SOME REMARKS ON KRASNOSELSKII'S FIXED POINT THEOREM

CEZAR AVRAMESCU AND CRISTIAN VLADIMIRESCU

Department of Mathematics
University of Craiova,
13 A.I. Cuza Street, 1100 Craiova, ROMANIA
E-mail addresses: cezaravramescu@hotmail.com
vladimirescu@ucluj.ro

Abstract. Let $M$ be a closed convex non-empty set in a Banach space $(X, \| \cdot \|)$ and let $P = Ax + Bx$ be a mapping such that: (i) $Ax + By \in M$ for each $x, y \in M$; (ii) $A$ is continuous and $AM$ compact; (iii) $B$ is a contraction mapping. The theorem of Krasnoselskii asserts that in these conditions the operator $P$ has a fixed point in $M$. In this paper some remarks about the hypothesis of this theorem are given. A variant for the cartesian product of two operators is also considered.

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1. INTRODUCTION

Two main results of fixed point theory are Schauder's theorem and the contraction mapping principle. Krasnoselskii combined them into the following result (see [5], p. 31 or [6], p. 501).

Theorem K (Krasnoselskii). Let $M$ be a closed convex bounded non-empty subset of a Banach space $(X, \| \cdot \|)$. Suppose that $A, B$ map $M$ into $X$ such that

i) $Ax + By \in M$, for all $x, y \in M$;

ii) $A$ is continuous and $AM$ is contained in a compact set;

iii) $B$ is a contraction mapping with constant $\alpha \in (0, 1)$.

Then there exists $x \in M$, with

$$x = Ax + Bx.$$
In [2] T.A. Burton remarks the difficulty to check the hypothesis i) and replaces it with the weaker condition:

\[ (x = Ax + By, \ y \in M) \Rightarrow (x \in M). \]

The proof idea is the following: for every \( y \in M \) the mapping \( x \rightarrow Bx + Ay \) is a contraction. Therefore, there exists \( \varphi : M \rightarrow M \) such that \( \varphi(y) = B\varphi(y) + Ay \), for every \( y \in M \). Then the problem is reduced to prove that \( \varphi \) admits a fixed point; to this aim Schauder’s theorem is used (see [6], p. 57).

In [3] the authors show that instead of Schauder’s theorem one can use Schaefer’s fixed point theorem (see [5], p. 29) which yields in normed spaces or more generally in locally convex spaces.

In the present paper, stimulated by the ideas contained in [2] and [3] we shall continue the analysis of Krasnoselskii’s result. We shall give in addition a variant of the result contained in [1] within we shall use Schaefer’s fixed point theorem.

2. General results

Let \((X, \|\cdot\|)\) be a Banach space, \( M \subset X \) be a convex closed (not necessary bounded) subset of \( X \). Let in addition \( A, B : M \rightarrow X \) be two operators; consider the equation

\[ x = Ax + Bx. \]

A way to proof that equation (1) admits solutions in \( M \) is to write (1) under the equivalent form

\[ x = Hx \]

and to apply a fixed point theorem to operator \( H \). There exist two possibilities to build the operator \( H \).

**Case 1.** The operator \( I - B \) admits a continuous inverse; then

\[ H = (I - B)^{-1} A. \]

**Case 2.** The operator \( B \) admits a continuous inverse; then

\[ H = B^{-1} (I - A). \]
If we want to apply Schauder’s theorem to operator $H$, then in the Case 1 we must suppose that $A$ fulfills hypothesis ii) from Theorem K and in the Case 2 we must suppose that $I - A$ fulfills this hypothesis.

If we would suppose $A, B : X \to X$, then it will exists another possibility i.e. to consider the equation

$$x = (A + T)x + (B - T)x,$$

where $T : X \to X$ is an arbitrary operator chosen such that the Case 1 or the Case 2 yields.

We state the following two general results.

