THE GOURSAT-IONESCU PROBLEM FOR HYPERBOLIC INCLUSIONS WITH MODIFIED ARGUMENT

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Abstract. In this paper we consider the Goursat-Ionescu Problem defined in Straburzyński’s sense, for hyperbolic inclusions with modified argument. An existence theorem for a local solution of this problem is proved and some properties of the set of its solutions are established.

Keywords: multifunction, hyperbolic inclusion, upper-semicontinuity, initial values, absolutely continuous in Carathéodory’s sense function.

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1. Introduction

Goursat’s Problem defined by E. Goursat [11] for a quasilinear hyperbolic equation consists in determining one of its solutions, provided that the values of the solution on two curve arcs having a common point, which may be taken as the origin of the system of coordinates, are known [4].

In his PhD Thesis (1927) [13], D.V. Ionescu studied − for the first time in the mathematical literature − boundary value problems of Darboux, Cauchy, Picard and Goursat types for second order partial differential equations with modified argument.

More recently, a series of authors studied the same problems for second order hyperbolic equations of various forms.

In this paper, we consider Goursat-Ionescu Problem in Straburzyński’s sense [18], for a hyperbolic inclusion with modified argument.

Let \(a, b, a', b', a_0, b_0\) be positive numbers with \(0 < a_0 \leq a'\), \(0 < b_0 \leq b'\) and \(y = g(x) : [0, a] \to \mathbb{R}, \ x = h(y) : [0, b] \to \mathbb{R}\) be nondecreasing functions of class \(C^1\) such that \(g(0) = h(0) = 0, 0 \leq g(x) \leq b, 0 \leq h(y) \leq a\). We denote:

\[
P = [-a', a] \times [-b', b], \quad \Delta = [0, a] \times [0, b], \quad \Delta_0 = [0, x_0] \times [0, y_0] \subseteq \Delta,
\]

\[
D = \{(s, t)/h(t) < s \leq a, \ g(s) < t \leq b\}, \quad P_0 = [-a_0, x_0] \times [-b_0, y_0] \subseteq P,
\]

\[
D_{xy} = \{(s, t)/h(t) < s \leq x, \ g(s) < t \leq y\}
\]

for \((x, y) \in \Delta, \ G = P - D, \ G_0 = P_0 - D_{x_0y_0}, \ G_0 \subseteq G.\)

Let \(\varphi : P \rightarrow \mathbb{R}^n\) be an absolutely continuous function in Carathéodory’s sense, \(\varphi \in C^* (P; \mathbb{R}^n) \ [1, §565 - §570].\)
We consider Goursat-Ionescu Problem for the hyperbolic inclusion with modified argument of the form
\[
\frac{\partial^2 z(x, y)}{\partial x \partial y} \in F(x, y, z(\alpha(x, y), \beta(x, y))), \quad (x, y) \in \overline{\Omega},
\]
where \( F : \Delta \times \Omega \to 2^{\mathbb{R}^n} \) is a multifunction with compact, convex and non-empty values, \( \Omega \subset \mathbb{R}^n \) is an open subset, \( \alpha \in C(\Delta; [0, a]) \), \( \beta \in C(\Delta; [0, b]) \).

Under suitable assumptions, we prove an existence theorem for a local solution of this problem, and that the set of solutions is compact in Banach space \( C(P_0; \mathbb{R}^n) \), where \( P_0 = [-a_0, x_0] \times [-b_0, y_0] \subseteq P \); moreover, as a function of the initial values, this set defines an upper-semicontinuous multifunction.

This study was suggested by papers which deal with the Goursat Problem [7], [18], with Goursat-Ionescu Problem for univalued hyperbolic equations [8], [9] and [19].

2. Preliminaries

The definitions and Theorem 2.1 in this section are taken from [1], [2], [3], [5]-[7], [14]-[17].

Definition 2.1. Let \( X \) and \( Y \) be two non-empty sets. A multifunction \( \Phi : X \to 2^Y \) is a function from \( X \) into the family of all non-empty subsets of \( Y \).

To each \( x \in X \), a subset \( \Phi(x) \) of \( Y \) is associated by the multifunction \( \Phi \). The set \( \bigcup_{x \in X} \Phi(x) \) is the range of \( \Phi \).

