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# WHEELER-FEYNMAN PROBLEM ON A COMPACT INTERVAL

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**Abstract.** In this paper the problem (1)+(2) is studied.
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## 1. Introduction

In the paper [1] and [3] the autor study the Weeler-Feynman problem on $R$. In this paper we consider the following Weeler-Feynman problem:

(1) $$x'(t) = f(t, x(t), x(t-h), x(t+h)), \quad t \in [a, b],$$

(2) $$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h],$$

where $t_0 \in [a, b], \quad a \leq t_0 - h, t_0 + h \leq b$ and $\varphi \in C^1[t_0 - h, t_0 + h]$

## 2. Remarks and examples

**2.1.** By a solution of (1) we understand a function $x \in C[a-h, b+h] \cap C^1[a, b]$ which satisfies the relation (1) for all $t \in [a, b]$.

**2.2.** Let $\alpha,\ \beta,\ \gamma \in R,\ \beta \neq 0,\ \gamma \neq 0,\ t_0 \in [a, b]$. We consider the following problem:

(3) $$x'(t) = \alpha x(t) + \beta x(t-h) + \gamma x(t+h)), \quad t \in [a, b],$$

(4) $$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h],$$

where $t_0 \in [a, b],\ a \leq [t_0 - h, t_0 + h] \leq b$.

We shall apply the method of steps on intervals $[t_0, b]$ and $[a, t_0]$ to find some "if and only" conditions for the existence of a solution of problem (3)+(4).

Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = \alpha\varphi(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_1(t) = \frac{1}{\gamma}[\alpha\varphi(t-h) + \beta\varphi(t-2h) - \varphi'(t-h)],\ t \in [t_0+h, t_0+2h]$$

Let $t \in [t_0+h, t_0+2h]$

$$x_1'(t) = \alpha x_1(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_2(t) = \frac{1}{\gamma}[\alpha x_1(t-h) + \beta\varphi(t-2h) - x_1'(t-h)],\ t \in [t_0+2h, t_0+3h]$$

By the same way the final step on $[t_0, b]$:

$$x_{n_b}(t) = \frac{1}{\gamma}[\alpha x_{n_b-1}(t-h) + \beta x_{n_b-2}(t-2h) - x'_{n_b-1}(t-h)],\ t \in [t_0+n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.
By the same way on $[a, t_0]$ we find $n_a = [\frac{t_0-a}{h}]$.
Let $n := max\{n_a, n_b\}$.

Let $\varphi \in C^{n+1}[t_0-h, t_0+h]$.

Let $x \in C^n[a-h, b+h] \cap C^{n+1}[a,b]$ be a solution of problem (3)+(4).
We have:

$$x^{(k+1)}(t) = \alpha x^{(k)}(t) + \beta x^{(k)}(t-h) + \gamma x^{(k)}(t+h),\ k \in 0,1,\ldots,n$$

For $t = t_0$, we have:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0,1,\ldots,n\}$$

Then the problem (3)+(4) has a solution if and only if:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0,1,\ldots,n\}.$$

**2.3.** For the case in which $\beta = 0$ or $\gamma = 0$ see [2].

## 3. The main result

In what follow we consider the problem (1)+(2). We need the following conditions.
Let $n_a := [\frac{t_0-a}{h}],\ n_b := [\frac{b-t_0}{h}],\ n := max\{n_a, n_b\}$.

Let $f \in C^{n+1}([a,b] \times R^3)$.

(C1):For all $u_1 \in [a,b]$, $u_2, u_4, u_5 \in R$, there exist a unique $u_3 \in R$, $u_3 = f_1(u_1, u_2, u_4, u_5)$, $f_1 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.
(C2):For all $u_1 \in [a,b]$, $u_2, u_3, u_5 \in R$, there exist a unique $u_4 \in R$, $u_4 = f_2(u_1, u_2, u_3, u_5)$, $f_2 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.

We have

**Theorem 1.** *Let $f \in C^{n+1}([a,b] \times R^3)$ satisfies (C1) and (C2). If $\varphi \in C^{n+1}[t_0 - h, t_0 + h]$, then the problem (1)+(2) has a unique solution if and only if $\varphi$ satisfies the following condition:*

$$\varphi^{(k+1)}(t_0) = [f(t, \varphi(t), \varphi(t-h), \varphi(t+h)]^{(k)}_{t=t_0}, \ , \ k \in \{0, 1, \dots, n\} \tag{5}$$

**Proof.** By the method of steps we construct the solution of (1) +(2) as follows. Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = f(t, \varphi(t), \varphi(t-h), x(t+h))$$

From (C2) we have

$$x(t) := x_1(t) = f_2(t-h, \varphi(t-h), \varphi(t-2h), \varphi'(t-h)), \ t \in [t_0 + h, t_0 + 2h] \ .$$

By the same method we find the final step:

$$x_{n_b}(t) = f(t-h, x_{n_b-1}(t-h), x_{n_b-2}(t-2h), x\prime_{n_b-1}(t-h)), \ t \in [t_0 + n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.

We must have:

$$\varphi(t_0 + h) = x_1(t_0 + h)$$

$$x_p(t_0 + (p+1)h) = x_{p+1}(t_0 + (p+1)h), \quad p \le n_b - 1$$

By the same way we have the solution on $[a, t_0]$ with the condition

$$\varphi(t_0 - h) = x_{-1}(t_0 - h)$$

$$x_{-p}(t_0 - (p+1)h) = x_{-(p+1)}(t_0 - (p+1)h), \quad p \le n_a - 1$$

where $n_a = [\frac{t_0 - a}{h}]$

So the solution is:

$$x(t) = \begin{cases} x_{-n_a}(t) & \text{dacă} \quad t \in [a, t_0 - n_a h] \\ x_{-k}(t) & \text{dacă} \quad t \in [t_0 - (k+1)h, t_0 - kh], 1 \le k \le n_a - 1 \\ \varphi(t) & \text{dacă} \quad t \in [t_0 - h, t_0 + h] \\ x_k(t) & \text{dacă} \quad t \in [t_0 + kh, t_0 + (k+1)h], \quad 1 \le k \le n_b - 1 \\ x_{n_b}(t) & \text{dacă} \quad t \in [t_0 + n_b h, b] \end{cases}$$

Let $n = max\{n_a, n_b\}$.

Now we prove the necesity of the condition (5). Let $x \in C[a-h, b+h] \cap C^1[a,b]$ a solution of the problem (1)+(2)

Then $x \in C^n[a-h, b+h] \cap C^{n+1}[a,b]$ is a solution.

We have:

$$x^{(k+1)}(t) = [f(t, x(t), x(t-h), x(t+h))]^{(k)}, \ t \in [a,b], \ k \in \{0, 1, \dots, n\}$$

For $t = t_0$, we have (5).

## References

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