**Proposition 1.** Suppose that

i) $M$ is a closed convex set;

ii) $I - B : X \to X$ is an injective operator;

iii) $(I - B)^{-1}$ is continuous;

iv) $A : M \to X$ is a continuous operator and $A(M)$ is contained into a compact set;

v) the following inequalities hold:

\begin{align*}
(5) & \quad A(M) \subset (I - B)(X) \\
(6) & \quad (I - B)^{-1}A(M) \subset M.
\end{align*}

Then the equation (1) has solutions in $M$.

Indeed, by hypotheses one can apply to operator $H$ Schauder’s fixed point theorem. Remark that the condition (5) which assures the existence of operator $H$ is automatically fulfilled if $I - B$ is a surjective operator.

**Proposition 2.** Suppose that

i) $M$ is a closed convex set;

ii) $B : X \to X$ is an injective operator;

iii) $B^{-1}$ is a continuous operator;

iv) $I - A : M \to X$ is a continuous operator and $(I - A)(M)$ is contained into a compact set;

v) the following inequalities hold:

\begin{align*}
(7) & \quad (I - A)(M) \subset B(X)
\end{align*}
Then the equation \((1)\) has solutions in \(M\).

The proof is like the proof of Proposition 1.

3. Particular results

Without loss of generality one can admit

\[(9)\]

\[A_0 = 0,\]

(or \(B_0 = 0\)); indeed, one can write \((1)\) under the form

\[x = A_1 x + B_1 x,\]

where \(A_1 x = A x + B_0,\) \(B_1 x = B x - B_0.\)

Clearly, the translated operators \(A_1, B_1\) keep many algebraic and topological properties of the operators \(A, B.\)

Suppose that \(B : X \rightarrow X\) is a contraction mapping with constant \(\alpha < 1.\)

By inequalities

\[(10)\]

\[(1 - \alpha) \|x - y\| \leq \|(I - B)x - (I - B)y\| \leq (1 + \alpha) \|x - y\|, \quad (\forall)\ x, y \in X.\]

it follows in the case \(B_0 = 0\)

\[
(1 - \alpha) \|x\| \leq \|(I - B)x\| \leq (1 + \alpha) \|x\|, \quad (\forall)\ x \in X.
\]

Admitting the hypotheses of Proposition 1 about \(A,\) let us set

\[h_{\rho} := \sup_{x \in B_{\rho}} \{\|Ax\|\},\]

where

\[B_{\rho} := \{x \in X, \|x\| \leq \rho\}.\]

One has the following

**Corollary 1.** Suppose that \(B : X \rightarrow X\) is a contraction mapping with constant \(\alpha \in (0, 1),\) \(B_0 = 0,\) \(A : B_{\rho} \rightarrow X\) is continuous and \(A (B_{\rho})\) is compact.

If

\[(11)\]

\[h_{\rho} \leq (1 - \alpha) \rho,\]

then the equation \((1)\) has solutions in \(B_{\rho}.\)
Indeed, as is known, \( I - B : X \to X \) is homeomorphism. The operator \( H \) given by (3) does exist and (11) assures the inclusion \( HM \subset M \), since (10) implies
\[
\left\| (I - B)^{-1} x \right\| \leq \frac{1}{1 - \alpha} \| x \|.
\]

Consider now the case when \( B \) is expansive, i.e. \( B \) satisfies the condition
\[
\| Bx - By \| \geq \beta \| x - y \|, \quad \text{for all } x, y \in X,
\]
with \( \beta > 1 \).

Clearly, if \( B \) is expansive then \( B \) is injective and \( B^{-1} \) is continuous (it is well known that if \( \dim X < \infty \), then every expansive mapping is surjective, so it is a homeomorphism; since \( B^{-1} \) maps \( X \) into \( X \) it is a contraction mapping having the constant \( \frac{1}{\beta} \); hence every expansive mapping \( B : \mathbb{R}^n \to \mathbb{R}^n \) admits an unique fixed point).

From the inequalities
\[
\|(I - B) x - (I - B) y\| \geq \|Bx - By\| - \|x - y\| \geq (\beta - 1) \|x - y\|,
\]
it follows also \( I - B \) is injective and \( (I - B)^{-1} \) is continuous, since
\[
\left\| (I - B)^{-1} x - (I - B)^{-1} y \right\| \leq \frac{1}{\beta - 1} \| x - y \|.
\]

Therefore, by Proposition 1 one gets the following corollary.

