WHEELER-FEYNMAN PROBLEM ON A COMPACT INTERVAL

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Abstract. In this paper the problem (1)+(2) is studied.

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1. Introduction

In the paper [1] and [3] the autor study the Weeler-Feynman problem on $R$. In this paper we consider the following Weeler-Feynman problem:

(1) 
$$x'(t) = f(t, x(t), x(t-h), x(t+h)), \quad t \in [a, b],$$

(2) 
$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0 + h],$$

where $t_0 \in [a, b]$, $a \leq t_0 - h$, $t_0 + h \leq b$ and $\varphi \in C^1[t_0 - h, t_0 + h]$

2. Remarks and examples

2.1. By a solution of (1) we understand a function $x \in C[a-h, b+h] \cap C^1[a, b]$ which satisfies the relation (1) for all $t \in [a, b]$.

2.2. Let $\alpha, \beta, \gamma \in R$, $\beta \neq 0$, $\gamma \neq 0$, $t_0 \in [a, b]$. We consider the following problem:

(3) 
$$x'(t) = \alpha x(t) + \beta x(t-h) + \gamma x(t+h), \quad t \in [a, b],$$

(4) 
$$x(t) = \varphi(t), \quad t \in [t_0 - h, t_0 + h],$$

where $t_0 \in [a, b]$, $a \leq t_0 - h$, $t_0 + h \leq b$.

We shall apply the method of steps on intervals $[t_0, b]$ and $[a, t_0]$ to find some "if and only" conditions for the existence of a solution of problem (3)+(4).

Let $t \in [t_0, t_0 + h]$ 
$$\varphi'(t) = \alpha \varphi(t) + \beta \varphi(t-h) + \gamma x(t+h)$$
Then:

\[ x(t) := x_1(t) = \frac{1}{\gamma} [\alpha \varphi(t - h) + \beta \varphi(t - 2h) - \varphi'(t - h)], \ t \in [t_0 + h, t_0 + 2h] \]

Let \( t \in [t_0 + h, t_0 + 2h] \)

\[ x'_1(t) = \alpha x_1(t) + \beta \varphi(t - h) + \gamma x(t + h) \]

Then:

\[ x(t) := x_2(t) = \frac{1}{\gamma} [\alpha x_1(t - h) + \beta \varphi(t - 2h) - x'_1(t - h)], \ t \in [t_0 + 2h, t_0 + 3h] \]

By the same way the final step on \([t_0, b] \): \[ x_{n_b}(t) = \frac{1}{\gamma} [\alpha x_{n_{b-1}}(t - h) + \beta x_{n_{b-2}}(t - 2h) - x'_{n_{b-1}}(t - h)], \ t \in [t_0 + n_b h, b] \]

where \( n_b = \lfloor \frac{b - t_0}{h} \rfloor \).

By the same way on \([a, t_0]\) we find \( n_a = \lfloor \frac{t_0 - a}{h} \rfloor \).

Let \( n := max\{n_a, n_b\} \).

Let \( \varphi \in C^{n+1}[t_0 - h, t_0 + h] \).

Let \( x \in C^n[a - h, b + h] \cap C^{n+1}[a, b] \) be a solution of problem (3)+(4).

We have:

\[ x^{(k+1)}(t) = \alpha x^{(k)}(t) + \beta x^{(k)}(t - h) + \gamma x^{(k)}(t + h), \ k \in \{0, 1, \ldots, n\} \]

For \( t = t_0 \), we have:

\[ \varphi^{(k+1)}(t_0) = \alpha \varphi^{(k)}(t_0) + \beta \varphi^{(k)}(t_0 - h) + \gamma \varphi^{(k)}(t_0 + h), \ k \in \{0, 1, \ldots, n\} \]

Then the problem (3)+(4) has a solution if and only if:

\[ \varphi^{(k+1)}(t_0) = \alpha \varphi^{(k)}(t_0) + \beta \varphi^{(k)}(t_0 - h) + \gamma \varphi^{(k)}(t_0 + h), \ k \in \{0, 1, \ldots, n\} \].

2.3. For the case in which \( \beta = 0 \) or \( \gamma = 0 \) see [2].

3. THE MAIN RESULT

In what follow we consider the problem (1)+(2). We need the following conditions.

