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FIXED POINTS FOR MAPPINGS SATISFYING CYCLICAL CONTRACTIVE CONDITIONS

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Abstract. This paper deals with mappings of the type $f : A_i \to A_{i+1}$, $i = 1, 2, \cdots, p+1$, with $A_{p+1} = A_p$, where the contractive assumptions are restricted to pairs $(x, y) \in A_i \times A_{i+1}$. Extensions of Banach's theorem are considered as well as an extension of Caristi's theorem. The paper concludes with a result about nonexpansive mappings in a Banach space setting.
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1. Introduction

It is well known and easy to prove that if X is a complete metric space and if $F : X \to X$ is *continuous* and satisfies

$$d\left(F(x), F^2(x)\right) \leq kd(x, F(x)) \quad \forall x \in X, \text{ where } k \in (0,1),$$

then F has a fixed point in X. The condition on F ensures that $\{F^n(x)\}$ is a Cauchy sequence for each $x \in X$, and continuity does the rest.

On the other hand, suppose there exist two nonempty closed subsets A and B of X such that the mapping $F : A \cup B \to A \cup B$ satisfies:

(1) $F(A) \subseteq B$ and $F(B) \subseteq A$.

(2) $d(F(x), F(y)) \leq kd(x, y) \quad \forall x \in A$ and $y \in B$, where $k \in (0,1)$.

Then it readily follows that for any $x \in A \cup B$

$$d\left(F(x), F^2(x)\right) \leq kd(x, F(x))$$

and this again implies that $\{F^n(x)\}$ is a Cauchy sequence. Consequently $\{F^n(x)\}$ converges to some point $z \in X$. However in view of (2) an infinite number of terms of the sequence $\{F^n(x)\}$ lie in A and an infinite number of terms lie in B. Therefore $z \in A \cap B$, so $A \cap B \neq \emptyset$. Now (1) implies $F : A \cap B \to A \cap B$ and (2) implies that F restricted to $A \cap B$ is a contraction mapping. Since Banach's contraction mapping principle applies to F on $A \cap B$ we have the following result.

Theorem 1.1. *Let A and B be two non-empty closed subsets of a complete metric space X, and suppose $F : X \to X$ satisfies (1) and (2) above. Then F has a unique fixed point in $A \cap B$.*

An interesting feature about the above observation is that continuity of F is no longer needed. Indeed, simple examples can be constructed showing that discontinuous mappings can satisfy all the assumptions. Also, it is possible to reformulate this result as a common fixed point theorem for two mappings.

Corollary 1.2. *Let A and B be two non-empty closed subsets of a complete metric space X. Let $f : A \to B$ and $g : B \to A$ be two functions such that*

$$(*) \qquad d(f(x), g(y)) \leq kd(x, y) \quad \forall x \in A \text{ and } y \in B,$$

where $k \in (0,1)$. Then there exists a unique $x_0 \in A \cap B$ such that

$$f(x_0) = g(x_0) = x_0.$$

Proof. Apply Theorem 1.1 to the mapping $F : A \cup B \to A \cup B$ defined by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in A; \\ g(x) & \text{if } x \in B. \end{cases}$$

Observe that F is well defined since (*) implies $f(x) = g(x)$ if $x \in A \cap B$. □

Obviously the reasoning of Theorem 1.1 can be extended to a collection of finite sets.

Theorem 1.3. *Let* $\{A_i\}_{i=1}^p$ *be nonempty closed subsets of a complete metric space, and suppose* $F : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ *satisfies the following conditions (where* $A_{p+1} = A_1$*):*

(1) $F(A_i) \subseteq A_{i+1}$ *for* $1 \le i \le p$*;*

(2) $\exists\ k \in (0,1)$ *such that* $d(F(x), F(y)) \le kd(x,y)$ $\forall x \in A_i$, $y \in A_{i+1}$ *for* $1 \le i \le p$.

Then F *has a unique fixed point.*

Proof. One only need to observe that given $x \in \cup_{i=1}^p A_i$, infinitely many terms of the Cauchy sequence $\{F^n(x)\}$ lie in each A_i. Thus $\cap_{i=1}^p A_i \ne \emptyset$, and the restriction of F to this intersection is a contraction mapping. □

The objective of this note is to extend the above reasoning to more general classes of mappings.

2. Contractive extensions

We first take up the question of whether Edelstein's classical result for contractive mappings can be similarly extended. Recall that a mapping $F : M \to M$ is said to be *contractive* if $d(F(x), F(y)) < d(x,y)$ whenever $x, y \in M$, $x \ne y$. Edelstein's result ([3]) asserts that a contractive mapping defined on a complete metric space has a unique fixed point if some Picard sequence $\{F^n(x)\}$, $x \in M$, has a convergent subsequence. This result extends as follows. (Again, no continuity assumption is needed.)