**Corollary 2.** Suppose that \( B \) is an expansive mapping, \( B0 = 0 \) and \( I - B \) is surjective. If \( A : \overline{B}_\rho \to X \) is continuous with \( A(\overline{B}_\rho) \) compact and
\[
h_\rho \leq \rho (\beta - 1),
\]
then the equation (1) has solutions in \( \overline{B}_\rho \).

One can renounce to surjectivity hypothesis of \( I - B \) replacing it by the condition
\[
A\overline{B}_\rho \subset (I - B) X.
\]

Consider now, as in Proposition 2 \( M = \overline{B}_\rho \) and set
\[
k_\rho := \sup_{x \in \overline{B}_\rho} \{(I - A) x\}.
\]

**Corollary 3.** Let \( B : X \to X \) be an expansive and surjective operator with \( B0 = 0 \) and \( I - A : M \to X \) a continuous operator with \( (I - A) \overline{B}_\rho \) compact.
If
\[ k_\rho \leq \beta \rho, \]
then the equation (1) has solutions in \( \overline{B_\rho} \).

Indeed, by (12) it follows for \( y = 0 \),
\[ \|B^{-1}x\| \leq \frac{1}{\beta} \|x\|. \]

Corollary 4. Suppose that \( A \) satisfies the hypotheses of Corollary 3. Let 
\( B : X \to X \) be a linear injective continuous Fredholm operator having null 
index. If
\[ \|B^{-1}\| \leq \frac{\rho}{\kappa_\rho}, \]
then the equation (1) has solutions in \( \overline{B_\rho} \).

An interesting particular case of Corollary 3 is the following.

Corollary 5. Suppose that
i) \( \dim X < \infty \);
ii) \( A : \overline{B_\rho} \to X \) is continuous;
iii) \( B : X \to X \) is an expansive mapping with constant \( \beta > 1 \).

Then, if the relation (17) yields, the equation (1) admits solutions in \( \overline{B_\rho} \).

Indeed, as we remarked, by hypothesis i) it results \( B \) is homeomorphism.

Applying to \( H \) given by (4) the fixed point theorem of Brouwer one obtains 
the result.

4. Results via Schaefer’s theorem

In what follows we give a particular form of Schaefer’s theorem which can 
be found in [5], p. 29.

Theorem S. (Schaefer) Let \( (X, \|\cdot\|) \) be a normed space, \( H \) be a continuous 
mapping of \( X \) into \( X \), which maps bounded sets of \( X \) into compact sets. Then 
either

I) the equation \( x = \lambda Hx \) has a solution for \( \lambda = 1 \)
or

II) the set of all such solutions \( x \), for \( 0 < \lambda < 1 \) is unbounded.

One can state now variants of Propositions 1, 2 by renouncing to complete-
ness of \( X \).

Proposition 3. Let \( (X, \|\cdot\|) \) be a normed space. Suppose that
i) $A : X \rightarrow X$ is a continuous operator with $A$ mapping bounded sets of $X$ into compact sets;

ii) $I - B$ is a homeomorphism;

iii) the set

$$\{ x \in X, \ (\exists) \ \lambda \in (0,1), \ x = \lambda Hx \}$$

is bounded, where $H$ is given by (3).

Then the equation (1) admits solutions in $X$.

**Proposition 4.** Let $(X, \|\cdot\|)$ be a normed space. Suppose that

i) $B : X \rightarrow X$ is a homeomorphism;

ii) $I - B$ is a continuous operator with $I - B$ mapping bounded sets of $X$ into compact sets;

iii) the set (19), where $H$ is given by (4), is bounded.

Then the equation (1) admits solutions in $X$.

5. Remarks

Reverting to Theorem K, let $x$ be an arbitrary solution for (1) and $y$ the unique fixed point of $B$.

