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FUNCTIONAL-DIFFERENTIAL EQUATIONS THAT APPEAR IN PRICE THEORY

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Abstract. Sufficient conditions are obtained for all positive solutions of:

$$\frac{dx(t)}{dt} = [f(x(t)) - g(x(t-\tau))]x(t)$$

to converges as $t \to \infty$ to a positive equilibrium solution.
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1. Introduction

In considering the dynamics of price, productions and consumption commodity, Bélair and Mackey [2] have studied the model

$$p'(t) = p(t)f(p_d, p_s)$$

where $p(t)$ is the function which means the price of commodity at the moment t, and p_d, p_s are the demand price respectively the supply price of this commodity.

Our purpose here is to study the following model:

$$x'(t) = [f(x(t)) - g(x(t-\tau))]x(t), \quad t \in R_+ \tag{1}$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0] \tag{2}$$

where $\tau > 0$, $f, g \in C(R_+, R_+)$ and $\varphi \in C([-\tau, 0], R_+^*)$.

2. Coincidence points and equilibrium solutions

We consider the equation (1), where f and $g \in C(R_+, R_+)$. Let E be the set of equilibrium solutions of (1) and $E_+ = \{r \in E | \ r > 0\}$. We also denote:

$$C(f,g) := \{t \in R_+ | \ f(t) = g(t)\}$$

$$C_+(f,g) := \{t \in C(f,g) | \ t > 0\}$$

We remark that:

$$E_+ = C_+(f,g)$$

We need the following well-known result:

Lemma. (Goebel's theorem) *Suppose that:*

(i) there exists $a \in]0,1[$ *such that:*

$$|f(x)-f(y)| \le a|g(x)-g(y)| \text{ for all } x,y \in R_+$$

(ii) g *is bijective.*
Then

$$E_+ = \{r^*\}$$

For more information see [1], [8].

3. A model in case of naive consumer

We consider the problem (1)+(2). The next result establishes sufficient conditions for every positive solution of equation (1) to oscillate about r^*.

We have:

Theorem 1. $f, g \in C(R_+, R_+)$ *and* $\varphi \in C([-\tau, 0], R_+^*)$.
We suppose that:
(i) there exists $a \in]0,1[$ *such that*

$$|f(x)-f(y)| \le a|g(x)-g(y)| \text{ for all } x,y \in R_+$$

(ii) g *is bijective*
(iii) f *is strictly decreasing*
(iv) g *is strictly increasing*
(v) there exists f' *and* g' *and* $|f'|$ *is bounded*
(vi) $\tau(f(0)+g(M)) \le 1$ *and* $\sum_{n=0}^{\infty} f(n\tau)$ *converges.*
Then
(a) the equation (1) has a unique positive equilibrium solution, r^*
(b) if x^* *is a solution of the problem (1)+(2) then there exists* $m, M \in R_+$, $0 < m < M$, *such that* $m \le x^*(t) \le M$ *for all* $t \in R_+$
(c) there exists a unique solution $x^*(t)$ *of the problem (1)+(2)*
(d) if x^* *is* r^**-nonoscillatory, then*

$$\lim_{t\to\infty} x(t) = r^*.$$

Proof. (a) follows from Lemma.

(b) We shall first show $x(t)$ is bounded from above. For the sake of contradiction, suppose this is not the case. Then there exists $T \in (0,1]$, and a sequence $t_j \to T$ such that $x(t_j) \to \infty$ and $x'(t_j) \ge 0$. The contradiction will come from the consideration of following two cases:

(1a) Suppose

$$\liminf_{j\to\infty} x(t_j - \tau) > 0$$

Then there exists $k > 0$ such that $x(t_j - \tau) \ge k$ for large j. This implies that

$$g(x(t_j - \tau)) \ge g(k)$$

and $f(x(t_j))$ is bounded. It follows from eq. (1) with t replaced by t_j that:

$$\lim_{j\to\infty} x'(t_j) = -\infty$$

This is impossible because $x'(t_j - \tau) \geq 0$.