Definition 2.2. Let us consider \( \Phi : X \to 2^Y \).

a) If \( A \subseteq X \), the image of \( A \) by \( \Phi \) is \( \Phi(A) = \bigcup_{x \in A} \Phi(x) \);

b) If \( B \subseteq Y \), the counterimage of \( B \) by \( \Phi \) is \( \Phi^{-1}(B) = \{x \in X | \Phi(x) \cap B \neq \emptyset\} \);

c) The graph of \( \Phi \), denoted \( \text{graph} \Phi \) is the set \( \text{graph} \Phi = \{(x, y) \in X \times Y | y \in \Phi(x)\} \).

Definition 2.3. Let now take \( \Phi : X \to 2^X \). An element \( x \in X \) with the property \( x \in \Phi(x) \) is called a fixed point of the multifunction \( \Phi \).

Definition 2.4. A univalued function \( \varphi : X \to Y \) is said to be a selection of \( \Phi : X \to 2^Y \) if \( \varphi(x) \in \Phi(x) \) for all \( x \in X \).

Definition 2.5. Let \( X \) and \( Y \) be two topological spaces. The multifunction \( \Phi : X \to 2^Y \) is upper-semicontinuous if, for any closed subset \( B \subseteq Y \), \( \Phi^{-1}(B) \) is closed in \( X \).

Definition 2.6. If \( (X, \mathcal{F}) \) is a measurable space and \( Y \) is a topological space, the multifunction \( \Phi : X \to 2^Y \) is measurable if \( \Phi^{-1}(B) \in \mathcal{F} \) for every closed subset \( B \subseteq Y \), \( \mathcal{F} \) being the \( \sigma \)-algebra of the measurable sets of \( X \), i.e. \( \Phi^{-1}(B) \) is measurable.

Theorem 2.1. [17]. Let \( X \) and \( Y \) be two compact metric spaces and \( \Phi : X \to 2^Y \) a multifunction with the property that \( \Phi(x) \) is a closed subset of \( Y \) for any \( x \in X \).
The following assertions are equivalent:
(i) the multifunction \( \Phi \) is upper-semicontinuous;
(ii) the graph of \( \Phi \) is a closed subset of \( X \times Y \);
(iii) any would be the sequences \( (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \), from \( x_n \rightarrow x, \ y_n \in \Phi(x_n), \ y_n \rightarrow y \) it follows \( y \in \Phi(x) \).

**Definition 2.7.** [5]-[6]. The function \( u : \Delta \rightarrow \mathbb{R}^n \) is absolutely continuous in Carathéodory’s sense [1, §565 - §570] iff \( u(x, y) \) is continuous on \( \Delta \), absolutely continuous in \( x \) (for any \( y \)), absolutely continuous in \( y \) (for any \( x \)), \( u_x(x, y) \) is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in \( y \) (for any \( x \)) and \( u_{xy} \) is Lebesgue-integrable on \( \Delta \).

We denote the class of absolutely continuous functions in Carathéodory’s sense by \( C^*(\Delta; \mathbb{R}^n) \) [5]-[6].

3. Results

In a similar way as in [2] and [19], we define the notion of a local solution for the Goursat-Ionescu Problem (1.1)+(1.2) and we prove an existence theorem for a local solution of this problem, together with some properties of the set of solutions, namely that this is a compact subset in Banach space \( C(\Delta_0; \mathbb{R}^n) \) and, as a function of initial values, it defines an upper-semicontinuous multifunction.

Let the following hypotheses be satisfied:

(\( H_0 \)) The curves \( C_1 : y = g(x), 0 \leq x \leq a, \) and \( C_2 : x = h(y), 0 \leq y \leq b \) are defined by nondecreasing functions of class \( C^1 \) such that \( g(0) = h(0), 0 \leq g(x) \leq b, 0 \leq h(y) \leq a. \)

(\( H_1 \)) \( F : \Delta \times \Omega \rightarrow \mathbb{R}^n \) is a multifunction with compact, convex, non-empty values in \( \mathbb{R}^n, \Omega \subset \mathbb{R}^n \) is an open subset, \( \Delta = [0, a] \times [0, b] \subset \mathbb{R}^2. \)

(\( H_2 \)) For any \((x, y) \in \Delta, \) the mapping \( z \rightarrow F(x, y, z) \) is upper-semicontinuous on \( \Omega; \)

(\( H_3 \)) For any \( z \in \Omega \) the mapping \((x, y) \rightarrow F(x, y, z) \) is Lebesgue measurable on \( \Delta; \)

(\( H_4 \)) \( \alpha \in C(\Delta; [0, a]) \) and \( \beta \in C(\Delta; [0, b]); \)