Setting

$$a := \inf_{x \in M} \{\|Ax\|\}, \ b := \sup_{x \in M} \{\|Ax\|\},$$

one has

$$\|x - y\| = \|Ax + Bx - By\| \leq \|Ax\| + \alpha \|x - y\| \leq b + \alpha \|x - y\|,$$

therefore

$$\|x - y\| \leq \frac{b}{1 - \alpha}.$$  

Similarly,

$$\|x - y\| \geq \|Ax\| - \alpha \|x - y\| \geq a - \alpha \|x - y\|,$$

hence

$$\|x - y\| \geq \frac{a}{1 + \alpha}.$$  

Finally, between the unique fixed point of $B$ and every fixed point of operator $A + B$ one has the relation

$$\frac{a}{1 + \alpha} \leq \|x - y\| \leq \frac{b}{1 - \alpha}.$$  

If \( B_0 = 0 \), then
\[
\frac{a}{1 + \alpha} \leq \|x\| \leq \frac{b}{1 - \alpha},
\]
relation true for each solution of equation (1).

6. **Theorems of Krasnoselskii’s type for a cartesian product of operators**

In [1] the problem of the existence of solutions \((x, y)\) for the system

\[
\begin{aligned}
\begin{cases}
  x = F(x, y) \\
  y = G(x, y)
\end{cases}
\end{aligned}
\]

One can make the same remarks like in previous sections. To obviate the repetition we deal only with the possibility of application Schaefer’s theorem to the problem (21).

Let \((X_1, \| \cdot \|_1), (X_2, \| \cdot \|_2)\) be two Banach spaces and let \(F : X_1 \times X_2 \to X_1\), \(G : X_1 \times X_2 \to X_2\) be two operators.

We state and prove the following result.

**Theorem A.** Suppose that:

i) \(F(x, y)\) is continuous with respect to \(y\), for every \(x \in X_1\) fixed;

ii) \[
\|F(x_1, y) - F(x_2, y)\|_1 \leq L \|x_1 - x_2\|_1, \]
for all \(x_1, x_2 \in X_1\) and \(y \in X_2\), with \(L \in (0, 1)\);

iii) there exists a constant \(C > 0\) such that
\[
\|F(0, y)\|_1 \leq C \|y\|_2, \]
for all \(y \in X_2\);

iv) \(G(x, y)\) is continuous on \(X_1 \times X_2\);

v) \(G\) is a compact operator.

Then either the system

\[
\begin{aligned}
\begin{cases}
  x = F(x, y) \\
  y = G(x, y)
\end{cases}
\end{aligned}
\]

admits a solution or the set of all such solutions for \(\lambda \in (0, 1)\) of the system

\[
\begin{aligned}
\begin{cases}
  x = \lambda F \left( \frac{x}{\lambda}, y \right) \\
  y = \lambda G(x, y)
\end{cases}
\end{aligned}
\]

is unbounded.
Proof. Firstly we prove that if $\lambda \in (0, 1]$ then $\lambda F\left(\frac{x}{\lambda}, y\right)$ is contraction mapping with respect to $x$, for every $y \in X.

Indeed, if $x \in X_1$, then $\frac{x}{\lambda} \in X_1.$
Evaluate for $x_1, x_2 \in X_1$ and $y \in X_2,
\[\|\lambda F\left(\frac{x_1}{\lambda}, y\right) - \lambda F\left(\frac{x_2}{\lambda}, y\right)\|_1 \leq \lambda \|F\left(\frac{x_1}{\lambda}, y\right) - F\left(\frac{x_2}{\lambda}, y\right)\|_1 \leq \lambda L \|\frac{x_1}{\lambda} - \frac{x_2}{\lambda}\|_1 = L \|x_1 - x_2\|_1.

Consider $y \in X_2$ arbitrary. Denote by $g\left(y\right)$ the unique solution of equation
$x = \lambda F\left(\frac{x}{\lambda}, y\right).

Therefore, for every $y \in X_2$ there exists an unique $g\left(y\right) \in X_1$ such that
$g\left(y\right) = \lambda F\left(\frac{g\left(y\right)}{\lambda}, y\right).

