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# FIXED POINT THEOREMS FOR OPERATORS ON CARTESIAN PRODUCT SPACES AND APPLICATIONS

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**Abstract.** We present some abstract fixed point theorems for operators $f : X^k \to X$. We give some results concerning data dependence of fixed points set for a this class of operators defined on cartesian product of metric spaces, which are (weakly) Picard operators.
**Keywords**: (Weakly) Picard operators, c-(weakly) Picard operators, (c)-comparison function, fixed point, data dependence.
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## 1. Introduction

Let $(X, d)$ be a metric space and $f : X^k \to X$. For $\bar{x} = (x_0, \ldots, x_{k-1}) \in X^k$, we can construct the sequences

$$\begin{array}{l} y_0 = f(x_0, \ldots x_{k-1}), \\ y_1 = f(y_0, \ldots, y_0), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ y_{n+1} = f(y_n, \ldots, y_n). \end{array} \tag{1}$$

$$x_{n+k} = f\left(x_n, \ldots, x_{n+k-1}\right), \quad n \in \mathbb{N}. \tag{2}$$

$$\begin{array}{c} \tilde{f} : X \to X \\ \tilde{f}\left(x\right) = f\left(x, \ldots, x\right)) \end{array} \tag{3}$$

$$\begin{array}{c} A_f : X^k \to X^k \\ (u_1, \ldots, u_k) \longmapsto (u_2, \ldots, u_k, f(u_1, \ldots, u_k)). \end{array} \tag{4}$$

For (1) and for (2) we have:

$$\begin{array}{rcl} y_{n+1} & = & \tilde{f}^n\left(y_0\right) \\ (x_{n+1}, \ldots, x_{n+k}) & = & A_f^n\left(x_0, \ldots, x_{k-1}\right) \end{array}$$

**Definition 1.1.** *(I.A. Rus* [8]*). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) Picard operator (PO) if $\exists\ x^* \in X$ sucht that:*

(a) $F_A = \{x^*\}$,
(b) $(A^n(x))_{n\in N}$ *converges (uniformly) to* $x^*$, $\forall\ x \in X$.

**Definition 1.2.** *(I.A. Rus [8]). Let (X,d) be a metric space. An operator $A : X \to X$ is (uniformly) weakly Picard operator (WPO) if:*

(a) *the sequence* $(A^n(x))_{n\in N}$ *converges (uniformly),* $\forall\ x \in X$,
(b) *the limit (which may depend on x) is a fixed point of A.*

If A is weakly Picard operator then we consider the following operator:

$$A^\infty : X \to X,$$
$$A^\infty(x) = \lim_{n\to\infty} A^n(x).$$

**Definition 1.3.** *(I.A. Rus [5]) Let (X,d) be a metric space. An operator $A : X \to X$ is c-(uniformly) weakly Picard operator (c-WPO) if:*

(a) *A is (uniformly) weakly Picard;*
(b) $\exists\ c > 0$ *such that.:*

$$d\left(x, A^\infty(x)\right) \le c \cdot d\left(x, A(x)\right), \tag{5}$$

$\forall\ x \in X$.

## 2. Comparison functions and (c)-comparison function

**Definition 2.1.** *(I.A. Rus [7]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if:*

(a) *$\varphi$ is monotone increasing:*
(b) $(\varphi^n(t))_{n\in N}$ *converges to 0, as $n \to \infty$, $\forall\ t$.*

We are interested to find that comparison functions which satisfies the condition:

$$\sum_{k=0}^{\infty} \varphi^k(t) < \infty. \tag{6}$$

V. Berinde in [2] give a necessary and sufficient result for the convergence of the series of decreasing positive terms. Using this result he introduce the following notion:

**Definition 2.2.** *(V. Berinde [1], [2]) A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called (c)-comparison function if the following condition hold:*

(a) *$\varphi$ is monotone increasing:*
(b) *there exist two numbers $k_0$, $\alpha$, $0 < \alpha < 1$, and a conv. series of nonnegative terms $\sum\limits_{i=0}^{\infty} v_k$ such that.:*

$$\varphi^{k+1}(t) \le \alpha\varphi^k(t) + v_k, \quad \forall t \quad and \quad k \ge k_0$$

**Theorem 2.1.** *(V. Berinde [1], [3]) If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function then:*

(i) *$\varphi(t) < t$, for each $t > 0$;*
(ii) *$\varphi$ is continuous in 0;*

(iii) *the series* $\sum_{k=0}^{\infty} \varphi^k(t)$ *converges* $\forall\, t \in \mathbb{R}_+$;

(iv) *the sum of the series (1), s(t), is monotone increasing and continous in 0;*

(v) $(\varphi^n(t))_{n\in N}$ *converges to 0, as* $n \to \infty$, $\forall\, t$.

