on $[-r, 0]$ with $\|\psi\| < \delta$ then $|x(t, 0, \psi)| < 1/\sqrt{3}$ for all $t \geq 0$.*

Proof. Write (4.1) as

$$\begin{aligned} x' &= -a(t)x(t-r) - b(t)x(t-r) + b(t)[x(t-r) - x^3(t-r)] \\ &= -c(t)x(t-r) + b(t)[x(t-r) - x^3(t-r)] \\ &= -c(t+r)x(t) + \frac{d}{dt}\int_{t-r}^{t} c(s+r)x(s)ds + b(t)[x(t-r) - x^3(t-r)] \end{aligned}$$

so that by the variation of parameters formula, followed by integration by parts we have

$$\begin{aligned} x(t) =& \psi(0)e^{-\int_0^t c(s+r)ds} - e^{-\int_0^t c(u+r)du}\int_{-r}^{0} c(u+r)\psi(u)du \\ &+ \int_0^t e^{-\int_s^t c(u+r)du} b(s)[x(s-r) - x^3(s-r)]ds + \int_{t-r}^{t} c(u+r)x(u)du \\ &- \int_0^t e^{-\int_s^t c(u+r)du} c(s+r)\int_{s-r}^{s} c(u+r)x(u)duds. \end{aligned}$$

Now $f(x) = x - x^3$ has a maximum of $2/3\sqrt{3}$ at $1/\sqrt{3}$. As $x - x^3$ increases on $(0, 1/\sqrt{3})$ we could work on any shorter interval. We will show here that if ψ is small enough and if $|\phi(t)| \le 1/\sqrt{3}$ then $|(P\phi)(t)| \le 1/\sqrt{3}$. We use the previous equation to define P as we did in (3.5) and have

$$\begin{aligned}
|(P\phi)(t)| \le \|\psi\| + \|\psi\| \int_{-r}^{0} c(u+r)du \\
+ \int_0^t e^{-\int_s^t c(u+r)du} |b(s)|(2\sqrt{3}/9)ds \\
+ 2 \sup_{t \ge 0} \int_{t-r}^{t} c(u+r)du(1/\sqrt{3}) \\
\le 1/\sqrt{3}
\end{aligned}$$

provided that (4.3) holds and $\|\psi\| < \delta$ where $\delta + \delta \int_{-r}^{0} c(u+r)du < 1 - \alpha$. Our mapping set for any such fixed ψ is

$$M = \{\phi : [-r, \infty) \to R | \phi_0 = \psi, |\phi(t)| \le 1/\sqrt{3}\}$$

and $P : M \to M$.

The contraction argument parallel to Theorem 2.1 uses the weight for the norm as

$$e^{-2d \int_0^t [|b(s)| + c(s+r)]ds}$$

since for $\phi, \eta \in M$ we have

$$|(P\phi)(t) - (P\eta)(t)| \le \int_0^t e^{-\int_s^t c(u+r)du} |b(s)||\phi(s) - \eta(s)|ds$$

$$+2 \int_{t-r}^{t} c(u+r)|\phi(s) - \eta(s)|ds.$$

5. Further borrowing

Here, we obtain an extension of the result in the last section by starting with a nonlinear term which does not necessarily satisfy the conditions of Theorem 3.1. Yet, when there is a suitable higher order term present, it may be used to stabilize the equation. Moreover, the coefficient corresponding to the 2/3 in (4.3) is reduced to a small fraction of that 2/3 as the order increases. This means that the penalty for merging the coefficients reduces as the order increases.

Consider the equation

$$x'(t) = -a(t)x^{2n+1}(t-r) - b(t)x^{2n+3}(t-r) \tag{5.1}$$

where n is a positive integer. Let

$$c(t) := a(t) + b(t) \geq 0.$$

Remark 3. As remarked in Section 4, we could work with

$$x'(t) = -a(t)g_1(x(t-r)) - b(t)g_2(x(t-r))$$

where $g_2(x)/g_1(x) \to 0$ as $|x| \to 0$, g_i odd, $g_1(x)-g_2(x)$ positive and increasing on $(0, L)$. But the impact of the next result depends on the reader seeing the coefficient of the first integral in Theorem 5.2. See Remark 6 for a summary.

Remark 4. Here, we enlarge on the process begun in the proof of Theorem 3.1. The order of these steps is critical, as is explained between (6.2) and (6.4). The following steps are taken in the proof of the next theorem.

