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COMPACTNESS THEOREMS FOR QUASI-AUTONOMOUS EVOLUTION PROBLEMS

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Abstract. This note deals with precompactness of trajectories of solutions of quasi-autonomous problems in Banach spaces in the compact resolvent case. We carry out new criterions which extend or supplement for instance fundamental results of Pazy, Dafermos-Slemrod or Webb.
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1. Introduction

Let A be an unbounded multivalued dissipative nonlinear operators in the Banach space X with compact resolvent and let $x^0 \in X$. We carry out general conditions leading to precompactness results for bounded solutions of the quasi-autonomous problem

$$QP\left(x^0\right) = \begin{cases} \dfrac{dx}{dt}(t) & \in Ax(t) + f(t), \quad t \in [0,+\infty[\\ x(0) & = x^0, \end{cases} \tag{1}$$

with $f \in \mathbb{L}^1_{\mathrm{loc}}([0,+\infty[\,,X)$. The concept of solution of $QP\left(x^0\right)$ will be always that of **mild solution** (see [5]).

The trajectory of a continuous function x on $[0,+\infty[$, denoted by traj(x) will be the set $x([0,+\infty[)$.

Definition 1.1. *We will say that $QP\left(x^0\right)$ has the **precompactness property** if either $QP\left(x^0\right)$ has no bounded solution or $QP\left(x^0\right)$ has a bounded solution with precompact trajectory in X.*

We will say also that $f \in \mathbb{L}^1_{\mathrm{loc}}([0,+\infty[\,,X)$, is **uniformly locally integrable** if we have,

$$\lim_{h\downarrow 0}\sup_{t\geq 0}\int_t^{t+h}\|f(\tau)\|\,d\tau = 0.$$

Let us recall now some fundamental precompactness results for the quasi-autonomous problem $QP\left(x^0\right)$. The notation $D(A)$ stands for the domain of A.

(R1) (**Dafermos-Slemrod,** see [6]) If A is m-dissipative with compact resolvent, and if $f \in \mathbb{L}^1([0,+\infty[\,,X)$, then for all $x^0 \in \overline{D(A)}$, the problem $QP(x^0)$ has the precompactness property.

(R2) (**Pazy in the linear case**, see [7] or [1]) If A is m-dissipative and generates a compact semigroup on $\overline{D(A)}$ and if f is uniformly locally integrable on $[0,+\infty[$, then for all $x^0 \in \overline{D(A)}$, the problem $QP(x^0)$ has the precompactness property.

(R3) (**Webb**, see [8]) If f takes its values in a compact subset of a Hilbert space X and if A is linear and generates an exponentially stable C^0-semigroup on X, then for all x^0, the problem $QP(x^0)$ has the precompactness property.

Let us notice that the result (R2) follows easily from the following relation for $h > 0$,

$$\sup_{t \geq h} \left\| x(t) - e^{hA} x(t-h) \right\| = o_h(1) ,$$

which is immediately deduced from Bénilan's integral inequalities (see [2]), and thus (R2) holds in the nonlinear case. Notice also that (R3) is true in an arbitrary Banach space.

In applications, almost all the precompactness results of trajectories come from one of these three general results or are deduced from the following situation: traj(x) is bounded in a Banach space $(Y, \|.\|_Y)$, and there is a compact injection from Y to X. Of course, it happens also that direct estimates allow to show that the solution x has a limit when t goes to infinity and then ensures the precompactness of traj(x). But in numerous applications the precompactness problem is a sharp question or remains an open problem. So we propose here new results which extend and supplement the fundamental results which one has just quoted.

This note is organized as follows. A preliminary result is given in Section 2. The kind of approximations using in our approach is defined in Section 3 and the statements of the results end this document in Section 4.

