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NEUTRAL FUNCTIONAL EQUATIONS IN DISCRETE TIME

C. CORDUNEANU*, MEHRAN MAHDAVI**

* The University of Texas at Arlington

** Bowie State University (Bowie, MD)

Abstract. The paper is dedicated to the investigation of some discrete–time functional equations of neutral type, such as (1). Existence results in various sequence spaces are obtained. Analogy with continuous–time neutral equations is pursued.
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1. Introduction

In a recent paper, [8], the first author investigated some global problems for neutral functional equations in discrete–time, of the form

$$\Delta f(n, x_n) = g(n, x_n), \tag{1}$$

where $\Delta u_n = u_{n+1} - u_n$, and f, g stand for maps from $I \times R^m$ into R^m, $I \subset \mathcal{Z} =$ the ring of integers. Some existence results have been obtained in [8] for (1), in the spaces $c_\ell(N, R^m)$, $N = \mathcal{Z}_+$, and $AP(\mathcal{Z}, R^m)$. The first is the space of convergent sequences in R^m, with the supremum norm, while the second is the space of almost periodic sequences (see, for instance, [3]), with values in R^m.

The aim of this paper is to obtain similar results for equation (1), or related equations, oftenly called *difference equations* of neutral type. We shall pursue here the analogy with the continuous–time case, discussed in our joint papers [11], [12] or in other sources. See also [14], where some generalizations of the results in [11], [12] are considered. A survey of some recent contributions to the theory of discrete–time functional equations, not necessarily of neutral type, can be found in [9], in which results concerning existence, stability, oscillations and other behavior are presented.

The simplest case of equation (1) is obtained when $f(n, x_n) = x_n = x(n)$, which means

$$\Delta x(n) = g(n, x(n)), \tag{2}$$

or

$$x(n+1) = \bar{g}(n, x(n)), \tag{3}$$

with $\bar{g}(n, x(n)) = x(n) + g(n, x(n))$.

Of course, (2) or (3) are usual difference equations of recurrent type. These equations have been broadly investigated in the existing literature.

One approach we shall emphasize consists in reducing the neutral equation (1) to the non–neutral type (2). Then, results available in the literature can be transferred to the neutral case.

Besides the spaces $c_\ell(N, R^m)$ and $AP(\mathcal{Z}, R^m)$ mentioned above, we shall also use the spaces $B(N, R^m)$ or $B(\mathcal{Z}, R^m)$, consisting of bounded sequences (as usual, with the supremum norm). Further sequence spaces like $\ell^p(N, R^m)$, $1 \leq p < \infty$, will also be necessary. Let us notice that $\ell^\infty(N, R^m) = B(N, R^m)$.

Instead of the notation used in (1), which is basically the classical notation, one may use an operator notation such as

$$\Delta F(x)(n) = (Gx)(n), \ n \in I \subset \mathcal{Z}, \tag{4}$$

where $x \to Gx$ is given by $(Gx)(n) = g(n, x_n)$, $n \in I =$ an interval in $\mathcal{Z}$, and similarly for the operator $x \to Fx$. One may consider, for instance, such operators as acting on the sequence space $s(N, R^m)$, consisting of all sequences $x = (x_1, x_2, ..., x_k, ...)$, $x_k \in R^m$, with the metric function

$$d(x, y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{1 + |x_k - y_k|}. \tag{5}$$

The space $s(N, R^m)$ contains many subspaces, among them those mentioned above: ℓ^p, $(1 \leq p \leq \infty)$, c_ℓ, $c_0 =$ the space of sequences convergent to zero (obviously, $c_0 \subset c_\ell$). For the case of the space $AP(\mathcal{Z}, R^m)$, one has to consider the larger space $s(\mathcal{Z}, R^m)$, defined in the same manner as $s(N, R^m)$.