**Definition 2.3.** *(V. Berinde* [1], [3]*)*$\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ *is called a k-dimensional (c)-comparison function if:*

(a) $\varphi(u) \le \varphi(v)$ , $\forall\, u, v \in \mathbb{R}^k$, $u \le v$;

(b) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *define by* $\psi(t) = \varphi(t, \ldots, t)$ *is a (c)-comparison function.*

## 3. Fixed point theorems

In this section we present some general fixed point theorems for the operator defined on cartesian product spaces.

**Theorem 3.1.** *(M. A. Şerban* [11]*) Let* $(X, d)$ *be a complete metric space,* $f : X^k \to X$. *Suppose that*

(i) $\exists\, \varphi_1 : \mathbb{R}_+^k \to \mathbb{R}_+$ *k-dimensional (c)-comparison function such that.:*

$$d\left(f\left(f(\bar{x}), \ldots, f(\bar{x})\right), f(\bar{x})\right) \le \varphi_1\left(d(x_1, f(\bar{x})), \ldots, d(x_k, f(\bar{x}))\right) \tag{7}$$

$\bar{x} = (x_1, \ldots, x_k) \in X^k$;

(ii) $\exists\, \varphi_2 : \mathbb{R}_+^{5k} \to \mathbb{R}_+$ *continuous function such that.*

$$\begin{aligned} d\left(f(\bar{x}), f(\bar{y})\right) \le\ & \varphi_2(d(x_1, f(\bar{x})), \ldots, d(x_k, f(\bar{x})),\\ & d(y_1, f(\bar{y})), \ldots, d(y_k, f(\bar{y})),\\ & d(x_1, f(\bar{y})), \ldots, d(x_k, f(\bar{y})),\\ & d(y_1, f(\bar{x})), \ldots, d(y_k, f(\bar{x})),\\ & d(x_1, y_1), \ldots, d(x_k, y_k)), \end{aligned} \tag{8}$$

$\forall\ \bar{x} = (x_1, \ldots, x_k), \bar{y} = (y_1, \ldots, y_k) \in X^k$;

(iii) *for* $r \in \mathbb{R}_+$, *if:* $r \le \varphi_2\left(\underbrace{0, \ldots, 0}_{k}, \underbrace{r, \ldots, r}_{2k}, \underbrace{0, \ldots, 0}_{2k}\right)$ *then* $r = 0$.

*Then:*

(a) $\tilde{f} : X \to X$, *defined by (3), is WPO and:*

$$d\left(\tilde{f}^n(x), \tilde{f}^\infty(x)\right) \le \psi^n\left(\tau_{d(x, \tilde{f}(x))}\right), \qquad \forall x \in X, \tag{9}$$

*where* $\psi(t) = \varphi_2(t, t, \ldots, t)$, $\tau_{d(x,\tilde{f}(x))} = \sup\left\{t : t - \psi(t) \le d(x, \tilde{f}(x))\right\}$;

(b) $\tilde{f} : X \to X$ *is c-WPO with constant:*

$$c = \tau_{d(x,\tilde{f}(x))}.$$

*If, additionaly, the following condition holds*

(iv) *for* $r \in \mathbb{R}_+$, *if :*

$$r \le \varphi_2\left(\underbrace{0, \ldots, 0}_{2k}, \underbrace{r, \ldots, r}_{3k}\right) \quad \text{then} \quad r = 0. \tag{10}$$

*then $\tilde{f}$ is Picard and c-Picard operator.*

**Proof.** We consider the sequence $(y_n)_{n\in\mathbb{N}}$, defined by (1), and we have that

$$y_n = \tilde{f}^n(y_0).$$

Condition (7) implies that

$$d\left(\tilde{f}^2(x), \tilde{f}(x)\right) \leq \psi\left(d(x, \tilde{f}(x))\right),$$

for any $x \in X$. Therefore we obtaine that:

$$d(y_n, y_{n+p}) \leq \psi^n\left(\tau_{d(y_0,y_1)}\right), \tag{11}$$

which shows that $(y_n)_{n\in\mathbb{N}}$ is fundamental, so is convergent to a point $x^* \in X$.