(\( H_5 \)) There exists a function \( k : \Delta \rightarrow \mathbb{R}_+, k \in \mathcal{L}^1(\Delta; \mathbb{R}_+) \) such that

\[ \|\zeta\| \leq k(x, y) \] for \( \forall \zeta \in F(x, y, z), \forall (x, y) \in \Delta, \forall z \in \Omega; \)

(\( H_6 \)) There exists a convex, compact set \( M \subset \Omega \) and a point \((x_0, y_0) \in [0, a] \times [0, b], \)

such that

\[ \int_0^{x_0} \int_0^{y_0} k(s, t)ds \ dt \leq d(M, C_\Omega), \]

where \( d(M, C_\Omega) \) is the distance from \( M \) to \( C_\Omega = \mathbb{R}^n - \Omega; \)

(\( H_7 \)) The function \( \varphi : P \rightarrow \mathbb{R}^n \) is absolutely continuous in Carathéodory’s sense, \( \varphi \in C^*(P; \mathbb{R}^n). \)

(\( H_8 \)) The values of function \( \lambda : \Delta \rightarrow \mathbb{R}^n, \) defined by

\[ \lambda(x, y) = \varphi(0, 0) + \int_0^x \varphi_x(s, g(s))ds + \int_0^y \varphi_y(h(t), t)dt, \]

belong to the set \( M \) for \((x, y) \in \Delta_0 = [0, x_0] \times [0, y_0] \subset \Delta. \)
Remark. It follows that the function \( \lambda \) defined by (3.1) is absolutely continuous in Carathéodory's sense [1, §565 - §570], \( \lambda \in C^*(\Delta; \mathbb{R}^n) \), due to hypotheses \((H_7)\), \( \varphi \in C^*(P; \mathbb{R}^n) \) and the integral is absolutely continuous.

**Definition 3.1.** The **Goursat-Ionescu Problem** for the hyperbolic inclusion with modified argument (1.1) means to determine a solution of this inclusion which satisfies the initial conditions (1.2).

**Definition 3.2.** It is defined a local solution of the Goursat-Ionescu Problem (1.1)+(1.2) as a function \( Z : P_0 \to \Omega, P_0 = [-a_0, x_0] \times [-b_0, y_0], \) with \( 0 < a_0 \leq a' \) and \( 0 < b_0 \leq b' \), which is absolutely continuous in Carathéodory’s sense defined on \( (\Delta; \mathbb{R}^n) \) and satisfies (1.1) a.e. for \((x, y) \in \mathcal{D}_{x_0 y_0}\) and also conditions (1.2) for \((x, y) \in G_0 = P_0 - D_{x_0 y_0} \subseteq G\).     

**Theorem 3.1.** Let the hypotheses \((H_0) - (H_s)\) be satisfied. Then:

(i) there exists at least a local solution \( Z \) of the Goursat-Ionescu Problem (1.1)+(1.2);

(ii) the set \( S_\lambda \) of local solutions \( Z \) is compact in the Banach space \( C(P_0; \mathbb{R}^n) \);

(iii) the multifunction \( \lambda \to S_\lambda \) is upper-semicontinuous on \( C^*(\Delta_0; \mathbb{R}^n) \) taking values in \( C(\Delta_0; \mathbb{R}^n) \).

**Proof.** (i) Let \( C^*(P_0; \mathbb{R}^n) \) be the set of absolutely continuous functions in Carathéodory’s sense defined on \( P_0 \) with values in \( \mathbb{R}^n \) [1]. We denote by \( \mathcal{Z}_M \) the set of functions \( Z : P_0 \to \mathbb{R}^n, Z \in C^*(P_0; \mathbb{R}^n) \), which satisfy the inequality

\[
\| \frac{\partial^2 Z(x, y)}{\partial x \partial y} \| \leq k(x, y), \text{ a.e. for } (x, y) \in \mathcal{D}_{x_0 y_0},
\]

and also conditions (1.2) for \((x, y) \in G_0 = P_0 - D_{x_0 y_0}\). The notation \( \mathcal{Z}_M \) is suitable because, by hypothesis \((H_s)\), \( \lambda(x, y) \in M \) for \((x, y) \in \Delta_0\). We remark that the absolute continuity in Carathéodory’s sense of \( Z \) assures the existence of the derivative \( \frac{\partial^2 Z(x, y)}{\partial x \partial y} \) a.e. for \((x, y) \in P_0 \) [1, §565 - §570].