2. Preliminary result

Proposition 2.1. *Suppose $A - \omega I$ m-dissipative in X and f locally uniformly integrable on $[0,+\infty[$. Denote by x the solution of $QP(x^0)$ with $x^0 \in \overline{D(A)}$. If traj(x) is precompact, then x is uniformly continuous on $[0,+\infty[$.*

Proof. By contradiction, we can suppose that there exist $\varepsilon > 0$ and a sequence $(t_n, h_n)_n$ with $t_n \uparrow +\infty$, $h_n \downarrow 0$, satisfying for all $n \in N$,

$$\|x(t_n + h_n) - x(t_n)\| \geq \varepsilon . \tag{2}$$

We have

$$\|x(t_n + h_n) - x(t_n)\| \leq \left\| e^{t_n A} x(t_n) - x(t_n) \right\| + \int_{t_n}^{t_n + h_n} e^{\omega\tau} \|f(\tau)\| \, d\tau$$

Since the trajectory is assumed to be precompact, the sequence $(x(t_n))_n$ converges (taking eventually a subsequence) towards some $l \in X$. Then Inequality (2) implies

$$\varepsilon \leq \left\| e^{t_n A} x(t_n) - e^{t_n A} l \right\| + \left\| e^{t_n A} l - l \right\| \\ + \|l - x(t_n)\| + \int_{t_n}^{t_n + h_n} e^{\omega\tau} \|f(\tau)\| \, d\tau.$$

Then applying assumption on f and taking the limit when n goes to infinity we obtain the following contradiction $0 < \varepsilon \leq 0$.■

We will carry out further (see Theorem 4.2) a partial converse of Proposition 2.1 in the quasi-autonomous case.

3. Special approximations

The approximations introduced in this Section can be defined in a more general framework but we will restrict here to the quasi-autonomous problem.

3.1. **Spaces $\mathcal{V}_\alpha(E)$ and $\mathbb{L}_\alpha(E)$.** We recall the notation given in the introduction, but more generally with $g \in \mathbb{L}^1_{\text{loc}}([0,+\infty[\,,E)$, where $E = X$ or $E = \mathbb{R}$, and $\alpha \geq 0$,

$$N_{\alpha,E}(g) = \lim \sup_{t\to+\infty} e^{-\alpha t} \int_0^t e^{\alpha\tau} \|g(\tau)\|_E \, d\tau.$$

Let $\alpha \geq 0$. We will denote by $\mathcal{V}_\alpha(E)$ the linear subspace of $\mathbb{L}^1_{\text{loc}}([0,+\infty[\,,E)$ consisted of bounded $g \in \mathbb{L}^1_{\text{loc}}([0,+\infty[\,,E)$ satisfying

$$\sup_{t\geq 0} \lim \sup_{h\downarrow 0} e^{-\alpha t} \int_0^t e^{\alpha\tau} \frac{\|g(\tau+h) - g(\tau)\|}{h} \, d\tau < +\infty.$$

The space $\mathcal{V}_0(E)$ is the space of functions of bounded variations on $[0,+\infty[$. For $\alpha > 0$, $\mathcal{V}_\alpha(E)$ contains the set of bounded Lipschitz maps on $[0,+\infty[$.

Definition 3.1. *The closure under the semi-norm $N_{\alpha,E}$ in $\mathbb{L}^1_{\text{loc}}([0,+\infty[\,,E)$ of the space $\mathcal{V}_\alpha(E)$ is denoted by $\mathbb{L}_\alpha(E)$.*

In other words, $f \in \mathbb{L}_\alpha(E)$, means that there exists a sequence $(g_n)_n$ of elements in $\mathcal{V}_\alpha(E)$ such that $\lim_n N_{\alpha,E}(f - g_n) = 0$.

Remark 3.2. *Notice that we have*

$$\mathbb{L}_\alpha(E) \subseteq \mathbb{L}_\beta(E)$$

for $\alpha \leq \beta$ and

$$\mathbb{L}_0(E) = \mathcal{V}_0(E) + \mathbb{L}^1([0,+\infty[\,,E).$$

For $\alpha > 0$, the vector space $\mathbb{L}_\alpha(E)$ contains for instance, each bounded uniformly continuous map on $[0,+\infty[$, each p-integrable function on $[0,+\infty[$, with $1 \leq p < +\infty$, any locally integrable periodic function, any bounded step function associated to some subdivision $\Lambda = (t_0, t_{1,\ldots})$ of $[0,+\infty[$ satisfying $\inf_{i\in\mathbb{N}} \delta_i > 0$, where $\delta_i = t_{i+1} - t_i$ and any locally integrable f such that there is a null subset $\Omega \subset [0,+\infty[$ satisfying

$$\lim_{\substack{t\to+\infty \\ t\notin\Omega}} f(t) = 0. \tag{3}$$

3.2. $\overline{BV}$**-approximations.** The two following definitions are fundamental in the sequel.