(i) Add and subtract the term $b(t)x^{2n+1}(t-r)$ to yield the term $-c(t)x^{2n+1}(t-r)$.

(ii) Two steps are now taken to enable us to use the variation of parameters formula:

(a) First, write

$$-c(t)x^{2n+1}(t-r) = -c(t+r)x^{2n+1}(t) + \frac{d}{dt}\int_{t-r}^{t} c(s+r)x^{2n+1}(s)ds.$$

(b) Finally, write

$$-c(t+r)x^{2n+1}(t) = -c(t+r)x(t) + c(t+r)[x(t) - x^{2n+1}(t)].$$

Theorem 5.1. *Let $a(t) + b(t) = c(t) \geq 0$, $L^2 = \frac{2n+1}{2n+3}$, and let*

$$(1-L^2)\sup_{t\geq 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2\sup_{t\geq 0}\int_{t-r}^{t} c(s+r)ds < 1.$$

Then there is a $\delta > 0$ such that if the continuous initial function ψ for (5.1) satisfies $\|\psi\| < \delta$ then $|x(t, 0, \psi)| < L$.

Proof. Write (5.1) as

$$x'(t) = -c(t+r)x(t) + c(t+r)[x(t) - x^{2n+1}(t)]$$

$$+\frac{d}{dt}\int_{t-r}^{t} c(s+r)x^{2n+1}(s)ds + b(t)[x^{2n+1}(t-r) - x^{2n+3}(t-r)], \tag{5.2}$$

As we have done twice before, use the variation of parameters formula, integrate by parts, and obtain

$$\begin{aligned} x(t) &= e^{-\int_0^t c(s+r)ds}\psi(0) - e^{\int_0^t c(u+r)du}\int_{-r}^0 c(u+r)du\psi^{2n+1}(u)du \\ &+ \int_0^t e^{-\int_s^t c(u+r)du}c(s+r)[x(s)-x^{2n+1}(s)]ds \\ &+ \int_0^t e^{-\int_s^t c(u+r)du}b(s)[x^{2n+1}(s-r)-x^{2n+3}(s-r)]ds \\ &+ \int_{t-r}^t c(u+r)x^{2n+1}(u)du \\ &- \int_0^t e^{-\int_s^t c(u+r)du}c(s+r)\int_{s-r}^s c(u+r)x^{2n+1}(u)duds. \end{aligned} \tag{5.3}$$

Now

$$f(x) = x^{2n+1} - x^{2n+3} \tag{5.4}$$

has a local maximum at

$$x = \sqrt{\frac{2n+1}{2n+3}} = L \tag{5.5}$$

and is increasing on $(0, L)$. Moreover,

$$f(L) = \left(\frac{2n+1}{2n+3}\right)^{\frac{2n+1}{2}} - \left(\frac{2n+1}{2n+3}\right)^{\frac{2n+3}{2}}. \tag{5.6}$$

Use (5.3) to define a mapping P as in (3.5) of a set

$$M = \{\phi : [-r,\infty) \to R | \phi_0 = \psi, |\phi(t)| \le L\} \tag{5.7}$$

into itself where ψ is a sufficiently small initial function. That mapping in M will be possible if

$$\begin{aligned} L(1-L^{2n}) + f(L)\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds \\ + 2L^{2n+1}\sup_{t\ge 0}\int_{t-r}^t c(s+r)ds < L. \end{aligned}$$

As $f(L)/L^{2n+1} = 1 - L^2$ this will reduce to

$$(1-L^2)\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2\sup_{t\ge 0}\int_{t-r}^t c(s+r)ds < 1. \tag{5.8}$$

The remainder of the proof is exactly as before.

Remark 5. The critical value for applications in (5.8) is

$$1 - L^2 = 1 - \frac{2n+1}{2n+3} = \frac{2}{2n+3}$$

and we note that this tends to zero as $n \to \infty$.

Example 2. Let

$$a(t) = 1 - 2\sin t, \qquad b(t) = 2\sin t, \qquad c(t) = 1.$$

Then

$$\int_0^t e^{-\int_s^t 1du}|b(s)|ds \le 2$$

so to satisfy (5.8) we need

$$2\frac{2}{2n+3} + 2r < 1$$

or

$$r < \frac{2n-1}{4n+6}.$$

The idea used here will also work for the nonlinear equation

$$x'(t) = -a(t)g(x(t-r)) \tag{5.9}$$

with the conditions appearing with (3.1). For this section we denote it by

$$x'(t) = -a(t)g(x(t-r).) \tag{5.10}$$

We can remove a portion of $a(t)$ which does not fit our theorem, provided that the removed portion has a sufficiently small average.