Theorem 2.1. *Let* $\{A_i\}_{i=1}^p$ *be nonempty closed subsets of a complete metric space, at least one of which is compact, and suppose* $F : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ *satisfies the following conditions (where* $A_{p+1} = A_1$*):*

(1) $F(A_i) \subseteq A_{i+1}$ *for* $1 \le i \le p$*;*

(2) $d(F(x), F(y)) < d(x,y)$ *whenever* $x \in A_i$, $y \in A_{i+1}$ *and* $x \ne y$, $(1 \le i \le p)$.

Then F *has a unique fixed point.*

Proof. Assume A_1 is compact, and let

$$d = dist\,(A_1, A_p) := \inf \{d\,(x, y) : x \in A_1, y \in A_p\}\,.$$

By compactness there exists $x_0 \in A_1$ and a sequence $\{u_n\} \subset A_p$ such that $\lim_n d\,(x_0, u_n) = d$. Assume $d > 0$. Then

$$d\left(F^{p+1}\,(x_0)\,, F^{p+1}\,(u_n)\right) < \cdots < d\,(F\,(x_0)\,, F\,(u_n)) < d\,(x_0, u_n)\,. \tag{2.1}$$

Since the sequence $\left\{F^{p+1}\,(u_n)\right\}_{n=1}^{\infty} \subset A_1$ and A_1 is compact, this sequence has a subsequence that converges to some $z \in A_1$. By (2.1) and continuity of the distance function it must be the case that

$$d\left(z, F^{p+1}\,(x_0)\right) \leq d.$$

However this implies

$$d\left(F^{p-1}\,(z)\,, F^{2p}\,(x_0)\right) < d$$

and since $F^{p-1}\,(z) \in A_p$ and $F^{2p}\,(x_0) \in A_1$ we have a contradiction. We conclude therefore that $d = 0$ and $A_1 \cap A_p \neq \emptyset$. Thus by (1), $A_1 \cap A_2 \neq \emptyset$.

We now consider the sets $A_1' = A_1 \cap A_2$, $A_2' = A_2 \cap A_3, \cdots, A_P' = A_P \cap A_1$. In view of condition (1) these sets are all nonempty (and closed) and A_1' is compact. Thus conditions (1) and (2) of the theorem hold for F and the family $\{A_i'\}_{i=1}^{p}$, and by repeating the argument just given we conclude

$$A_1' \cap A_p' \neq \emptyset.$$

This in turn implies $A_1 \cap A_2 \cap A_3 \neq \emptyset$. Continuing step-by-step we conclude

$$A := \cap_{i=1}^{p} A_i \neq \emptyset.$$

Since A is compact and the restriction of F to A is contractive, we conclude that F has a unique fixed point in A. Uniqueness follows from the fact that any fixed point of F necessarily lies in A by condition (1). □

We now take up the question of whether condition (2) of Theorem 1.3 can be replaced by contractive conditions which typically arise in extensions of Banach's theorem. The answer is affirmative, but the arguments now become a little more subtle. We begin with a condition introduced by Geraghty [4]. Let $\mathcal{S}$ denote the class of those function $\alpha : \mathbb{R}^+ \to [0, 1)$ that satisfy the simple condition:

$$\alpha\,(t_n) \to 1 \Rightarrow t_n \to 0.$$

Theorem 2.2. ([4]) *Let X be a complete metric space, let $f : X \to X$ and suppose there exists $\alpha \in \mathcal{S}$ such that*

$$d(f(x), f(y)) \leq \alpha(d(x,y))\, d(x,y) \quad \forall x, y \in X.$$

Then f has a unique fixed point $z \in X$, and $\{f^n(x)\}$ converges to z for each $x \in X$.

Theorem 2.3. *Let $\{A_i\}_{i=1}^{p}$ be nonempty closed subsets of a complete metric space, let $\alpha \in \mathcal{S}$, and suppose $f : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ satisfies the following conditions (where $A_{p+1} = A_1$)*

(1) *$f(A_i) \subseteq A_{i+1}$ for $1 \leq i \leq p$;*

(2) *$d(f(x), f(y)) \leq \alpha(d(x,y))\, d(x,y)$ $\forall x \in A_i$, $y \in A_{i+1}$ for $1 \leq i \leq p$.*

Then f has a unique fixed point.