Definition 3.3. *Let $\alpha \geq 0, .\omega \geq 0$. We say that $BV(\omega, \alpha)$ holds for (A, f, x^0), if $A + \omega I$ is m-dissipative with compact resolvent, $x^0 \in D(A)$, and if we have*

$$\limsup_{h \downarrow 0} \frac{1}{h} \int_0^h \|f(\tau)\| \, d\tau < +\infty, \text{ and,} \tag{4}$$

$$V_{f,\alpha} = \sup_{t \geq 0} \limsup_{h \downarrow 0} e^{-\alpha t} \int_0^t e^{\alpha \tau} \frac{\|f(\tau + h) - f(\tau)\|}{h} \, d\tau < +\infty. \tag{5}$$

Of course the above property BV is too particular to be applicable directly in practice. The following definition proposes a suitable concept of approximation of (A, f, x^0).

Definition 3.4. *Let $\alpha, \beta \geq 0$. We say that $\overline{BV}(\omega, \alpha, \beta)$ holds for (A, f, x^0), if $A + \omega I$ is m-dissipative with compact resolvent, and if there is a sequence $(A_n, f_n, x_n^0)_n$ such that:*

(a) for each n, (A_n, f_n, x_n^0) satisfies at once $BV(\omega, \alpha)$, $f_n \in \mathbb{L}^\infty([0, +\infty[, E)$ and $\lim_n x_n^0 = x^0$;

(b) there exists $\eta_n \in \mathbb{L}^1_{loc}([0, +\infty[)$ satisfying,

$$\begin{array}{lll} -[x - y, -\xi_x(s) + \xi_y^n(t)] & \leq \|f(s) - f(t)\| + \eta_n(s) + \eta_n(t) - \omega \|x - y\| \\ \text{and} \quad \lim_n N_{\beta+\omega, \mathbb{R}}(\eta_n) & = 0. \end{array} \tag{6}$$

for all $(x, \xi_x(s)) \in Ax + f(s)$, $(y, \xi_y^n(t)) \in A_n y + f_n(t)$, and almost all $s, t \in [0, +\infty[$.

*We will say that the sequence $(A_n, f_n, x_n^0)_n$ is a (ω, α, β)-**approximation of** (A, f, x^0) or a $\overline{BV}$-**approximation** (when the parameters (ω, α, β) are unspecified).*

Example 3.5. *Relation (5) means $f \in \mathcal{V}_\alpha(X)$ when f is bounded. It holds with $\alpha = 0$, for instance if f is of bounded variations on $[0, +\infty[$, as well as, with $\alpha > 0$ if f is Lipschitz on $[0, +\infty[$, or if f is a bounded step function on $[0, +\infty[$, associated to some subdivision $\Lambda = (t_0, t_{1,\dots})$ of $[0, +\infty[$ satisfying $\inf_{i \in \mathbb{N}} \delta_i > 0$, where $\delta_i = t_{i+1} - t_i$. Thus the condition $BV(\theta, 0, 0)$ holds for the family $A(t)x = Ax + f(t)$ pointed by $x^0 \in D(A)$, if A is m-dissipative densely defined with compact resolvents and f of bounded variations on $[0, +\infty[$. With the same assumptions on A, the family $A(t)x = Ax + f_1(t) + f_2(t)$ pointed by any $x^0 \in X$, with $f_1 \in \mathbb{L}^1([0, +\infty[, X)$ and f_2 of bounded variations on $[0, +\infty[$, satisfies $\overline{BV}(\theta, 0, 0, 0)$. Indeed (6) holds with $\eta_n = \|f_1 - g_n\|$ and $A_n(t) = A + g_n(t) + f_2(t)$, where $(g_n)_n$ is a sequence of Lipschitz-continuous functions with compact supports converging to f_1 in $\mathbb{L}^1([0, +\infty[, X)$.*

4. Results

The following Theorems 4.1, 4.2 are deduced from nonlinear methods in the Semigroup Theory (see[[2], [4], [5]]).