We have \( \mathcal{Z}_M \subseteq C^*(P_0; \mathbb{R}^n) \). Then, by hypothesis \((H_6)\) and inequality (3.2), for any \( Z \in \mathcal{Z}_M \), it follows that \( Z(x, y) \in \Omega \).

Indeed, integrating \( \frac{\partial^2 Z(x, y)}{\partial x \partial y} \) on \( \mathcal{D}_{xy} \) we obtain

\[
Z(x, y) = \varphi(0, 0) + \int_0^x \varphi'(s, g(s))ds + \int_0^y \varphi'(s, h(t))dt + \int \int_{\mathcal{D}_{xy}} \frac{\partial^2 Z(s, t)}{\partial s \partial t}ds dt = 
\]

\[
= \lambda(x, y) + \int \int_{\mathcal{D}_{xy}} \frac{\partial^2 Z(s, t)}{\partial s \partial t}ds dt.
\]

Using hypothesis \((H_6)\), inequality (3.2) and (3.3) it results

\[
\|Z(x, y) - \lambda(x, y)\| = \| \int \int_{\mathcal{D}_{xy}} \frac{\partial^2 Z(s, t)}{\partial s \partial t}ds dt \| \leq \int \int_{\mathcal{D}_{xy}} \| \frac{\partial^2 Z(s, t)}{\partial s \partial t} \| ds dt \leq 
\]

\[
\leq \int \int_{\mathcal{D}_{xy}} k(s, t)ds dt \leq \int_0^{x_0} \int_0^{y_0} k(s, t)ds dt \leq d(M, C_\Omega).
\]
From the hypothesis \((H_8)\), \(\lambda(x,y) \in M\) for \((x,y) \in \Delta_0 = [0,x_0] \times [0,y_0]\) and we have
\[
d(Z(x,y), \lambda(x,y)) = \|Z(x,y) - \lambda(x,y)\| \leq d(M, C_\alpha),
\]
which shows that \(Z(x,y) \in \Omega\) for \((x,y) \in \Delta_0\).

The set of functions \(Z_M\) is convex and compact in \(C(P_0; \mathbb{R}^n)\). The convexity follows by the definition of this set, and its compactness from the Arzelà-Ascoli Theorem, using hypotheses \((H_0)\), \((H_6)\), \((H_7)\), \((H_8)\).

We denote by \(\mathcal{G}\) the set of the triples \((\lambda, Z, U) \in C^*(\Delta_0; \mathbb{R}^n) \times Z_M \times Z_M\) with the property that \(Z\) and \(U\) satisfy the membership relation
\[
\|\partial^2 U(x,y) / \partial x \partial y\| \in F(x, y, Z(\alpha(x,y), \beta(x,y))), \text{ a.e. for } (x,y) \in \overline{D}_{x_0,y_0}.
\]

We prove that, for each \(\lambda \in C^*(\Delta_0; \mathbb{R}^n)\) with \(\lambda(x,y) \in M\) for \((x,y) \in \Delta_0\), the set of those pairs \((Z,U)\) such that \((\lambda, Z, U) \in \mathcal{G}\) is non-empty and the set \(\mathcal{G}\) is closed.

Indeed, let us take \(Z \in Z_M\). From Theorem 1 [2], there exists a \(\mu\)-measurable (under the \(\mu\)-Lebesgue measure) multifunction \(\Gamma : \Delta_0 \rightarrow 2^\mathbb{R}^n\) with compact, non-empty values in \(\mathbb{R}^n\) such that
\[
\Gamma(x,y) \subset F(x,y, Z(\alpha(x,y), \beta(x,y))), \forall (x,y) \in \Delta_0.
\]

Then, by Theorem 2 or Theorem 3 [3], there exists a measurable selection \(\gamma\) of \(\Gamma\), i.e. a measurable univalued function \(\gamma : \Delta_0 \rightarrow \mathbb{R}^n\) with \(\gamma(x,y) \in \Gamma(x,y)\) for \((x,y) \in \Delta_0\).