(1b) Suppose

$$\liminf_{j\to\infty} x(t_j - \tau) = 0$$

By passing to a sequence, if necessary, we may assume

$$\lim_{j\to\infty} x(t_j - \tau) = 0 \tag{3}$$

Note that because $\varphi(t) > 0$ for $t \in [-\tau, 0]$, it follows by (3) that $T > \tau$. Integrate eq. (1) from $t_j - \tau$ to obtain

$$x(t_j) - x(t_j - \tau) = \int_{t_j-\tau}^{t_j} f(x(t))x(t)dt - \int_{t_j-\tau}^{t_j} g(x(t-\tau))x(t)dt \leq \int_{t_j-\tau}^{t_j} f(x(t))x(t)dt$$

$$x(t_j) \leq x(t_j - \tau) + \int_{t_j-\tau}^{t_j} f(x(t))x(t)dt \tag{4}$$

An application of Gronwall's lemma to (4) leads to:

$$x(t_j) \leq x(t_j - \tau) \exp \int_{t_j-\tau}^{t_j} f(x(t))dt$$

This is impossible because

$$\lim_{j\to\infty} x(t_j) = \infty$$

Next we claim

$$\liminf_{t\to\infty} x(t) \neq 0$$

Suppose that is not the case. Then, there exists a sequence $t_j \to \infty$ such that

$$x(t_j) \to 0 \text{ and } x'(t_j) \leq 0$$

It follows from eq. (1) that

$$g(x(t_j - \tau)) \geq f(x(t_j)) \to f(0) \text{ as } j \to \infty$$

and so there exists $k > 0$ such that $x(t_j - \tau) \geq k$ for all large j. By integrating eq. (1) from $t_j - \tau$ to t_j, we obtain:

$$\ln \frac{x(t_j)}{x(t_j - \tau)} = \int_{t_j-\tau}^{t_j} [f(x(t)) - g(x(t-\tau))]dt \tag{5}$$

This is impossible because

$$\lim_{j\to\infty} \ln \frac{x(t_j)}{x(t_j - \tau)} = -\infty$$

while the right hand side of eq. (5) is bounded.

(c) An application of method of steps to equation (1) leads to:

$$A(x)(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ \varphi(0) + \displaystyle\int_0^t [f(x(s)) - g(\varphi(s-\tau))]x(s)ds, & t \in [0, \tau] \end{cases}$$

$$A : B(\widetilde{\varphi}(t); M) \to C([-\tau, \tau], R_+)$$

where

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ \varphi(0), & t \in [0, \tau] \end{cases}$$

$$M = \max_{[-\tau,0]} \varphi(t) \exp\left(\tau \sum_{n-0}^{\infty} f(n\tau)\right)$$

We show that $B(\widetilde{\varphi}(t); M) \in I(A)$

$$|A(x)(t) - \varphi(0)| = \left| \int_0^t [f(x(s)) - g(\varphi(s-\tau))]x(s)ds \right| \leq$$

$$\leq M \int_0^t \left(f(0) + g\left(\max_{t\in[-\tau,0]} \varphi(t) \right) \right) ds \leq M\tau \left(f(0) + g\left(\max_{t\in[-\tau,0]} \varphi(t) \right) \right)$$

From (vi) we have that

$$\tau \left(f(0) + g\left(\max_{t\in[-\tau,0]} \varphi(t) \right) \right) \leq 1$$

Now we can consider the operator:

$$A : B(\widetilde{\varphi}(t); M) \to B(\widetilde{\varphi}(t); M)$$

From (v) we have that the operator A is contraction with respect to Bieletscki norm, satisfactory chosen. From Contraction Principle we have:

$$A(x)(t) = \begin{cases} x_1(t) & t \in [0, \tau] \\ x_1(\tau) + \displaystyle\int_\tau^t [f(x(s)) - g(x_1(s-\tau))]x(s)ds & x \in [\tau, 2\tau] \end{cases}$$

$$A : B(\widehat{\varphi}(t); M) \to C([-\tau, 2\tau], R_+)$$

$$|A(x)(t) - x_1(\tau)| = \left| \int_\tau^t [f(x(s)) - g(x_1(s-\tau))]x(s)ds \right| \leq$$

$$\leq M\tau \left(f(0) + g\left(\max_{t\in[0,\tau]} x_1(t) \right) \right)$$

But

$$\tau \left(f(0) + g\left(\max_{t\in[0,\tau]} x_1(t) \right) \right) \leq 1$$

In this conditions we have that the operator:

$$A : B(\widehat{\varphi}(t); M) \to M(\widehat{\varphi}(t); M)$$

has a unique fixed point x_2.