2. A quasilinear functional equation

Let us consider the equation similar to (1),

$$(Lx)(n) = (Gx)(n), \ n \in N, \tag{6}$$

where L stands for a linear operator on a Banach sequence space E, taken among the spaces mentioned above. Simply, we can rewrite the equation (6) as

$$Lx = Gx, \ x \in E. \tag{7}$$

Since E is by assumption a Banach sequence space, it is well–known that the necessary and sufficient condition for th einvertibility of L on E can be written as

$$|Lx|_E \geq m|x|_E, \ \forall x \in E, \tag{8}$$

for some $m > 0$. In this case L^{-1} is defined on the whole space E, and it is continuous whenever L is continuous (Banach). Therefore, (7) is equivalent to the equation

$$x = L^{-1}(Gx), \ x \in E, \tag{9}$$

which presents the advantage of being in the appropriate form for the use of fixed point theorems. In this case, the Banach contraction mapping theorem can be directly applied to (9), if we assume G to be Lipschitz continuous on E:

$$|Gx - Gy|_E \leq K|x - y|_E, \tag{10}$$

for any $x, y \in E$. Since from (8) one has $|L^{-1}| \le m^{-1}$, there results that $L^{-1}G$ is also Lipschitz on E, with constant Km^{-1}. Hence, equation (9), or equivalently (7), has a unique fixed point in E, provided

$$K < m^{-1}. \tag{11}$$

Therefore, under condition (9), (10), (11), the equation (7) is uniquely solvable in any space E listed above. We point out the fact that $E = s$ is not acceptable, since s is a Fréchet space (a linear complete metric space, with translation invariant metric).

By particularizing the space E and the operators L and G, one can obtain without difficulty existence results for discrete–time functional equations. Such results can be found, for instance, in [2], [16] (just to mention some early results).

Let us consider now the system

$$x_k = \sum_{j=1}^{\infty} a_{kj} x_j + f_k(x), \ k \ge 1, \tag{12}$$

on the sequence space $\ell^2(N, R^m)$.

In other words, we assume $x = (x_1, x_2, ..., x_k, ...) \in \ell^2$, and in order to deal with (12) in ℓ^2, we shall also assume

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |a_{kj}|^2 \le \alpha^2 < \infty. \tag{13}$$

Condition (13) guarantees the boundedness on ℓ^2 of the linear operator $x \to Lx$, where

$$(Lx)_k = \sum_{j=1}^{\infty} a_{kj} x_j, \ k \ge 1. \tag{14}$$

Since (12) can be rewritten as

$$x = Lx + fx, \tag{15}$$

with $fx = (f_1, f_2, ..., f_k, ...)$, one must also assume that $fx \in \ell^2$ for each $x \in \ell^2$, i.e.,

$$\sum_{j=1}^{\infty} |f_j(x)|^2 < \infty, \tag{16}$$

for each $x \in \ell^2$. We actually need a Lipschitz continuity condition for f, which follows from the assumption

$$|f_k(x) - f_k(y)| \le \beta_k |x - y|_2, \ k \ge 1, \tag{17}$$

with

$$\sum_{k=1}^{\infty} \beta_k^2 = \beta^2 < \infty. \tag{18}$$

Namely, from (17) and (18) on eobtains

$$|fx - fy|_2 \le \beta |x - y|_2. \tag{19}$$

For the system (12), all conditions required for existence and uniqueness of solution to (7) are verified. Indeed, (15) can be rewritten as

$$(I - L)x = fx \tag{20}$$

and $\alpha < 1$ assures the existence of $(1 - L)^{-1}$. Moreover, condition (11) becomes in our case $\beta < 1 - \alpha$. Hence, taking into account the fact that α and β are positive numbers, it suffices to assume

$$\alpha + \beta < 1 \tag{21}$$

which also implies $\alpha < 1$.

Summarizing the above discussion in regard to the system (12), we can state the following result:

Proposition 1. *Consider the discrete–time functional system* (12), *under the following assumptions:*

1) a_{kj} *satisfy* (13);
2) $f_k(x)$ *satisfy* (15), (17) *and* (18);
3) *Inequality* (21) *holds.*

Then the system (12) *has a unique solution in* $\ell^2(N, R^m)$.

Remark 1. Since $\ell^2(N, R^m) \subset c_0(N, R^m)$, there results for the solution $x = (x_1, x_2, ..., x_k, ...)$ of (12)

$$\lim |x_k| = 0 \text{ as } k \to \infty, \tag{22}$$

which means that, under our assumptions, only a finite number of coordinats really matters for practical purposes.

Remark 2. With minor changes in our hypotheses, the existence and uniqueness of solution to (12) can be obtained in any space $\ell^p(N, R^m)$, $1 \le p < \infty$.