Proof. The strategy is to prove that $\cap_{i=1}^{p} A_i \neq \emptyset$ and apply Geraghty's theorem to f restricted to $\cap_{i=1}^{p} A_i \neq \emptyset$. For convenience of notation, if $j > p$, define $A_j = A_i$ where $i = j \bmod p$ and $1 \leq i \leq p$.

The argument we give a slight modification of the proof of Geraghty's theorem given in [6]. Let $x_0 \in A_1$ and let $x_n = f^n(x_0)$, $n = 1, 2, \cdots$.

Step 1.$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Proof. Condition (2) implies that $\{d(x_n, x_{n+1})\}$ is monotone decreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = r \geq 0$. Assume $r > 0$. Then again by condition (2)

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \alpha(d(x_n, x_{n+1})), \quad n = 1, 2, \cdots.$$

Letting $n \to \infty$ we see that $\alpha(d(x_n, x_{n+1})) \to 1$. But since $\alpha \in \mathcal{S}$, this in turn implies $d(x_n, x_{n+1}) \to 0$. Thus it must be the case that $r = 0$.

Step 2. $\{x_n\}$ is a Cauchy sequence.

Proof. Suppose there exists $\rho > 0$ such that given any $N \in \mathbb{N}$ there exist $n > m \geq N$ with $n - m = 1 \bmod p$ such that $d(x_n, x_m) \geq \rho > 0$. By the triangle inequality

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m).$$

Since $n-m=1 \bmod p$ x_m and x_n lie in different adjacently labelled sets A_i and A_{i+1} for some $1 \leq i \leq p$, so by the contractive condition

$$\begin{aligned}[1-\alpha\left(d\left(x_n, x_m\right)\right)] \rho &< [1-\alpha\left(d\left(x_n, x_m\right)\right)] d\left(x_n, x_m\right) \\ &\leq d\left(x_n, x_{n+1}\right)+d\left(x_m, x_{m+1}\right).\end{aligned}$$

Letting $n, m \rightarrow \infty$ with $n-m=1 \bmod p$ we conclude $\alpha\left(d\left(x_n, x_m\right)\right) \rightarrow 1$. But since $\alpha \in \mathcal{S}$ this implies $d\left(x_n, x_m\right) \rightarrow 0$, which is a contradiction. Therefore, given any $\varepsilon>0$ there exists $N \in \mathbb{N}$ such that if $n, m \geq N$ and $n-m=1 \bmod p$, $d\left(x_n, x_m\right) \leq \varepsilon / p$. By Step 1 it is possible to choose $N_1 \in \mathbb{N}$ so that $d\left(x_n, x_{n+1}\right) \leq \varepsilon / p$ if $n \geq N_1$. Now let $n, m \geq \max \{N, N_1\}$ with $m>n$. Then there exists $k \in\{1,2, \cdots, p\}$ such that and $n-m=k \bmod p$. Thus $n-m+j=1 \bmod p$, where $j=p-k+1$, so

$$d\left(x_n, x_m\right) \leq d\left(x_m, x_{n+j}\right)+d\left(x_{n+j}, x_{n+j-1}\right)+\cdots+d\left(x_{n+1}, x_n\right) \leq \varepsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence, and consequently that $\cap_{i=1}^p A_i \neq \emptyset$. By Geraghty's theorem f has a unique fixed point in $\cap_{i=1}^p A_i$, and by condition (1) any fixed point of f must lie in this intersection. $\square$

Next we look at the well known Boyd-Wong condition [1]. A statement of the Boyd-Wong theorem can be obtained by taking $A_i=A_j$ for all i, j in the following.

Theorem 2.4. *Let $\{A_i\}_{i=1}^p$ be nonempty closed subsets of a complete metric space and suppose $f: \cup_{i=1}^p A_i \rightarrow \cup_{i=1}^p A_i$ satisfies the following conditions (where $A_{p+1}=A_1$):*

(1) $f\left(A_i\right) \subseteq A_{i+1}$ for $1 \leq i \leq p$;

(2) $d(f(x), f(y)) \leq \psi(d(x, y))$ $\forall x \in A_i$, $y \in A_{i+1}$ for $1 \leq i \leq p$, where $\psi: \mathbb{R}^{+} \rightarrow[0, \infty)$ is upper semi-continuous from the right and satisfies $0 \leq \psi(t)<t$ for $t>0$.

Then f has a unique fixed point.

Proof. We follow the same strategy as in the preceding argument. For $j>p$ define $A_j=A_i$ if $j=i \bmod p$. Let $x_0 \in A_1$ and let $x_n=f^n\left(x_0\right), n=1,2, \cdots$.