Let the function \(U : P_0 \rightarrow \mathbb{R}^n\) be defined by
\[
U(x,y) = \begin{cases} 
\lambda(x,y) - \int_{\overline{D}_{x_0,y_0}} \gamma(s,t) ds dt, & (x,y) \in \overline{D}_{x_0,y_0}, \\
\varphi(x,y), & (x,y) \in G_0 = P_0 - D_{x_0,y_0}.
\end{cases}
\]

Then, the set of those pairs \((Z,U)\) such that \((\lambda, Z, U) \in \mathcal{G}\) is non-empty because
\[
\gamma(x,y) \in \Gamma(x,y) \subset F(x,y,Z(\alpha(x,y), \beta(x,y))), \text{ a.e. for } (x,y) \in \Delta_0,
\]
\[
\frac{\partial^2 U(x,y)}{\partial x \partial y} = \gamma(x,y) \in \Gamma(x,y) \subset F(x,y, Z(\alpha(x,y), \beta(x,y))), \text{ a.e. for } (x,y) \in \overline{D}_{x_0,y_0},
\]
\[
\|\frac{\partial^2 U(x,y)}{\partial x \partial y}\| = \|\gamma(x,y)\| \leq k(x,y), \forall (x,y) \in \overline{D}_{x_0,y_0},
\]

by hypothesis \((H_8)\) for \(\zeta = \gamma(x,y)\).

For the proof that \(\mathcal{G}\) is closed, we consider a sequence of elements in \(\mathcal{G}\), \(\{ (\lambda_n, Z_n, U_n) \}_{n \in \mathbb{N}}\), convergent to \((\lambda, Z, U)\) in the space \(C^*(\Delta_0; \mathbb{R}^n) \times C(P_0; \mathbb{R}^n) \times L^1(P_0; \mathbb{R}^n)\). We must check that \((\lambda, Z, U) \in \mathcal{G}\), what implies, by the definition of set \(\mathcal{G}\), that conditions (1.2) and (3.10) are satisfied by \(Z\) and \(U\).
The set \( \left\{ \frac{\partial^2 U_n(x,y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}} \) is relatively weakly compact in \( L^1(\Delta_0; \mathbb{R}^n) \) by the Dunford-Pettis Criterion [10]. It follows that \( \left\{ \frac{\partial^2 U_n(x,y)}{\partial x \partial y} \right\}_{n \in \mathbb{N}} \) is weakly convergent to a function \( V \in L^1(\Delta_0; \mathbb{R}^n) \). For each \((x,y) \in P_0\), we have

\[
(3.12) \quad U(x,y) = \left\{
\begin{array}{l}
 w - \lim_{n \to \infty} U_n(x,y) = w - \lim_{n \to \infty} \left[ \lambda_n(x,y) + \int_{\mathcal{D}_x} \frac{\partial^2 U_n(s,t)}{\partial s \partial t} \, ds \, dt \right], (x,y) \in \overline{D}_{xo0}, \\
\varphi(x,y), \quad (x,y) \in G_0 = P_0 - D_{xo0}.
\end{array}
\right.
\]

From the weak convergence \( \frac{\partial^2 U_n(x,y)}{\partial x \partial y} \to V(x,y), \quad (x,y) \in \overline{D}_{xo0} \), using the Corollary of Mazur’s Theorem [12], it follows that there exists a sequence of convex combinations \( \{W_r\}_{r \in \mathbb{N}} \) of the set \( \left\{ \frac{\partial^2 U_r}{\partial x \partial y}, \frac{\partial^2 U_{r+1}}{\partial x \partial y}, \ldots \right\} \), strongly convergent to \( V \) in \( L^1(\Delta_0; \mathbb{R}^n) \). Then, we can extract a subsequence from the sequence \( \{W_r\}_{r \in \mathbb{N}} \) which converges a.e. to \( V : W_r \to V \) a.e. for \((x,y) \in \Delta_0\).

Since \( F(x,y,Z) \) is convex and compact for all \((x,y) \in \Delta \) and for all \( Z \in \Omega \), we obtain from the previous results and from Lemma 2 [2] that

\[
(3.13) \quad V(x,y) \in \cap_{r=1}^{\infty} \text{conv} \left(\bigcup_{n=r}^{\infty} \frac{\partial^2 U_n(x,y)}{\partial x \partial y} \right) \subset \bigcup_{n=r}^{\infty} \text{conv} \left( F(x,y,Z_n(\alpha(x,y),\beta(x,y))) \right) \subset F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \overline{D}_{xo0},
\]

from which it follows that \( \mathcal{G} \) is closed.