We'll proof that $x^* \in F_f$. We have

$$\begin{aligned} & d(x^*, f(x^*,\ldots,x^*)) \leq d(x^*, y_n) + d(y_n, f(x^*,\ldots,x^*)) \leq \\ \leq\ & d(x^*, y_n) + \varphi_2(d(y_{n-1}, y_n),\ldots,d(y_{n-1}, y_n), \\ & \qquad d(x^*, f(x^*,\ldots,x^*)),\ldots,d(x^*, f(x^*,\ldots,x^*)), \\ & \qquad d(y_{n-1}, f(x^*,\ldots,x^*)),\ldots,d(y_{n-1}, f(x^*,\ldots,x^*)), \\ & \qquad d(x^*, y_n),\ldots,d(x^*, y_n), d(x^*, y_{n-1}),\ldots,d(x^*, y_{n-1})), \end{aligned}$$

making $n \to \infty$ we obtaine:

$$d(x^*, f(x^*,\ldots,x^*)) \leq$$
$$\leq \varphi_2\left((\underbrace{0,\ldots,0}_{k}, \underbrace{d(x^*, f(x^*,\ldots,x^*)),\ldots,d(x^*, f(x^*,\ldots,x^*))}_{2k}, \underbrace{0,\ldots,0}_{2k}\right)$$

which implies $d(x^*, f(x^*,\ldots,x^*)) = 0$. The estimation (9) can be obtained from (11) making $p \to \infty$.

The fact that $\tilde{f}$ is c-WPO can be deduced from the (9) choosing $n = 0$.

Using condition (iv) we obtaine the uniqueness of the fixed point and therefore that $\tilde{f}$ is Picard and c-Picard operator. □

**Theorem 3.2.** *Let $(X, d)$ be a complete metric space, $f : X^k \to X$ . Suppose that $\exists$ $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ such that.:*

(i) *$\varphi$ k-dimensional (c)-comparison function;*

(ii) *$\forall\ x_0,\ldots,x_{k-1},x_k \in X$ we have*

$$\begin{aligned} & d(f(x_0,\ldots,x_{k-1}), f(x_1,\ldots,x_k)) \leq \\ & \leq \varphi(d(x_0,x_1),\ldots,d(x_{k-1},x_k)); \end{aligned} \tag{12}$$

(iii) *$\forall\ r \in \mathbb{R}_+$ we have:*

$$\varphi(r,0,\ldots,0) + \varphi(0,r,0,\ldots,0) + \ldots + \varphi(0,\ldots,0,r) \leq \varphi(r,\ldots,r).$$

*Then:*

(a) *operator* $\tilde{f} : X \to X$, *defined by (3), is PO:*

$$d(y_n, x^*) \leq \sum_{i=n}^{\infty} \psi^i (d(y_0, y_1)),$$

*where* $(y_n)_{n\in\mathbb{N}}$ *is defined by (1),* $\forall\, \bar{x} = (x_0, \ldots, x_{k-1}) \in X^k$ *and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\psi(r) = \varphi(r, \ldots, r).$$

(b) *operator* $A_f : X^k \to X^k$, *defined by (4), is PO and:*

$$d(x_n, x^*) \leq k \cdot \sum_{i=0}^{\infty} \psi^{\left[\frac{n}{k}\right]+i} (d_0), \tag{13}$$

*where* $(x_n)_{n\in\mathbb{N}}$ *is defined by (2),* $d_0 = \max\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\}$;

