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BROWNIAN MOTION AND FRACTAL PROCESSES USING CONTRACTION METHOD IN PROBABILISTIC METRIC SPACES

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Abstract. In this paper we show how can be generalized the random scaling law such that the Brownian motion satisfies it. Using contraction method in probabilistic metric spaces, we can weak the first moment condition for the existence and uniqueness of fractal process.

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A first theory of selfsimilar fractal sets and measures was developed in Hutchinson [1]. Falconer, Graf, Mouldin and Williams, and Arbeiter randomized each step in the approximation process to obtain self-similar random fractal sets and measures. Recently Hutchinson and Rüschendorf [2] gave a simple proof for the existence and uniqueness of random fractal sets, measures and fractal functions using probability metrics defined by expectation. In these works a finite first moment condition is essential.

In this paper, using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of fractal process.

1. Invariant sets in E-spaces

Let Δ^+ denote the set of all distribution functions F with $F(0) = 0$, and let X be a nonempty set. A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $\mathcal{F} : X \times X \to \Delta^+$ is a mapping with the next properties:
1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;
2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;
3^0. $F_{x,y}(s + t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$,
and T is a t-norm.

The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0, 1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from X is said to be *fundamental* if $\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$ for all $t > 0$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$ if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

The notion of *E-space* was introduced by Sherwood [5] in 1969. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F} : \mathcal{E} \times \mathcal{E} \to \Delta^+$ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete, where $T_m(x, y) = \max\{x + y - 1, 0\}$.

The next result was proved in [3]:

Theorem 1.1. *Let* $(\mathcal{E}, \mathcal{F})$ *be a complete E- space,* $N \in \mathbb{N}^*$*, and let* $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ *be contractions with ratio* $r_1, ...r_N$*, respectively. Suppose that there exists an element* $z \in \mathcal{E}$ *and a real number* γ *such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{1}$$

for all $i \in \{1, .., N\}$ *and for all* $t > 0$*. Then there exists a unique nonempty closed bounded and compact subset* K *of* $\mathcal{E}$ *such that*

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 1.1. *Let* $(\mathcal{E}, \mathcal{F})$ *be a complete E- space, and let* $f : \mathcal{E} \to \mathcal{E}$ *be a contraction with ratio* r*. Suppose there exists* $z \in \mathcal{E}$ *and a real number* γ *such that*

$$P(\{\omega \in \Omega |\ \rho(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t}\ \textit{for all}\, t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ *such that* $f(x_0) = x_0$.

2. Scaling law and Brownian motion

Denote (X, d) a complete separable metric space. Let $g : I \to X$, where $I \subset \mathbb{R}$ is a closed bounded interval, $N \in \mathbb{N}$ and let $I = I_1 \cup I_2 \cup \cdots \cup I_N$ be a partition of I into disjoint subintervals. Let $\Phi_i : I \to I_i$ be increasing Lipschitz maps with $p_i = Lip\Phi_i$. We have $\sum_{i=1}^{N} p_i \geq 1$ and, if the Φ_i are affine, then $\sum_{i=1}^{N} p_i = 1$. If $g_i : I_i \to X$, for $i \in \{1, ..., N\}$ define $\sqcup_i g_i : I \to X$ by

$$(\sqcup_i g_i)(x) = g_j(x) \quad \text{for} \quad x \in I_j.$$

A *scaling law* $\mathbb{S}$ is an N-tuple $(S_1, ..., S_N)$, $N \geq 2$, of Lipschitz maps $S_i : X \to X$. Denote $r_i = LipS_i$.

A random scaling law $\mathbb{S} = (S_1, S_2, ..., S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = dist\mathbb{S}$ for the probability distribution determined by $\mathbb{S}$ and $\overset{d}{=}$ for the equality in distribution.

Let $\mathbb{S} = (S_1, ..., S_N)$ be a random scaling law and let $G = (G_t)_{t\in I}$ be a stochastic process or a random function with state space X. The trajectory of the process G is the function $g : I \to X$. The trajectory of the random function $\mathbb{S}g$ is defined up to probability distribution by

$$\mathbb{S}g \overset{d}{=} \sqcup_i S_i \circ g^{(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}, g^{(1)}, ..., g^{(N)}$ are independent of one another and $g^{(i)} \overset{d}{=} g$, for $i \in \{1, ..., N\}$. We say g or $\mathcal{G}$ *satisfies the scaling law* $\mathbb{S}$, or is a *random fractal function*, if

$$\mathbb{S}g \overset{d}{=} g,$$

The Brownian motion can be characterized as the fixed point of a scaling law.