Theorem 5.2. *Let the conditions with (3.1) hold for (5.10) and suppose that*

$$a(t) = c(t) - b(t) \tag{5.11}$$

where $c(t) \ge 0$ *and continuous, while*

$$\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2\sup_{t\ge 0}\int_{t-r}^t c(u+r)du < 1. \tag{5.12}$$

Then the zero solution of (5.1) is uniformly stable.

Proof. Write the equation as

$$\begin{aligned}
x'(t) &= -c(t)g(x(t-r)) + b(t)g(x(t-r)) \\
&= -c(t+r)g(x(t)) + \frac{d}{dt}\int_{t-r}^{t} c(s+r)g(x(s))ds + b(t)g(x(t-r)) \\
&= -c(t+r)x(t) + c(t+r)[x(t) - g(x(t))] \\
&+ \frac{d}{dt}\int_{t-r}^{t} c(s+r)g(x(s))ds + b(t)g(x(t-r)).
\end{aligned}$$

Use the variation of parameters formula with an initial function ψ and integrate the neutral term by parts as we have done before and obtain

$$x(t) = \psi(0)e^{-\int_0^t c(s+r)ds} - e^{-\int_0^t c(u+r)du}\int_{-r}^{0} c(u+r)g(\psi(u))du$$

$$+\int_0^t e^{-\int_s^t c(u+r)du}c(s+r)[x(s)-g(x(s))]ds + \int_0^t e^{-\int_s^t c(u+r)du}b(s)g(x(s-r))ds$$

$$+\int_{t-r}^{t} c(u+r)g(x(u))du - \int_0^t e^{-\int_s^t c(u+r)du}c(s+r)\int_{s-r}^{s} c(u+r)g(x(u))duds.$$

Now $f(x) = x - g(x)$ has a maximum on $[0, L]$ at L. Take

$$M = \{\phi : [-r, \infty) \to R | \phi_0 = \psi, |\phi(t)| \le L, \phi \in C\}.$$

Define a mapping P on M using the last equation in x, as before. We have

$$|(P\phi)(t)| \le \|\psi\| + g(\|\psi\|)\sup_{t\ge 0}\int_{t-r}^{t} c(u+r)du$$

$$+L - g(L) + g(L)\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2g(L)\int_{t-r}^{t} c(u+r)du.$$

In order to say that $P : M \to M$ we need

$$L - g(L) + g(L)\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2g(L)\sup_{t\ge 0}\int_{t-r}^{t} c(u+r)du < L.$$

When we subtract L from each side and divide by $g(L)$ we arrive at

$$\sup_{t\ge 0}\int_0^t e^{-\int_s^t c(u+r)du}|b(s)|ds + 2\sup_{t\ge 0}\int_{t-r}^{t} c(u+r)du < 1. \tag{5.13}$$

Theorem 3.1 was a result in which the second integral in (5.12) yielded stability by averaging the values of c. Theorem 4.1 showed how the fixed point method averaged the coefficients of terms of different powers. We now see that the integral in (5.12) can average an oscillating large term to make its effect smaller than a small constant term.

Example 3. Consider the scalar equation

$$x'(t) = -(1 - k\cos^2 pt)g(x(t-r))$$

where $1 < k < 2$, g satisfies the conditions with (3.1), and p is a large positive integer. Notice that, while $a(t)$ can change sign, for large p the integral of $a(t)$ becomes negative; on average, $a(t)$ is negative. Conditions of Theorem 3.1 would not be satisfied.

To show the behavior of solutions, referring to Theorem 5.2, we take

$$c(t) = 1, \qquad b(t) = k\cos^2 pt,$$

so that (5.12) asks that

$$\sup_{t\geq 0}\int_0^t e^{-(t-s)}k\cos^2 psds + 2r < 1.$$

We will show that for p large enough and r small enough, then (5.12) is satisfied. Now

$$k\cos^2 pt = (k/2)(1+\cos 2pt)$$

and

$$\begin{aligned}
&(k/2)\int_0^t e^{-(t-s)}(1+\cos 2ps)ds \\
&\leq (k/2) + (k/2)e^{-t}\int_0^t e^s\cos 2psds \\
&= \frac{k}{2} + \frac{ke^{-t}}{2(1+4p^2)}[e^s(\cos 2ps + 2p\sin 2ps)\Big|_0^t \\
&= \frac{k}{2} + \frac{ke^{-t}}{2(1+4p^2)}[e^t(\cos 2pt + 2p\sin 2pt) - 1] \\
&= \frac{k}{2} + \frac{k}{2(1+4p^2)}[\cos 2pt + 2p\sin 2pt - e^{-t}] \\
&\leq \frac{k}{2} + \frac{k(1+2p)}{2(1+4p^2} \\
&\to \frac{k}{2}
\end{aligned}$$

as $p \to \infty$. Thus, for p large enough and r small enough, then (5.12) is satisfied.