(c) *if* $\psi$ *is positive semihomogeneouse* $\Longrightarrow \tilde{f}$ *is c-PO with*

$$c = \sum_{i=0}^{\infty} \psi^i (1);$$

(d) *if* $\psi$ *is positive semihomogeneouse* $\Longrightarrow A_f$, *is c-PO with*

$$c = k \cdot \sum_{i=0}^{\infty} \psi^i (1).$$

**Proof. (a)** Using conditions (ii) şi (iii) we obtaine:

$$\begin{aligned}
&d\left(\tilde{f}(x), \tilde{f}(y)\right) = d(f(x, \ldots, x), f(y, \ldots, y)) \leq \\
&\leq d(f(x, \ldots, x), f(x, \ldots, x, y)) + \ldots + d(f(x, y, \ldots, y), f(y, \ldots, y)) \leq \\
&\leq \varphi(0, \ldots, 0, d(x, y)) + \varphi(0, \ldots, 0, d(x, y), 0) + \ldots + \varphi(d(x, y), 0, \ldots, 0) \leq \\
&\leq \varphi(d(x, y), \ldots, d(x, y))) = \psi(d(x, y))),
\end{aligned}$$

for any $x, y \in X$, which proves that $\tilde{f} : X \to X$ is a $\psi-$contraction with $\psi$ a (c)-comparison function. This implies that $F_{\tilde{f}} = F_f = \{x^*\}$ and $\tilde{f}$ is a Picard operator (see V. Berinde [2]).

**(b)** We consider the sequence $(x_n)_{n\in\mathbb{N}}$, defined by (2). From condition (ii) we have

$$d(x_k, x_{k+1}) = d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \varphi(d_0, \ldots, d_0) < d_0$$

$$\begin{aligned}
d(x_{k+1}, x_{k+2}) &= d(f(x_1, \ldots, x_k), f(x_2, \ldots, x_{k+1})) \leq \\
&\leq \varphi(d(x_1, x_2), \ldots, d(x_k, x_{k+1})) \leq \varphi(d_0, \ldots, \psi(d_0)) \leq \\
&\leq \varphi(d_0, \ldots, d_0) < d_0
\end{aligned}$$

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$$\begin{aligned}
d(x_{2k-1}, x_{2k}) &\leq \varphi(d(x_{k-1}, x_k), \ldots, d(x_{2k-2}, x_{2k-1})) \leq \\
&\leq \varphi(d_0, \psi(d_0), \ldots, \psi(d_0)) < d_0
\end{aligned}$$

$$\begin{aligned}
d(x_{2k}, x_{2k+1}) &\leq \varphi(d(x_k, x_{k+1}), \ldots, d(x_{2k-1}, x_{2k})) \leq \\
&\leq \varphi(\psi(d_0), \psi(d_0), \ldots, \psi(d_0)) = \psi^2(d_0) < \psi(d_0)
\end{aligned}$$

We can prove by induction that:

$$d\left(x_n, x_{n+1}\right) \leq \psi^{\left[\frac{n}{k}\right]}(d_0) < \psi^{\left[\frac{n}{k}\right]-1}(d_0), \qquad n \geq k,$$

and thus:

$$d\left(x_{n+p}, x_n\right) \leq k \cdot \sum_{i=0}^{\infty} \psi^{\left[\frac{n}{k}\right]+i}(d_0), \qquad n \geq k, \quad p \in \mathbb{N},$$

which implies that $(x_n)_{n\in\mathbb{N}}$ is Cauchy sequence, so it is convergent. Let $x^* = \lim\limits_{n\to\infty} x_n$, using condition (ii) and the fact that function $\varphi$ is continuous in 0 we obtaine $x^* \in F_f$, the uniqueness of $x^*$ can be deduced from condition (iii). The fact that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to the unique fixed point of the operator $f$ implies that the sequence $(y^n)_{n\in\mathbb{N}} \subset X^k$, defined by

$$\begin{aligned}
&y^0 = (x_0, x_1, ..., x_{k-1})\\
&y^1 = (x_1, x_2, ..., x_k)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&y^n = (x_n, x_{n+1} \ldots, x_{n+k-1}),
\end{aligned}$$

is convergent to the unique fixed point of the operator $A_f : X^k \to X^k$, so $A_f$ is a Picard operator.

**(c)** The conclusion (c) can be obtained from the conlusion that $\tilde{f}$ is a $\psi-$contraction and $\psi$ a (c)-comparison function positive semihomogeneouse.