Remark 3. Condition (16) will be satisfied if there exists only one element $\bar{x} \in \ell^2(N, R^m)$, such that (16) takes place for $x = \bar{x}$. On behalf of (17) and (18), and the elementary inequality

$$|f_j(x)|^2 \le 2(|f_j(\bar{x})|^2 + \beta_j^2 |x - y|_2^2),$$

one derives the condition (16) for any $x \in \ell^2$.

Remark 4. Condition (21) represents itself a condition of contractibility of the operator $L + f$ in the right hand of (15). Hence, the iterative process defined by

$$x^{(p+1)} = Lx^{(p)} + fx^{(p)}, \ p \ge 0, \tag{23}$$

with $x^{(0)} \in \ell^2$ arbitrary, is convergent in ℓ^2.

On the other hand, if we take into account the way this condition hs been obtained (discussion preceding Proposition 1), there results the convergence of the iterative process defined by

$$x^{(p+1)} = Lx^{(p+1)} + fx^{(p)},\ p \geq 0. \tag{24}$$

This alternate process may play a role in certain cases.

3. Existence in the space $B(N, R^m)$

We shall return to the equation (1), in order to provide conditions guaranteeing the existence of solution in the space $B(N, R^m)$. Let us notice that (1) can be rewritten as

$$f(n+1, x_{n+1}) = f(n, x_n) + g(n, x_n),\ n \geq 0. \tag{25}$$

Starting with an arbitrary $x_0 \in R^m$, the sequence $\{x_n\}$, $n \geq 1$, will be uniquely determined if the equation/system $f(n, u) = c \in R^m$ has a unique solution, say u_n, for every $n \geq 1$. This property can be secured in many ways. If we accept this as a hypothesis, then (1) has a unique solution in $s(N, R^m))$ for every initial data $x_0 \in R^m$. The real problem is to secure the existence in $B(N, R^m)$, or in another sequence space (which will be part of $s(N, R^m)$, anyway).

We shall formulate now another condition of $f(n, u)$, namely

$$|f(n, u)| \geq h(|u|),\ n \geq 1, \tag{26}$$

where $h(r)$, $0 \leq r < \infty$ is monotonically increasing, while $h^{-1}(r)$ has sublinear growth:

$$h^{-1}(r) \leq \alpha r + \beta,\ r \geq 0, \tag{27}$$

with $\alpha, \beta > 0$. This condition generalizes condition (15) in the paper [8], and shall be used in conjunction with other conditions, such as

$$|g(n, u)| \leq c_n |u|,\ n = 0, 1, ..., \tag{28}$$

where the series

$$\sum_{n=0}^{\infty} c_n \tag{29}$$

is convergent.

From (25) and (26) we derive

$$h(|x_{n+1}|) \leq \alpha(|f(0, x_0)| + \sum_{k=0}^{n} |g(k, x_k)|) + \beta,$$

which, on behalf of (27), leads to

$$|x_{n+1}| \leq \alpha |f(0, x_0)| + \beta + \alpha \sum_{k=0}^{n} |g(k, x_k)|. \tag{30}$$

Relying on (29), (30) yields for $n > 0$

$$|x_{n+1}| \le \alpha|f(0,x_0)| + \beta + \alpha\sum_{k=0}^{n} c_k|x_k|, \tag{31}$$

which constitutes a Gronwall's type inequality [13].

The inequality (31) implies

$$|x_{n+1}| \le (\alpha|f(0,x_0)| + \beta)\exp\left\{\alpha\sum_{k=0}^{n} c_k\right\},\ n \ge 0,$$

which leads to the estimate

$$|x_{n+1}| \le (\alpha|f(0,x_0)| + \beta)\exp\left\{\alpha\sum_{k=0}^{\infty} c_k\right\}, \tag{32}$$

for $n \ge 0$. Hence, the solution of (1), under above hypotheses, is in $B(N, R^m)$.

Let us summarize the discussion regarding equation (1) in the following

Proposition 2. *Consider equation* (1), *and assume the following conditions hold true:*

1) $f : N \times R^m \to R^m$ *is such that the equation*

$$f(n,u) = v \in R^m \tag{33}$$

has unique solution $u_n \in R^m$, *for each* $v \in R^m$;

2) f *satisfies a condition of the form* (26), *with* h *monotone and sublinear* (*see condition* (27));

3) $g : N \times R^m \to R^m$ *verifies* (28), *with* c_k *satisfying* (29).

Then equation (1) *has a unique solution in* $B(N, R^m)$, *for each initial datum* $x_0 \in R^m$.