<u>Step</u> 1. $\lim_{n \rightarrow \infty} d\left(x_n, x_{n+1}\right)=0$.

Proof. By (2) the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing and bounded below, so $\lim_{n\to\infty} d(x_n x_{n+1}) = r \geq 0$. Thus

$$d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) \Rightarrow r \leq \psi(r) \Rightarrow r = 0.$$

Step 2. $\{x_n\}$ is a Cauchy sequence.

Proof. Suppose not. Then there exists $\varepsilon > 0$ such that for any $k \in \mathbf{N}$, there exists $m_k > n_k \geq k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon.$$

Furthermore it may be assumed that for each k, m_k is chosen to be the smallest number greater that n_k for which the above is true. In view of Step 1,

$$\lim_{k\to\infty} d(x_{m_k}, x_{m_k-1}) = 0.$$

Since

$$\begin{aligned}\varepsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \varepsilon\end{aligned}$$

and we conclude $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon$. Also, since

$$d(x_{m_k}, x_{n_k}) - d(x_{m_k+1}, x_{m_k}) \leq d(x_{m_k+1}, x_{n_k}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}),$$

we conclude also that $\lim_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \varepsilon$. There exists $j, 0 \leq j \leq p-1$, such that $m_k - n_k + j = 1 \bmod p$ *for infinitely many* k. If $j = 0$ we have, for such k,

$$\begin{aligned}d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k+1}) + \psi(d(x_{m_k}, x_{n_k})) + d(x_{n_k+1}, x_{n_k}).\end{aligned}$$

Letting $k \to \infty$ gives $\varepsilon \leq \psi(\varepsilon)$, which contradicts $\psi(t) < t$ for $t > 0$. The case $j \neq 0$ similar. $\square$

The preceding two theorems are just examples to illustrate the methodology. Other extensions of Banach's theorem can be similarly recast. It would be nice to find a more abstract formulation that would unify all of these results. Of course it would even be nicer to have applications.

3. Caristi's theorem

The preceding ideas lead also to an analogous extension of Caristi's theorem [2], which is simply the following theorem when $A_i = A_j$ for all i, j.

Theorem 3.1. *Let $A_1, A_2, \cdots, A_p, A_{p+1} = A_1$ be nonempty closed subsets of a complete metric space X, and suppose $f : X \to X$ satisfies the following conditions.*

(1) $f(A_i) \subseteq A_{i+1}$ *for* $1 \leq i \leq p$.

(2) $d(x, f(x)) \leq \varphi_i(x) - \varphi_{i+1}(f(x))$ $\forall x \in A_i$ $(1 \leq i \leq p)$, *where each* $\varphi_i : A_i \to \mathbb{R}$ *is lower semicontinuous and bounded below.*

Then f has a fixed point.

Proof. Let $x_1 \in A_1$ and $x_n = f^{n-1}(x_1)$. By condition (2)

$$\varphi_1(x_1) \geq \cdots \geq \varphi_n(x_n) \geq \cdots, \quad n = 1, 2, \cdots,$$

where of course $\varphi_i = \varphi_j$ if $i = j \bmod p$. Therefore $\lim_{i\to\infty} \varphi_i(x_i) = r$. Now fix $x_n \in A_n$, let $k \in \mathbb{N}$, and let $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, f(x_n)) + d(f(x_n), f(x_{n+1})) + \cdots + d(f(x_{m-2}), x_m) \\ &\leq \varphi_n(x_n) - \varphi_{n+1}(f(x_n)) + \varphi_{n+1}(f(x_n)) - \varphi_{n+2}(f(x_{n+1})) \\ &+ \cdots + \varphi_{m-1}(f(x_{m-2})) - \varphi_m(x_m) \\ &= \varphi_n(x_n) - \varphi_m(x_m). \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence, and in turn that $A := \cap_{i=1}^p A_i \neq \emptyset$. We now have the following situation. $f : A \to A$ and

$$d(x, f(x)) \leq \min_{1\leq i\leq p} [\varphi_i(x) - \varphi_{i+1}(f(x))] \text{ for all } x \in A.$$

Thus

$$\begin{aligned} pd(x, f(x)) &\leq \varphi_1(x) - \varphi_2(f(x)) + \varphi_2(x) - \varphi_3(f(x)) + \cdots \\ &+ \varphi_p(x) - \varphi_1(f(x)) \\ &= \sum_{i=1}^{p} [\varphi_i(x) - \varphi_i(f(x))]. \end{aligned}$$