**(d)** To prove that $A_f : X^k \to X^k$ is a c-Picard operator we consider the metric space $\left(X^k, \sigma\right)$ where

$$\sigma\left((x_1, \ldots, x_k), (y_1, \ldots, y_k)\right) = \max\left\{d\left(x_1, y_1\right), \ldots, d\left(x_k, y_k\right)\right\}.$$

For $y^0 = (x_0, \ldots, x_{k-1})$ and $y^* = (x^*, \ldots, x^*)$ we have:

$$\sigma\left(y^0, y^*\right) = \max\left\{d\left(x_1, x^*\right), \ldots, d\left(x_{k-1}, x^*\right)\right\}$$

Using relation (13) we obtaine:

$$\sigma\left(y^0, y^*\right) \leq k \cdot \max_{j=\overline{0,k-1}} \left\{\sum_{i=0}^{\infty} \psi^{\left[\frac{j}{k}\right]+i}\left(d_0\right)\right\} = k \cdot \sum_{i=0}^{\infty} \psi^i\left(d_0\right).$$

Since $\psi$ is positive semihomogeneouseeste then:

$$\sigma\left(y^0, y^*\right) \leq k \cdot \sum_{i=0}^{\infty} \psi^i\left(1\right) \cdot d_0 = k \cdot \sum_{i=0}^{\infty} \psi^i\left(1\right) \cdot \sigma\left(y^0, A_f(y^0)\right),$$

thus operator $A_f$ is a c-Picard operator with $c = k \cdot \sum\limits_{i=0}^{\infty} \psi^i\left(1\right)$.$\square$

## 4. Data dependence of the fixed point set

In this section we present some results concerning data dependence of the fixed point set for two operators $f_1, f_2 : X^k \to X$ from the perspective of I.A.Rus, S. Mureşan result.

**Theorem 4.1.** *(I.A.Rus, S. Mureşan* [6]*) Let $(X,d)$ be a metric space. and $A_1, A_2 : X \to X$ two operator such that.:*

(i) *$A_i$ is $c_i-$WPO, $i = \{1,2\}$;*
(ii) *$\exists\ \eta > 0$ s. t.: $d(A_1(x), A_2(x)) \leq \eta, \qquad \forall x \in X$*

*Then:*

$$H(F_{A_1}, F_{A_2}) \leq \eta \cdot \max\{c_1, c_2\},$$

*where $H$ is Hausdorff-Pompeiu metric on $P(X)$.*

Using Theorem 4.1 we can give two data dependence of the fixed point set results.

**Theorem 4.2.** *Let $(X,d)$ be a complete metric space, $f_1, f_2 : X^k \to X$. Suppose that*

(i) *$\exists\ \varphi, \phi : \mathbb{R}_+^k \to \mathbb{R}_+$ k-dimensional (c)-comparison functions such that.:*

$$d(f_1(f_1(\bar{x}), \ldots, f_1(\bar{x})), f_1(\bar{x})) \leq \varphi(d(x_1, f_1(\bar{x})), \ldots, d(x_k, f_1(\bar{x})))$$

$$d(f_2(f_2(\bar{x}), \ldots, f_2(\bar{x})), f_2(\bar{x})) \leq \phi(d(x_1, f_2(\bar{x})), \ldots, d(x_k, f_2(\bar{x})))$$

$\bar{x} = (x_1, \ldots, x_k) \in X^k$;

(ii) *$\exists\ \varphi_1, \varphi_2 : \mathbb{R}_+^{5k} \to \mathbb{R}_+$ continuous function such that*

$$\begin{aligned} d(f_i(\bar{x}), f_i(\bar{y})) \leq\ & \varphi_i(d(x_1, f_i(\bar{x})), \ldots, d(x_k, f_i(\bar{x})), \\ & d(y_1, f_i(\bar{y})), \ldots, d(y_k, f_i(\bar{y})), \\ & d(x_1, f_i(\bar{y})), \ldots, d(x_k, f_i(\bar{y})), \\ & d(y_1, f_i(\bar{x})), \ldots, d(y_k, f_i(\bar{x})), \\ & d(x_1, y_1), \ldots, d(x_k, y_k)), \end{aligned}$$

$\forall\ \bar{x} = (x_1, \ldots, x_k), \bar{y} = (y_1, \ldots, y_k) \in X^k$, $i = \{1,2\}$;

(iii) *for $r \in \mathbb{R}_+$, if: $r \leq \varphi_i\left(\underbrace{0, \ldots, 0}_{k}, \underbrace{r, \ldots, r}_{2k}, \underbrace{0, \ldots, 0}_{2k}\right)$ then $r = 0$, $i = \{1,2\}$;*