Let $(\Omega, \mathcal{K}, P)$ be a probability space. A *Brownian motion* is a stochastic process $X^\alpha = (X_t^\alpha)_{t\in\mathbb{R}}$ characterised by $X_0^\alpha(\omega) = 0 a.s.$ and

$$X^\alpha(t + h) - X^\alpha(t) \overset{d}{=} N(0, \alpha h), \quad \text{for} \quad t > 0 \, and \, h > 0,$$

where $N(0, \alpha h)$ denote the normal distribution with mean 0 and variance αh.

For each $\alpha > 0$, let $B^\alpha : [0, 1] \to \mathbb{R}$ denote the *constrained Brownian motion* given by

$$B^\alpha(0) = 0 \; a.s., \quad \text{and} \quad B^\alpha(1) = 1 \; a.s..$$

For fix $p \in \mathbb{R}$ consider the Brownian motion $B^\alpha \Big|_{B^\alpha(\frac{1}{2})=p}$ constrained by $B^\alpha(\frac{1}{2}) = p$.

Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterized by $S_1(0) = 0$, $S_1(1) = S_2(0) = p$, $S_2(1) = 1$. If $r_1 = LipS_1 = |p|, \quad r_2 = LipS_2 = |1 - p|$, then

$$B^\alpha|_{B^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_1 \circ B^{\frac{\alpha}{2r_1^2}}(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$B^\alpha|_{B^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_2 \circ B^{\frac{\alpha}{2r_1^2}}(2t - 1), \quad t \in [\frac{1}{2}, 1].$$

Let $I = [0, 1]$, and define $\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2}$, and $\Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s+1}{2}$. It follows that

$$B^\alpha|_{B^\alpha(\frac{1}{2})}(t) \overset{d}{=} \sqcup_i S_i \circ B^{\frac{\alpha}{2r_i^2}} \circ \Phi_i^{-1}(t), \quad t \in [0, 1].$$

Now let p^α be random point with distribution $N(0, \frac{\alpha}{2})$. For each $\alpha > 0$ let us define the random scaling law $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$ in the same manner (S_1, S_2) was previously defined from the point p.

Let $r_i^\alpha = Lip_i^\alpha$ for $i = 1, 2$ and let $r^\alpha = \max\{r_1^\alpha, r_2^\alpha\}$. It follows for each $\alpha > 0$ that

$$B^\alpha \overset{d}{=} \sqcup_i S_i^\alpha \circ B^{\frac{\alpha}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as above, and then after conditioning on $\mathbb{S}$, $B^{\frac{\alpha}{2r_1^2}(1)} \overset{d}{=} B^{\frac{\alpha}{2r_1^2}}$ and $B^{\frac{\alpha}{2r_2^2}(2)} \overset{d}{=} B^{\frac{\alpha}{2r_2^2}}$ are chosen independently of one another.

Thus the family of constrained Brownian motion $\{B^\alpha | \alpha > 0\}$ satisfies the family of scaling laws $\mathbb{S} = \{\mathbb{S}^\alpha | \alpha > 0\}$.

3. Fractal stochastic process

In this section we generelize the notion of random scaling law. Let p^α be a random point in $\mathbb{R}$ with distribution $N(0, \frac{\alpha}{2})$ and denote $I = [a, b]$. Let $S_1^\alpha, S_2^\alpha : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterized by $S_1^\alpha(a) = a$, $S_1^\alpha(b) = S_2^\alpha(a) = p^\alpha$, $S_2^\alpha(b) = b$. Define $\Phi_i : I \to I_i, \quad i = 1, 2$ increasing Lipschitz maps, such that $I_1 \cup I_2 = I$ and $\overset{\circ}{I}_1 \cap \overset{\circ}{I}_2 = \emptyset$.