For $t \in [(n-1)\tau, n\tau]$ we have:

$$A(x)(t) = \begin{cases} x_{n-1}(t) \quad t \in [(n-2)\tau, (n-1)\tau] \\ x_{n-1}((n-1)\tau) + \int_{(n-1)\tau}^{t} [f(x(s)) - g(x_{n-1}(s-\tau))]x(s)ds \\ \qquad t \in [(n-1)\tau, n\tau] \end{cases}$$

But

$$\tau \left(f(0) + g\left(\max_{t\in[(n-2)\tau,(n-1)\tau]} x_{n-1}(t) \right) \right) \leq 1$$

and we have that the operator A has a unique fixed point x_n.

From (v) we have that the operator A is contraction with respect to Bieletscki norm, satisfactory chosen. From the Contraction Principle we have:

$$A : B(\breve{\varphi}(t); M) \to B(\breve{\varphi}(t); M)$$

where

$$\breve{\varphi}(t) = \begin{cases} \varphi(t) & t \in [-\tau, 0] \\ \varphi(0) & t \in [0, \tau] \\ \dots & \\ x_{n-1}((n-1)\tau) & t \in [(n-1)\tau, n\tau] \\ \dots & \end{cases}$$

has a unique fixed point x^*.

(d) We rewrite (1) in the form

$$\frac{dy(t)}{dt} = F(y(t), y(t-\tau)) - F(0,0) \tag{6}$$

where $F(y(t), y(t-\tau)) = [f(y(t)+r^*) - g(y(t-\tau)+r^*)](y(t)+r^*)$ and $y(t) = x(t) - r^*$.

It is now sufficient to show that $y(t) \to 0$ as $t \to \infty$. An application of mean-value theorem to (6) leads to

$$\frac{dy(t)}{dt} = -a(t)y(t) - b(t)y(t-\tau) \tag{7}$$

where

$$-a(t) = \frac{\partial F}{\partial y(t)}(u(t), v(t))$$

$$-b(t) = \frac{\partial F}{\partial y(t-\tau)}(u(t), v(t))$$

and $(u(t), v(t))$ lies on the line segment joining (0,0) and $(y(t), y(t-\tau))$. It is found that

$$a(t) = g(y(t-\tau) + r^*) - f(y(t) + r^*) - f'(y(t) + r^*)(y(t) + r^*)$$

$$b(t) = g'(y(t-\tau) + r^*)(y(t) + r^*)$$

Note that $a(t)$ and $b(t)$ are positive and are bounded away from zero. The existence of solutions of (7) for all $t \geq 0$ is a consequence of boundedness of $x(t)$ for all $t \geq 0$. Is nonoscillatory then $|y(t)| > 0$ for $t > T$. If $y(t) > 0$ for $t > T$ then we have from

(7) that $y'(t) < 0$ and so $\lim_{t\to\infty} y(t)$ exists. Since $y(t) > 0$ eventually, $\lim_{t\to\infty} y(t) = l \geq 0$. We claim that $I = 0$. Then there exists $t_0 > 0$ such that

$$y(t) \geq \frac{l}{2} \text{ for } t \geq t_0$$

We have directly from (7) that

$$\frac{dy(t)}{dt} \leq -a(t)\frac{l}{2}$$

leading to

$$y(t) - y(t_0) \leq -\frac{l}{2}\int_{t_0}^{t} a(s)ds$$

which implies that $y(t) \to \infty$ as $t \to \infty$; but this contradicts the eventual positivity of y. Thus $\lim_{t\to\infty} y(t) = l = 0$.

If $y(t) < 0$ for $t > T$, the arguments are again similar. Thus the result follows from

$$\lim_{t\to\infty} y(t) = 0$$

Remark. In the case of the model studied by A.M. Farahani and E.A. Grove [3] where $f(t) = \dfrac{a}{b+t^n}$, $n \in [1,\infty]$ we remark that $\sum_{n=0}^{\infty} f(n\tau)$ converges.

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