Remark 1. In [8], instead of assumption 1) we have chosen $f(n,u) = f(u)$ for each $n \in N$, with f a homeomorphism of the space R^m. The condition 2) in Proposition 1 was formulated directly on f.

Remark 2. From Proposition 2 we can derive an existence result in the space $c_\ell(N, R^m)$ of convergent sequences by adding more hypotheses on the function $f(n,u)$. Indeed, from (25) one obtains

$$f(n+p, x_{n+p}) - f(n, x_n) = \sum_{k=n+1}^{n+p} g(k, x_k),\ p \ge 1,$$

and taking into account (28), (29), we see that

$$\lim_{n\to\infty} f(n, x_n) \text{ exists.} \tag{34}$$

Now, let's postulate the existence of the limit

$$\lim_{n\to\infty} f(n,u) = f_\infty(u), \tag{35}$$

uniformly with respect to u, in any bounded set of R^m. Then, from

$$f(n+p, x_n+p) - f(n+p, x_n) = f(n+p, x_{n+p}) - f(n, x_n) + f(n, x_n) - f(n+p, x_n),$$

relying on (34) and (35), we obtain

$$|f(n+p, x_{n+p}) - f(n+p, x_n)| < \varepsilon, \tag{36}$$

provided $n \geq N(\varepsilon),\ p = 1, 2, 3, ...$

The next assumption on f will be the existence of a continuous increasing function $\lambda(r)$, $0 \leq r < \infty$, $\lambda(0) = 0$, such that

$$|f(n, u) - f(n, v)| \geq \lambda(|u - v|),\ n \in N, \tag{37}$$

Then, combining (36) and (37) we obtain

$$|x_{n+p} - x_n| < \lambda^{-1}(\varepsilon),\ n \geq N(\varepsilon),\ p \geq 1. \tag{38}$$

Inequality (38) shows that $\lim x_n$ exists as $n \to \infty$, which means that the solution $x = (x_0, x_1, x_2, ...) \in c_\ell(N, R^m)$.

4. Almost periodicity of bounded solutions

In [8] and [10, Appendix] we have discussed the almost periodicity of bounded solutions (supposed to exist!) of discrete–time functional equations of the form (1), with $f(n, x) = f(x)$, respectively of continuous–time functional equations of the form $(Vx)(t) = (Wx)(t)$, $t \in R$, where V and W are operators on the space $AP(R, R^m)$. The latter is susceptible of adaptation to the equations in discrete–time case.

We only sketch here the approach to almost periodicity of bounded (on $\mathcal{Z}$) solution to the abstract equation

$$(Vx)(n) = (Wx)(n),\ n \in \mathcal{Z}, \tag{39}$$

whee V and W stand for operators acting on the space $AP(\mathcal{Z}, R^m)$. The following hypotheses will be dealt with:

1) $V : AP(\mathcal{Z}, R^m) \to AP(\mathcal{Z}, R^m)$ is strongly monotone, i.e., there exists a positive number γ, such that

$$< (Vx)(n) - (Vy)(n), x(n) - y(n) > \geq \gamma |x(n) - y(n)|^2, \tag{40}$$

for $n \in \mathcal{Z}$, and each $x, y \in AP(\mathcal{Z}, R^m)$;

2) $W : AP(\mathcal{Z}, R^m) \to AP(\mathcal{Z}, R^m)$ is compact, i.e., takes bounded sets into compact ones.

These basic assumptions, together with the existence of a bounded solution, lead to the almost periodicity of such solutions.

Let us point out that compactness in the space $AP(\mathcal{Z}, R^m)$ means uniform boundedness and equi–almost–periodicity.

The operators V and W can be chosen in a variety of ways. The simplest choice for V is $V = I =$ the identity operator on the space $AP(\mathcal{Z}, R^m)$. But $(Vx)(n) = x(n) - x^3(n)$ is another valid choice (at least in the scalar case $m = 1$). The compactness in $AP(\mathcal{Z}, R^m)$ is more difficult to be checked.

Some recent results of Pennequin [15] are concerned with almost periodicity for discrete–time equations of the form

$$A(n, x_n, x_{n+1}, ..., x_{n+p}) = \theta, \; n \in N, \tag{41}$$

where $p \geq 1$ is a fixed integer. Several references are given in [15], where almost periodicity is taken in a broader sense (Besicovitch).

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