Theorem 4.1. *Suppose that* $\overline{BV}(\theta,\omega,\omega,0)$ *holds for* (A,f,x^0) *. Then* $QP(A,f,x^0)$ *has the precompactness property.*

Theorem 4.2. *Let* $\alpha>0,$ *and suppose that* $\overline{BV}(\theta,\omega,\alpha,\alpha)$ *holds for* (A,f,x^0) *. Then,* $QP(A,f,x^0)$ *has the precompactness property if its solution is uniformly continuous on* $[0,+\infty[$.

Theorem 4.2 can be viewed as a special form for evolution problems of the Ascoli-Arzelà-Fréchet-Kolmogorov' s Theorem and gives a special converse of the preliminary result stated in paragraph 2.We will see that these Theorems have various applications, improve the Ball-Slemrod's Theorem and supplement the Webb's Theorem ((R3) recalled in Introduction) and the Pazy result (R2).

For the sequel we make the following assumption $(HA1)$.

Assumptions $(HA1)$**:** $A+\omega I$ $(\omega\geq 0)$ is a densely defined m-dissipative operator on X with compact resolvent and we consider the quasi-autonomous problem $QP(x^0)=QP(A,x^0,f)$, for some $x^0\in X$.

Corollary 4.3. *Let* $f=f_1+f_2$, *with* f_1 *of bounded variations on* $[0,+\infty[$, *and* $f_2\in\mathbb{L}^1([0,+\infty[,X)$ *then* $QP(x^0)$ *has the precompactness property.*

Corollary 4.4. *Let* $\alpha>0,$ *and* $f\in\mathbb{L}_\alpha(X)$ *be assumed uniformly locally integrable. Then the precompactness property holds for* $QP(x^0)$, *if and only if the solution* x *of* $QP(x^0)$ *is uniformly continuous on* $[0,+\infty[$ *whenever it is bounded.*

Corollary 4.5. *In* $(HA1)$ *suppose* $\omega>0$ *and* $f\in\mathbb{L}_\omega(X)$. *Then the precompactness property holds for* $QP(x^0)$.

Corollary 4.6. *Suppose* $\omega>0$ *in* $(HA1)$. *Assume* $f=f_1+f_2$, *where* f_1 *takes its values in a compact subset of* X, *and where* $f_2\in\mathbb{L}_\omega(X)$. *Assume* A *to be linear. Then the precompactness property holds for* $QP(x^0)$.

Corollary 4.7. *Suppose that* X *is a Hilbert space and that we have* $\omega>0$ *in* $(HA1)$. *Let* $f\in\mathbb{L}_\alpha(X)$ *for some* $\alpha\geq 0$ *and suppose that* f *takes its values (essentially) in a compact subset of* X. *Suppose that the solution* x *of* $QP(x^0)$ *is weakly uniformly continuous on* $[0,+\infty[$. *Then* $QP(x^0)$ *has the precompactness property.*

The previous corollary concerns in particular operators $A=A_1+A_2$, where A_1 is a linear m-dissipative densely defined operator and A_2 is a continuous nonlinear operator bounded on bounded subsets of X. Indeed, in this case (X being reflexive) when A generates a C^0 semigroup, then for all initial values x^0 and all locally integrable functions f, the solution of $QP(A,x^0,f)$ is weakly uniformly continuous on $[0,+\infty[$. Let us notice also that corollaries 4.5, 4.6 and 4.7 supplement the Webb's Theorem even in the linear case.

Next we will give applications of this new approach to some second order equations generating semigroups with compact resolvent. We will answer open problems about Control (in particular Stabilization problems) of wave equations, beam equations...

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