Let \( n_a := \lfloor \frac{t_0 - a}{h} \rfloor, \ n_b := \lfloor \frac{b - t_0}{h} \rfloor, \ n := max\{n_a, n_b\} \).

Let \( f \in C^{n+1}([a, b] \times R^3) \).

(C1): For all \( u_1 \in [a, b], \ u_2, u_4, u_5 \in R, \) there exist a unique \( u_3 \in R, \ u_3 = f_1(u_1, u_2, u_4, u_5), \ f_1 \in C^{n+1}([a, b] \times R^3) \), such that, \( u_5 = f(u_1, u_2, u_3, u_4) \).

(C2): For all \( u_1 \in [a, b], \ u_2, u_3, u_5 \in R, \) there exist a unique \( u_4 \in R, \ u_4 = f_2(u_1, u_2, u_3, u_5), \ f_2 \in C^{n+1}([a, b] \times R^3) \), such that, \( u_5 = f(u_1, u_2, u_3, u_4) \).

We have
Theorem 1. Let \( f \in C^{n+1}([a,b] \times R^3) \) satisfies (C1) and (C2). If \( \varphi \in C^n[a - h, t_0 + h] \), then the problem (1)+(2) has a unique solution if and only if \( \varphi \) satisfies the following condition:

\[
(5) \quad \varphi^{(k+1)}(t_0) = [f(t, \varphi(t), \varphi(t - h), \varphi(t + h))]^{(k)}_{t = t_0}, \quad k \in \{0, 1, \ldots, n\}
\]

Proof. By the method of steps we construct the solution of (1)+(2) as follows. Let \( t \in [t_0, t_0 + h] \)

\[
\varphi'(t) = f(t, \varphi(t), \varphi(t - h), x(t + h))
\]

From (C2) we have

\[
x(t) := x_1(t) = f_2(t - h, \varphi(t - h), \varphi(t - 2h), \varphi'(t - h)), \quad t \in [t_0 + h, t_0 + 2h].
\]

By the same method we find the final step:

\[
x_n(t) = f(t - h, x_{n-1}(t - h), x_{n-2}(t - 2h), x_n(t - h)), \quad t \in [t_0 + n_1h, b]
\]

where \( n_b = \lfloor \frac{b - t_0}{h} \rfloor \).

We must have:

\[
\varphi(t_0 + h) = x_1(t_0 + h)
\]

\[
x_p(t_0 + (p + 1)h) = x_{p+1}(t_0 + (p + 1)h), \quad p \leq n_b - 1
\]

By the same way we have the solution on \([a, t_0]\) with the condition

\[
\varphi(t_0 - h) = x_{-1}(t_0 - h)
\]

\[
x_{-p}(t_0 - (p + 1)h) = x_{-(p+1)}(t_0 - (p + 1)h), \quad p \leq n_a - 1
\]

where \( n_a = \lfloor \frac{t_0 - a}{h} \rfloor \)

So the solution is:

\[
x(t) = \begin{cases} 
  x_{-n_a}(t) & \text{dac\‘a} \ t \in [a, t_0 - n_a h] \\
  x_{-k}(t) & \text{dac\‘a} \ t \in [t_0 - (k + 1)h, t_0 - kh], 1 \leq k \leq n_a - 1 \\
  \varphi(t) & \text{dac\‘a} \ t \in [t_0 - h, t_0 + h] \\
  x_k(t) & \text{dac\‘a} \ t \in [t_0 + kh, t_0 + (k + 1)h], 1 \leq k \leq n_b - 1 \\
  x_{n_b}(t) & \text{dac\‘a} \ t \in [t_0 + n_b h, b]
\end{cases}
\]

Let \( n = \max \{n_a, n_b\} \).

Now we prove the necessity of the condition (5). Let \( x \in C[a - h, b + h] \cap C^1[a, b] \) a solution of the problem (1)+(2).

Then \( x \in C^n[a - h, b + h] \cap C^{n+1}[a, b] \) is a solution.

We have:

\[
x^{(k+1)}(t) = [f(t, x(t), x(t - h), x(t + h))]^{(k)}_{t = t_0}, \quad t \in [a, b], \ k \in \{0, 1, \ldots, n\}
\]

For \( t = t_0 \), we have (5).
REFERENCES

