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APPLICATIONS OF AN EQUIVALENCE RELATION AT THE DETERMINATION OF SOME RELATIONS BETWEEN CAPACITIES AND OF THEIRS VALUES

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Abstract. In this paper we shall present some application of an equivalence relation defined on R^n. This relation is important because it leads to a simplification of many proofs in which intervene relations between different types of capacities.
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1. Definitions

In what follows we shall work in the space R^n and we shall denote by:
$B(a,r) = \{x \in R^n : d(x,a) < r\}$ - the open ball,
d - the Euclidean distance.

Definition 1.1. *Let φ_1, φ_2 be nonnegative functions defined in a neighborhood of $0 \in R^n$, without the origin. We say that φ_1 and φ_2 are equivalent when $x \to 0$, and we denote by $\varphi_1 \sim \varphi_2$, if there exists two numbers $r > 0$, $Q > 0$ such that:*

$$\frac{1}{Q}\varphi_1(x) \le \varphi_2(x) \le Q\varphi_1(x), (\forall) x : |x| < r \tag{1}$$

An analogous definition can be given when $x \to \infty$; in this case, $\varphi_1 \sim \varphi_2$ means that the inequalities (1) have place in all the space.

Remark 1.1. *The relation "$\sim$" is an equivalence relation.*

Definition 1.2. *A continuous function $h(r)$ defined on $[0, r_0)$, $(r_0 > 0)$, nondecreasing and such that $\lim_{r \to 0} h(r) = 0$ is called a measure function.*

Definition 1.3. *Let $E \subset R^n$ be a bounded set, $\delta > 0$ and h a measure function. The Hausdorff h - measure of E, denoted by $H_h(E)$, is the number*

$$H_h(E) = \lim_{\delta \to 0} \inf \sum_i h(\rho_i)$$

inf being considered over all coverings of E with a countable number of spheres of radii $\rho_i \le \delta$.

Remark 1.2. *The Hausdorff h - measure is a capacity.*

Definition 1.4. *Let consider the function* $f : D(\subset R^n) \to \overline{R}$. f *is called a* $\delta-$*class Lipschitz function if:*

$$|f(x+\alpha) - f(x)| \leq M\,|\alpha|^{\delta}, x \in D, \alpha \in R^n, x+\alpha \in D, M > 0 \tag{2}$$

Definition 1.5. *Let consider* $E \subset R^n$ *and h - a measure function. E has a positive inferior h -density in a point* $a \in E$ *, denoted by* $\underline{D_h(a)}$,*if*

$$\varliminf_{r\to 0} \frac{H_h(E \frown B(a,r))}{h(2r)} > 0$$

2. Lemmas

Lemma 2.1. *If h = h(r) is a measure function which satisfies:*

$$h(2r) \leq Qh(r), 0 \leq r \leq 1/2,\, Q > 0 \tag{3}$$

and $E \subset R^n$ *is a Cantor set, then:*

$$\frac{1}{Q} \varliminf_{j\to\infty} 2^{nj} h(l_j) \leq H_h(E) \leq Q \varliminf_{j\to\infty} 2^{nj} h(l_j), \tag{4}$$

where Q is a constant which doesn't depend on E.

For details, see $[AM]$.

Lemma 2.2. *The Borelian set* $E \subset R^n$ *has a positive inferior h -density in every point* $x \in E$ *if:*

i. $0 < H_h(E) < \infty$,

ii. $h(2r) \leq Qh(r), (\forall) 0 < r < r_0$ *(*r_0 *small enough),* $Q \in [1, 2^n]$,

iii. E is a Cantor set: $E = \bigfrown_{k=1}^{\infty} E_k$, *where* E_k *contains* 2^{nk} *n-dimensional intervals with the lengths* l_k *and*

$$2l_{k+1} < l_k,\ c_1 < 2^{nk} h(l_k) < c_2,\, c_1,\, c_2 \in R. \tag{5}$$

For details, see $[W]$.

In what follows, we denote by $B_{\alpha,p}(E)$ the Bessel capacity of a set $E \subset R^n$.

Lemma 2.3. *If h is a measure function such that:*

$$\int_0^R \left[r^{\alpha-np} h(r)\right]^{\frac{1}{p-1}} \frac{dr}{r} = +\infty,$$

E is a Borel set, which has a positive inferior h - density in every point $x \in E$ *and* $0 < H_h(E) < +\infty$, *then* $B_{\alpha,p}(E)$.