Remark 6. Compare (5.12) with (4.3) and (5.8). That crucial coefficient of the integral containing $|b(s)|$ is 1 in (5.12), 2/3 in (4.3), and $(1 - L^2)$ in (5.8). The higher order term made a significant contribution to stability.

6. Equations with more terms

It is quickly verified that the technique here works with any number of terms. In the equation

$$x'(t) = -a(t)x^{2n+1}(t-r) - b(t)x^{2n+3}(t-r) - p(t)x^{2n+5}(t-r) \tag{6.1}$$

we simply get an additional term

$$\int_0^t e^{-\int_s^t c(u+r)du} p(s)[x^{2n+1}(s-r) - x^{2n+5}(s-r)]ds \tag{6.2}$$

and $c(t) = a(t) + b(t) + p(t)$. The only step which requires caution is that the term $-c(t)x^{2n+1}(t-r)$ must be converted to the neutral term

$$-c(t+r)x^{2n+1}(t) + \frac{d}{dt}\int_{t-r}^t c(s+r)x^{2n+1}(s)ds; \tag{6.3}$$

do not add in the linear term and then convert it to the neutral term. If that order is inverted, then one obtains integrals of the form

$$\int_0^t e^{-\int_s^t c(u+r)du} c(s)...ds \tag{6.4}$$

which are not readily estimated.

This process also shows that the terms need not enter as consecutive odd exponents in x; if the coefficient in the next higher order term is insufficient to show stability, we can continue to still higher order terms.

References

[1] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.

[2] T. A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc. **124**(1996), 2383-2390.

[3] T. A. Burton, *Stability by fixed point theory or Liapunov theory: A comparison*, Fixed Point Theory, **4**(2003), 15-32.

[4] T. A. Burton, *Liapunov functionals, fixed points, and stability by Krasnoselskii's theorem*, Nonlinear Studies, **9**(2002), 181-190.

[5] T. A. Burton, *Perron-type stability theorems for neutral equations*, Nonlinear Analysis **55**(2003), 285-297.

[6] T. A. Burton, *Fixed points and stability of a nonconvolution equation*, Proc. Amer. Math. Soc., to appear.

[7] T. A. Burton and Tetsuo Furumochi, *Fixed points and problems in stability theory for ordinary and functional differential equations*, Dynamic Systems and Appl. **10**(2001), 89-116.

[8] T. A. Burton and Tetsuo Furumochi, *Krasnoselskii's fixed point theorem and stability*, Nonlinear Analysis **49**(2002), 445-454.

[9] T. A. Burton and Tetsuo Furumochi, *A note on stability by Schauder's theorem*, Funkcialaj Ekvacioj **44**(2001), 73-82.

[10] T. A. Burton W. E. Mahfoud, *Stability criteria for Volterra equations*, Trans. Amer. Math. Soc. **279**(1983), 143-174.

[11] L. Hatvani, *On the asymptotic stability for nonautonomous functional differential equations by Lyapunov functionals*, Trans. Amer. Math. Soc. **354**(2002), 3555-3571.

[12] N. N. Krasovskii, Stability of Motion, Stanford Univ. Press, Stanford, 1963.

[13] T. Krisztin, *On stability properties for one-dimensional functional differential equations*, Funkcialaj Ekvacioj **34**(1991), 241-256.

[14] O. Perron, *Die Stabilitätsfrage bei Differential Gleichungen*, Math. Z. **32**(1930), 703-728.

[15] Bo Zhang, *Asymptotic stability in functional differential equations by Liapunov functionals*, Trans. Amer. Math. Soc. **347**(1995), 1375-1382.

[16] Bo Zhang, *Formulas of Liapunov functions for systems of linear ordinary and delay differential equations*, Funkcialaj Ekvacioj **44**(2001), 253-278.