Indeed, (3.13) shows that \( V(x,y) \in F(x,y,Z(\alpha(x,y),\beta(x,y))) \) a.e. for \((x,y) \in \overline{D}_{xo0}\), and we obtain \( \frac{\partial^2 U(x,y)}{\partial x \partial y} = V(x,y) \) from (3.12); then, using (3.6) and (3.13) we have

\[
(3.14) \quad V(x,y) = \frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y,Z(\alpha(x,y),\beta(x,y))), \quad \text{a.e. for } (x,y) \in \overline{D}_{xo0},
\]

and also

\[
(3.15) \quad U(x,y) = \varphi(x,y) \quad \text{for } (x,y) \in G_0 = P_0 - D_{xo0};
\]

hence \( U \) satisfies initial conditions (1.2) for \((x,y) \in G_0\).
Let us take \( \lambda \in C^* (\Delta; \mathbb{R}^n) \) with \( (x,y) \in M \) for \( (x,y) \in \Delta_0 \). To each \( Z \in \mathcal{Z}_M \) we associate the set \( \Phi (Z) \subset \mathcal{Z}_M \) as follows:

\[
(3.16) \quad U \in \Phi (Z) \iff U \in \mathcal{Z}_M, \quad \frac{\partial^2 U(x,y)}{\partial x \partial y} \in F(x,y, Z(\alpha(x,y), \beta(x,y))), \text{ a.e. } (x,y) \in \Delta_0.
\]

We thus define a multifunction \( \Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M} \). The set \( \Phi (Z) \) is convex, compact and non-empty. It can be seen that \( \Phi (Z) \) is convex since \( F(x,y, Z(x,y)) \) is convex by hypothesis \((H)\). We have \( \Phi (Z) \subset \mathcal{Z}_M \) but \( \mathcal{Z}_M \) is compact. The multifunction \( \Phi \) has a closed graph because graph \( \Phi \) is the set \( \mathcal{G} \) for each fixed \( \lambda \) and \( \mathcal{G} \) is closed. It follows that \( \Phi (Z) \) is compact in \( C(P_0; \mathbb{R}^n) \) as a closed subset of the compact set \( \mathcal{Z}_M \). The set \( \Phi (Z) \) is non-empty since there exists \( U \), defined by (3.8) with the property \( U \in \Phi (Z) \).

The multifunction \( \Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M} \) having a closed graph, is upper-semicontinuous by Theorem 2.1. Taking into account all the properties of \( \Phi \), the Kakutani-Ky Fan fixed point Theorem [10, 17] can be applied.

Indeed, \( \Phi : \mathcal{Z}_M \to 2^{\mathcal{Z}_M} \) is defined on \( \mathcal{Z}_M \) which is a convex, compact and non-empty set; it is also upper-semicontinuous and its set-values \( \Phi (Z) \) are convex, closed and non-empty in \( \mathcal{Z}_M \). From Kakutani-Ky Fan fixed point Theorem it follows that the multifunction \( \Phi \) has at least a fixed point, i.e. there exists at least an element \( Z \in \mathcal{Z}_M \) such that \( Z \in \Phi (Z) \), hence \( Z = U \); but \( U \) is of the form (3.8), therefore this fixed point \( Z \) is a solution of Goursat-Ionescu Problem (1.1)+(1.2).

ii) We denote by \( S_\lambda \) the set of solutions to Problem (1.1)+(1.2), a notation showing that any solution \( Z \) depends on the function \( \lambda \) defined by (3.1). The set \( S_\lambda \) contains at least one element. The set \( S_\lambda \) is compact, non-empty in the Banach space \( C(P_0; \mathbb{R}^n) \), being the set of the fixed points of multifunction \( \Phi \).

iii) The graph \( \mathcal{H} \) of the multifunction \( \lambda \to S_\lambda \), defined on \( C^* (\Delta_0; \mathbb{R}^n) \) with values in \( 2^{\mathcal{Z}_M} \), \( S_\lambda \subset \Phi (\mathcal{Z}_M) \subset 2^{\mathcal{Z}_M} \), is closed in \( C^* (\Delta_0; \mathbb{R}^n) \times \mathcal{Z}_M \) since \( \mathcal{H} \) is the image of the compact set \( \mathcal{H}_1 \) of the triples \( (\lambda, Z, U) \in \mathcal{G} \) with \( Z = U \), through the projection mapping \( (\lambda, Z, U) \to (\lambda, Z) \). The mapping \( \lambda \to S_\lambda \) is \( \lambda \)-in general \( \lambda \)-a multifunction because several solutions of the Problem (1.1)+(1.2) can exist, which are fixed points of mapping \( \Phi \) corresponding to the same function \( \lambda \). Because the mapping \( \lambda \to S_\lambda \) has a closed graph \( \mathcal{H} \) by Theorem 2.1, it follows that \( \lambda \to S_\lambda \) is upper-semicontinuous on \( C^* (\Delta_0; \mathbb{R}^n) \), what completes the proof.

References