The *generalized random scaling law* is a family of scaling laws

$$\mathbb{S} = \{\mathbb{S}^\alpha | \alpha > 0\}.$$

If $f^{\omega,\alpha}(t) = f^\omega(\alpha, t) :]0, \infty[\times I \to \mathbb{R}$ is a stochastic process, then the stochastic process $(\mathbb{S}f)^\alpha$ is defined up to probability distribution by

$$(\mathbb{S}f)^\alpha \overset{d}{=} \sqcup_i S_i^\alpha \circ f^{\frac{\alpha}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as before, and then after conditioning on $\mathbb{S}$, $f^{\frac{\alpha}{2r_1^2}(1)} \overset{d}{=} f^{\frac{\alpha}{2r_1^2}}$ and $f^{\frac{\alpha}{2r_2^2}(2)} \overset{d}{=} f^{\frac{\alpha}{2r_2^2}}$ are chosen independently of one another.

The family of stochastic processes or random functions f^α *satisfies the generalized scaling law* $\mathbb{S}$ or is a *fractal stochastic process* if

$$(\mathbb{S}f)^\alpha \overset{d}{=} f^\alpha.$$

The next theorem is essentially proved in [2]:

Theorem 3.1. *(Hutchinson-Rüschendorf, 2000) Let* $\mathbb{S} = \{\mathbb{S}^\alpha | \alpha > 0\}$ *be a generalized scaling law. Then there exists a family of stochastic processes (or random functions)* $f^{\omega,\alpha}(t) = f^\omega(\alpha, t) :]0, \infty[\times I \to \mathbb{R}$ *with*

$$\sup_\alpha \alpha^{\frac{1}{2}} E_\omega \int_I |f^\omega(\alpha, t) dt < \infty$$

which satisfies $\mathbb{S}$.

Using contraction method in probabilistic metric spaces, we can weak the first moment condition in the above theorem:

Theorem 3.2. *Denote* $\mathcal{E}^\alpha$ *the set of random functions* $g^\alpha : \Omega \times I \to \mathbb{R}$ *with the next property: there exists* $h^\alpha \in \mathcal{E}^\alpha$ *and a positive number* γ *such that*

$$P(\{\omega \in \Omega | \sup_\alpha \alpha^{-\frac{1}{2}} \int_I |h^\alpha(x)| dx \geq t\}) \leq \frac{\gamma}{t}$$

for all $t > 0$.

Then there exists a family of stochastic processes $g* \in \mathcal{E}^\alpha$ *satisfying* $\mathbb{S}$.

Proof. Let $f : \mathcal{E}^\alpha \to \mathcal{E}^\alpha$ defined by

$$f(g^\alpha) = (\mathbb{S}g)^\alpha = \sqcup_i S_i^\alpha \circ g^{\frac{\alpha}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as in the previous section, and then after conditioning on $\mathbb{S}$, $g^{\frac{\alpha}{2r_i^2}(i)} \overset{d}{=} g^{\frac{\alpha}{2r_i^2}}$, $i = 1, 2$ are chosen independently of one another.

We first claim that, if $g^\alpha \in \mathcal{E}^\alpha$ then $f(g^\alpha) \in \mathcal{E}^\alpha$ also. For this, choose $g^{\frac{\alpha}{2r_i^2}(i)} \overset{d}{=} g^{\frac{\alpha}{2r_i^2}}$, $i = 1, 2$, independently of one another and $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$. Then, for $t > 0$,

$$P(\{\omega \in \Omega | \sup_\alpha \alpha^{-\frac{1}{2}} \int_I |(\mathbb{S}h)^\alpha(x)|dx \geq t\}) \leq$$

$$\leq P(\{\omega \in \Omega | \frac{1}{2} \sup_\alpha \alpha^{-\frac{1}{2}} \sum_{i=1}^{2} r_i^\alpha \int_{I_i} |h^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}(x)|dx \geq t\}) \leq \frac{\gamma\sqrt{2}}{t}.$$

To establish the contraction property let us consider $g_1^\alpha, g_2^\alpha \in \mathcal{E}^\alpha$. Because

$$F_{f(g_1^\alpha),f(g_2^\alpha)}(t) \geq F_{g_1^\alpha,g_2^\alpha}(\frac{t}{\sqrt{2}})$$

for all $t > 0$, f is a contraction. Then we can apply Corollary 1.1 and existence and uniqueness follows.

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