Define $T : X_2 \rightarrow X_2$ by
\begin{equation}
(22) \quad Ty := K\left(g\left(y\right), y\right), \text{ for every } y \in X_2.
\end{equation}

We show that the hypotheses of Schaefer’s theorem are fulfilled.
Indeed, if $(y_n)_n$ is a sequence converging to $y$ in $X_2$ as $n \rightarrow \infty$, then
\[\|g\left(y_n\right) - g\left(y\right)\|_1 = \|\lambda F\left(\frac{g\left(y_n\right)}{\lambda}, y_n\right) - \lambda F\left(\frac{g\left(y\right)}{\lambda}, y\right)\|_1 \leq \lambda \|F\left(\frac{g\left(y_n\right)}{\lambda}, y_n\right) - F\left(\frac{g\left(y\right)}{\lambda}, y\right)\|_1 + \lambda L \left\|\frac{g\left(y_n\right)}{\lambda} - \frac{g\left(y\right)}{\lambda}\right\|_1.

Hence,
\[\left(1 - L\right) \|g\left(y_n\right) - g\left(y\right)\|_1 \leq \lambda \|F\left(\frac{g\left(y\right)}{\lambda}, y_n\right) - F\left(\frac{g\left(y\right)}{\lambda}, y\right)\|_1\]
and $\lambda \|F\left(\frac{g\left(y\right)}{\lambda}, y_n\right) - F\left(\frac{g\left(y\right)}{\lambda}, y\right)\|_1 \rightarrow 0$ as $n \rightarrow \infty$ and the continuity of $g$ follows immediately.
So, one has
\[ \| T y_n - T y \|_2 = \| G ( g ( y_n ), y_n ) - G ( g ( y ) , y ) \|_2 \]
and since hypothesis iv) and \( y = \lim_{n \to \infty} y_n \) it results the continuity of \( T \).

Let \( M_2 \in X_2 \) be a bounded set. We prove that \( TM_2 \subset X_2 \) is compact.
Indeed, for \( y \in M_2 \) we have succesively
\[
\| g ( y ) \|_1 = \lambda \left\| F \left( \frac{g ( y )}{\lambda}, y \right) \right\|_1 \\
\leq \lambda \left\| F \left( \frac{g ( y )}{\lambda}, y \right) - F ( 0, y ) \right\|_1 + \lambda \| F ( 0, y ) \|_1 \\
\leq \lambda L \left\| \frac{g ( y )}{\lambda} - 0 \right\|_2 + \lambda \| F ( 0, y ) \|_1 \\
\leq L \| g ( y ) \|_1 + \lambda C \| y \|_2.
\]

It results
\[ \| g ( y ) \|_1 \leq \frac{\lambda C}{1 - L} \| y \|_2 , \text{ for all } y \in M_2. \]
Therefore the set \( g ( M_2 ) \) is bounded in \( X_1 \). Since the set \( g ( M_2 ) \times M_2 \) is bounded in \( X_1 \times X_2 \) and from hypothesis v) it follows that the set
\[ T ( M_2 ) = G ( g ( M_2 ) \times M_2 ) \]
is compact in \( X_2 \).
By applying Schaefer’s theorem one gets either the equation
\[ y = Ty \]
admits solutions in \( X_2 \) or the set of all solutions for \( \lambda \in ( 0, 1 ) \) of the equation
\[ y = \lambda Ty \]
is unbounded.
Equivalently, either the system
\[
\begin{cases}
  x = F ( x, y ) \\
  y = G ( x, y )
\end{cases}
\]
adopts a solution \( ( g ( y_0 ), y_0 ) \) or the set of all such solutions \( ( g ( y_0 ), y_0 ) \) for \( \lambda \in ( 0, 1 ) \) of the system
\[
\begin{cases}
  x = \lambda F \left( \frac{x}{\lambda}, y \right) \\
  y = \lambda G ( x, y )
\end{cases}
\]
is unbounded. □

REFERENCES