(iv) *there exists $\eta > 0$ such that:*

$$d(f_1(x, \ldots, x), f_2(x, \ldots, x)) \leq \eta,$$

*for all $x \in X$.*

*Then:*

$$H\left(F_{\tilde{f}_1}, F_{\tilde{f}_2}\right) \leq \eta \cdot \max\left\{\tau^1_{d(x,\tilde{f}_1(x))}, \tau^2_{d(x,\tilde{f}_2(x))}\right\}$$

*where*

$$\tau^1_{d(x,\tilde{f}_1(x))} = \sup\left\{t : t - \varphi(t, ..., t) \leq d(x, \tilde{f}_1(x))\right\},$$

$$\tau^2_{d(x,\tilde{f}_1(x))} = \sup\left\{t : t - \phi(t, ..., t) \leq d(x, \tilde{f}_2(x))\right\}.$$

**Proof.** Conditions (i)-(iii) implies that $f_1, f_2$ are c-WPO with $c_1 = \tau^1_{d(x,\tilde{f}_1(x))}$ and $c_2 = \tau^2_{d(x,\tilde{f}_1(x))}$ and from condition (iv) and Theorem 4.1 we obtaine the conclusion of this theorem.□

**Theorem 4.3.** *Let $(X, d)$ be a complete metric space, $f_1, f_2 : X^k \to X$ . Suppose that $\exists\ \varphi_1, \varphi_2 : \mathbb{R}^k_+ \to \mathbb{R}_+$ such that.:*

(i) *$\varphi_i$ are k-dimensional (c)-comparison functions, $i = \{1, 2\}$, with $\psi_1, \psi_2$ positive semihomogeneouse, where*
$$\psi_i(r) = \varphi_i\left(r, \ldots, r\right), \qquad i = \{1, 2\}\,;$$

(ii) *$\forall\ x_0, \ldots, x_{k-1}, x_k \in X,\ i = \{1, 2\}$, we have*
$$d\left(f_i(x_0, \ldots, x_{k-1}), f_i(x_1, \ldots, x_k)\right) \leq$$
$$\leq \varphi_i\left(d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\right);$$

(iii) *$\forall\ r \in \mathbb{R}_+,\ i = \{1, 2\}$, we have:*
$$\varphi_i\left(r, 0, \ldots, 0\right) + \varphi_i\left(0, r, 0, \ldots, 0\right) + \ldots + \varphi_i\left(0, \ldots, 0, r\right) \leq \varphi_i\left(r, \ldots, r\right);$$

(iv) *there exists $\eta > 0$ such that:*
$$d\left(f_1\left(x, \ldots, x\right), f_2\left(x, \ldots, x\right)\right) \leq \eta,$$
*for all $x \in X$.*

*Then:*
$$d\left(x_1^*, x_2^*\right) \leq \eta \cdot \max\left\{\sum_{i=0}^{\infty} \psi_1^i\left(1\right), \sum_{i=0}^{\infty} \psi_2^i\left(1\right)\right\},$$
*where $F_{f_1} = \{x_1^*\}$, respectively $F_{f_2} = \{x_2^*\}$.*

**Proof.** Conditions (i)-(iii) implies that $f_1, f_2$ are c-PO with $c_1 = \sum\limits_{i=0}^{\infty} \psi_1^i\left(1\right)$ and $c_2 = \sum\limits_{i=0}^{\infty} \psi_2^i\left(1\right)$ and from condition (iv) and Theorem 4.1 we obtaine the conclusion of this theorem.□

## 5. Applications

In this section we present the most used corollaries of the abstract fixed point theorems in proving the existence and uniqueness of the solution for integral equations and differential equations.

**Corollary 5.1.** *Let $(X, d)$ be a complete metric space and $f : X^k \to X$ such that there exist $q_i \in \mathbb{R}_+$, $i = \overline{1, k}$, with $\alpha = \sum\limits_{i=1}^{k} q_i < 1$, such that:*
$$d\left(f(\bar{x}), f(\bar{y})\right) \leq \sum_{i=1}^{k} q_i d(x_i, y_i),$$
*for any $\bar{x} = (x_1, \ldots, x_k)$, $\bar{y} = (y_1, \ldots, y_k) \in X^k$. Then*

(a) *the operator $\tilde{f} : X \to X$, defined by (3), is a Picard operator , ( $F_f = \{x^*\}$);*