3. Results

We could divide the following results in two parts:
i. theorems related to the values of the Hausdorff h -measure of some sets;
ii. theorems concerning the relations between different types of capacities.

The first class contains the following three theorems and the second class, the last two.

Theorem 3.1. *Let consider a set of contractions* $\{\psi_j\}_{j=1,\dots,m}$ *on* R^n, *with the contraction ratios* $r_j < 1$ *and* s *the number determined by:* $\sum_{j=1}^{m} r_j^s = 1$. *If* E *is the residual set of the Apollonian packing and* h *a measure function such that:*

$$h(t) \sim t^s, \tag{6}$$

then: $0 < H_h(E) < +\infty$.

Theorem 3.2. *If* $s \geq \frac{\log 3}{\log(1+2\cdot 3^{-1/2})}$, *$E$ is the residual set of the Apollonian packing and h is any measure function that satisfies (6), then there exist* $Q > 0$ *such that:* $H_h(E) < (2 \cdot 3^{-1/2})^s Q$.

For details, see $[B3]$.

Theorem 3.3. *Let consider:* $f : [0,1] \longrightarrow \overline{R}$ *- a* $\delta-$*class Lipschitz function,* Γ, *its graph and h - a measure function which satisfies (6).*
If $(\delta \in [0,1]$ *and* $p \geq 2)$ *or* $(\delta > 1$ *and* $p \geq 1)$ *then:* $H_h(\Gamma) < +\infty$.

For details, see $[B4]$.

Theorem 3.4. *If* $n, p \in N^*, p \neq 1, 0 < \alpha < +\infty, 0 < w \leq n, \alpha p \leq n$, *then:*
1. There exist a compact set $E \subset R^n$ *and*
i. a measure function h, such that:
$H_h(E) > 0 \Rightarrow B_{\alpha,p}(E) > 0$, *if* $\alpha p > w$;
ii. a measure function h, such that:
$H_h(E) < +\infty \Rightarrow B_{\alpha,p}(E) = 0$, *if* $\alpha p \leq w$.
2. There exist a compact set $E \subset R^n$ *and*
i. a measure function h, such that:
$H_h(E) > 0 \Rightarrow B_{\alpha,p}(E) > 0$, *if* $\alpha p \geq w$;
ii. a measure function h, such that:
$H_h(E) < +\infty \Rightarrow B_{\alpha,p}(E) = 0$, *if* $\alpha p < w$.

For details, see $[AM]$, $[B1]$, $[B2]$.

In the paper $[C]$, was denoted:

$$\log_m \frac{1}{r_m} = \underbrace{\log \circ \log \circ ... \circ \log}_{m\, times} \frac{1}{r_m}$$

and introduced the function:

$$h_{\alpha p, p-1, m, \beta}(r) = r^{n-\alpha p} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta}, \tag{7}$$

$m, p \in N^*, \alpha p \le n, 0 < \beta \le p-1, 0 < r < r_m, \log_m \frac{1}{r_m} > 1.$

Theorem 3.5. *There is a compact set* $E \subset R^n$*, which satisfies the following property: if* $0 < H_{h_{\alpha p,p-1,m,\beta}}(E) < +\infty, m, p \in N^*, \alpha p \le n,\ 0 < \beta \le p-1,$ *then* $B_{\alpha,p}(E) = 0$.

Proof. We consider E a Cantor set which satisfies the hypothesis of lemma 2 and h, the function introduced in (7). The interval lengths are chosen to satisfy:

(8) $$c_1 < 2^{nk} h_{\alpha p,p-1,m,\beta}(l_k) < c_2,\ c_1,\ c_2 > 0.$$

First,we prove that $h_{\alpha p,p-1,m,\beta}$ satisfies the hypothesis of the lemma 1, that is, there exist $Q > 0$, such that:

(9) $$h_{\alpha p,p-1,m,\beta}(2r) \le Q h_{\alpha p,p-1,m,\beta}(r), 0 < r \le 1/2.$$

(9) is equivalent with:

(10)
$$(2r)^{\alpha p-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{2r}\right)^{1-p} \left(\log_m \frac{1}{2r}\right)^{-\beta} \le Q r^{\alpha p-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta}$$

We look for $Q \in [1, 2^n]$ of the form:

(11) $$Q = Q_m^{\beta} \prod_{k=1}^{m-1} Q_k^{p-1},\ Q_k > 1,\ k = 1, ..., m.$$

Now, (10) can be written:

(10')
$$2^{\alpha p-n} \prod_{k=1}^{m-1} \left(\log_k \frac{1}{2r}\right)^{1-p} \left(\log_m \frac{1}{2r}\right)^{-\beta} \le Q_m^{\beta} \prod_{k=1}^{m-1} \left[Q_k^{p-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta}\right]$$

If $Q \in [1, 2^n]$ and $\alpha p \le n$,to have (10') is sufficient to prove:

(12) $$\log_k \frac{1}{2r} \le\ Q_k \log_k \frac{1}{r},\ Q_k > 1,\ k = 1, ..., m$$

We prove this assertion. To do this, we suppose that (12) take place and $0 < \beta \le p-1$. Thus:

$$\begin{cases} \left(\log_k \frac{1}{2r}\right)^{1-p} \le\ Q_k^{p-1} \left(\log_k \frac{1}{r}\right)^{1-p},\ k = 1, ..., m-1 \\ \left(\log_m \frac{1}{2r}\right)^{-\beta} \le\ Q_k^{\beta} \left(\log_k \frac{1}{r}\right)^{-\beta} \end{cases} \Rightarrow$$

$$\prod_{k=1}^{m-1} \left(\log_k \frac{1}{2r}\right)^{1-p} \left(\log_m \frac{1}{2r}\right)^{-\beta} \le Q_m^{\beta} \prod_{k=1}^{m-1} \left[Q_k^{p-1} \left(\log_k \frac{1}{r}\right)^{1-p} \left(\log_m \frac{1}{r}\right)^{-\beta}\right]$$

and $2^{\alpha p-n} \le 1$ because $\alpha p \le n$.

From the last two relations, it results (10').

Now, we shall prove that for $r > 0$,small enough, that following relation is true:

(13) $$\log \frac{1}{r} \sim \log \frac{1}{2r}$$

Indeed,

$$\lim_{r\to 0} \frac{\log \frac{1}{r}}{\log \frac{1}{2r}} = \lim_{r\to 0} \frac{\log r}{\log r + \log 2} = 1$$

Then, there exist $Q_1 > 0$ such that:

$$\frac{1}{Q_1}\log\frac{1}{2r} \le \log\frac{1}{r} \le Q_1 \log\frac{1}{2r}$$

But, the function $f(r) = log\frac{1}{r}$, $0 < r < r_0$ is decreasing and thus:

$$\log\frac{1}{2r} \le \log\frac{1}{r}$$

From the two previous relations it results that Q_1 must be greater or equal with zero.

Now, we use the induction to prove (12).

For $k = 1$, the relation was proved. We suppose that it is true for $k-1$ ($k \in N^* - \{1\}$) and we prove it for k, i.e.:

$$\log_k\frac{1}{r} \sim \log_k\frac{1}{2r}$$

for $r > 0$, small enough.

$$\lim_{n\to 0}\frac{\log_k \frac{1}{r}}{\log_k \frac{1}{2r}} = \lim_{r\to 0}\frac{\left[\log_{k-1}\frac{1}{r}\right]'}{\log_{k-1}\frac{1}{r}} \cdot \frac{\log_{k-1}\frac{1}{2r}}{\left[\log_{k-1}\frac{1}{2r}\right]'} = \lim_{r\to 0}\frac{\prod_{j=1}^{k-1}\log_j\frac{1}{r}}{\prod_{j=1}^{k-1}\log_j\frac{1}{2r}}$$

This limit is a finite one, because the fraction terms are comparable.

The proof of (12) is complete.

The hypothesis of lemma 2 are satisfied. Then, E has a inferior positive h -density in every point x.

Using (8) and lemma 1, it results that $0 < H_{h_{\alpha p,p-1,m,\beta}}(E) < +\infty$.

$$\int_0^{r_0}\left[r^{\alpha p-n}h_{\alpha p,p-1,m,\beta}(r)\right]^{\frac{1}{p-1}}\frac{dr}{r} = \int_0^{r_0}\left[\prod_{k=1}^{m-1}\left(\log_k\frac{1}{r}\right)^{-1}\left(\log_m\frac{1}{r}\right)^{-\frac{\beta}{p-1}}\right]\frac{dr}{r} = \infty,$$

for r_0 small enough.

From lemma 3, we obtain: $B_{\alpha,p}(E) = 0.\square$

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