Now define $\Phi : A \to \mathbb{R}$ by taking

$$\Phi(x) = p^{-1} \sum_{i=1}^{p} \varphi_i(x), \quad x \in A.$$

Then Φ is lower semicontinuous and bounded below, and moreover

$$d(x, f(x)) \leq \Phi(x) - \Phi(f(x)) \text{ for each } x \in A.$$

The conclusion now follows from Caristi's theorem. □

4. Nonexpansive mappings

We do not know of results analogous to Theorems 2.2 and 2.3 for nonexpansive mappings. However assumption (1) can arise in a reasonably natural way in the study of nonexpansive mappings. We illustrate this by giving a short proof of the following theorem (which is stated in a more general but less elegant way in [7]). Recall that if X is a Banach space and $K \subseteq X$, then $T : K \to X$ is said to be *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for each $x, y \in K$.

Theorem 4.1. ([7]) *Let K be a nonempty weakly compact convex subset of a Banach space X, suppose $T : K \to K$ is nonexpansive, and suppose for each $x \in K$ there exists a positive integer $N(x)$ and an $\alpha(x) \in (0,1)$ such that*

$$\left\| T^{N(x)}(x) - T^{N(x)}(y) \right\| \leq \alpha(x) \|x - y\| \text{ for each } y \in K.$$

Then T has a unique fixed point.

Proof. By weak compactness we may suppose K is *minimal* with respect to being nonempty, closed, convex, and T-invariant. It is well known (cf., [5], p. 124) that in this case each point $w \in K$ must be diametral, that is,

$$\sup\{\|w - y\| : y \in K\} = diam(K).$$

We shall show that the assumption $diam(K) > 0$ implies the existence of a nondiametral point, and thereby to the conclusion that K contains a single point which is fixed under T.

Let $x \in K$, let $N = N(x)$, and let $r = \alpha(x)\, diam(K)$. Define

$$
\begin{aligned}
S_1 &:= \left\{ z \in K : \left\| z - T^{iN}(x) \right\| \leq r \text{ for almost all } i \geq 1 \right\}; \\
S_2 &:= \left\{ z \in K : \left\| z - T^{iN+1}(x) \right\| \leq r \text{ for almost all } i \geq 1 \right\}; \\
&\vdots \\
S_N &:= \left\{ z \in K : \left\| z - T^{(i+1)N-1}(x) \right\| \leq r \text{ for almost all } i \geq 1 \right\}.
\end{aligned}
$$

Observe that $T^N(x) \in S_1$, so $S_1 \neq \emptyset$. Also, if $z \in S_1$ then

$$\left\| T(z) - T^{iN+1}(x) \right\| \leq \left\| z - T^{iN}(x) \right\|;$$

hence $T(z) \in S_2$. Similarly $T(S_i) \subseteq S_{i+1}$, $1 \leq i \leq N-1$ and $T(S_N) \subseteq S_1$. Also it is easy to see that the sets $\{S_i\}_{i=1}^N$ are convex. Now let $i \geq j \geq N$, say $j = N + k$ and $i = N + k + s$. Then

$$\left\| T^i(x) - T^j(x) \right\| \leq \left\| T^{N+s}(x) - T^N(x) \right\| \leq r.$$

Thus $T^i(x)$ lies in the closed ball $B\left(T^j(x); r\right)$ for all $i \geq j \geq N$. Therefore the family $\left\{ B\left(T^i(x); r\right) \cap K \right\}_{i=N}^{\infty}$of closed convex sets has the finite intersection property, so by weak compactness there exists a point $z \in K$ such that

$$z \in \cap_{i=N}^{\infty} B\left(T^i(x); r\right).$$

Clearly this implies $z \in S := \cap_{i=1}^N S_i \cap K$. Since $\overline{S}$ is nonempty closed convex and T-invariant, by minimality of K it must be the case that $\overline{S} = K$. In particular $\overline{S}_1 = K$. Hence if $u \in K$ and if $\varepsilon > 0$ is chosen so that $r + \varepsilon < diam(K)$, then

$$\left\| u - T^{iN}(x) \right\| \leq r + \varepsilon < diam(K)$$

for i sufficiently large. In particular, if $\{u_1, u_2, \cdots, u_k\} \subset K$ then $\cap_{i=1}^k B(u_i; r) \cap K \neq \emptyset$. Thus the family of weakly compact sets

$$\{B(u; r + \varepsilon) \cap K : u \in K\}$$

also has the finite intersection property. Let

$$z \in (\cap_{u \in K} B(u; r + \varepsilon)) \cap K.$$

Then $\|z - u\| \leq r + \varepsilon < diam(K)$ for each $u \in K$. This means that z is a non-diametral point of K – a contradiction. $\square$

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