(b) *the sequence* $(y_n)_{n\in\mathbb{N}}$ *, defined by (1), is convergent,* $y_n \to x^*$*,* $n \to \infty$*, and we have:*

$$d(y_n, x^*) \le \frac{\alpha^n}{1-\alpha} \cdot d_0,$$

*for any arbitrary* $\bar{x} = (x_1, \ldots, x_k) \in X^k$ *where* $d_0 = \max\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\}$*;*

(c) $\tilde{f} : X \to X$ *is a c-Picard operator with the constant:* $c = \frac{1}{1-\alpha}$*;*

(d) *the operator* $A_f : X^k \to X^k$*, defined by (4), is a Picard operator and we have:*

$$d(x_n, x^*) \le k \cdot d_0 \cdot \frac{\alpha^{\left[\frac{n}{k}\right]}}{1-\alpha},$$

*where* $(x_n)_{n\in\mathbb{N}}$ *is defined by (2), and ;*

(e) *the operator* $A_f : X^k \to X^k$ *is a c-Picard operator with the constant* $c = k \cdot \frac{1}{1-\alpha}$*.*

**Proof.** The conclusion can be obtained from Theorem 3.1 for

$$\varphi_2(r_1, ..., r_{5k}) = \sum_{i=1}^{k} q_i r_{4i+1}.$$

Here

$$(r_1, ..., r_5) = \sum_{i=1}^{k} q_i r_i.$$

The functions $\varphi_1$ and $\varphi_2$ satisfy the condition (i)-(iv) from the Theorem 3.1 and therefore we have the conclusion.□

**Corollary 5.2.** *Let* $(X, d)$ *be a complete metric space and* $f : X^k \to X$ *such that there exist* $q_i \in \mathbb{R}_+$*,* $i = \overline{1,k}$*, with* $\alpha = \sum_{i=1}^{k} q_i < 1$ *such that:*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \le \sum_{i=1}^{k} q_i d(x_{i-1}, x_i),$$

*for any* $x_0, x_1, \ldots, x_k \in X$*. Then*

(a) *the operator* $\tilde{f} : X \to X$*, defined by (3), is a Picard operator , (* $F_f = \{x^*\}$*);*

(b) *the sequence* $(y_n)_{n\in\mathbb{N}}$ *, defined by (1), is convergent,* $y_n \to x^*$*,* $n \to \infty$*, and we have:*

$$d(y_n, x^*) \le \frac{\alpha^{n+1}}{1-\alpha} \cdot \max_{i=\overline{1,k}} \{d(x_i, f(\bar{x}))\},$$

*for any arbitrary* $\bar{x} = (x_1, \ldots, x_k) \in X^k$*;*

(c) $\tilde{f} : X \to X$ *is a c-Picard operator with the constant:* $c = \frac{1}{1-\alpha}$*;*

(d) *the operator* $A_f : X^k \to X^k$*, defined by (4), is a Picard operator and we have:*

$$d(x_n, x^*) \le k \cdot d_0 \cdot \frac{\alpha^{\left[\frac{n}{k}\right]}}{1-\alpha},$$

*where* $(x_n)_{n\in\mathbb{N}}$ *is defined by (2), and* $d_0 = \max\{d(x_0, x_1), \ldots, d(x_{k-1}, x_k)\}$*;*

(e) *the operator* $A_f : X^k \to X^k$ *is a c-Picard operator with the constant* $c = k \cdot \frac{1}{1-\alpha}$.

**Proof.** The conclusion can be obtained from Theorem 3.2 for

$$\varphi(r_1, ..., r_k) = \sum_{i=1}^{k} q_i r_i.$$

The function $\varphi$ satisfies the condition (i)-(iii) from the Theorem 3.2 and thus we have the conclusion.□

**Remark 5.1.** *In Corollaries 5.1 and 5.2 we can replace the "sum" condition with the "max" condition, i.e.*

$$d(f(\bar{x}), f(\bar{y})) \leq \alpha \cdot \max_{i=\overline{1,k}} \{d(x_i, y_i)\}$$

*and*

$$d(f(x_0, \ldots, x_{k-1}), f(x_1, \ldots, x_k)) \leq \alpha \cdot \max_{i=\overline{1,k}} \{d(x_{i-1}, x_i)\},$$

*and we will obtaine the same theorems.*

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