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BIFURCATION RESULTS WITH MONOTONE NONLINEARITIES

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Abstract. Let X be a reflexive uniformly convex Banach space, X^* be its dual space and let $J : X \to X^*$ be the normalized duality mapping. Consider the eingenvalue problem

$$Jx + \mu Ax + R(\mu, x) = 0$$

where A and $R(\mu, \cdot)$ are (weakly) continuous mappings, generally nonlinear.

When A is a linear map, bounded from bellow, and R is nonlinear and asymptotical zero we can prove local and global bifurcation properties similar to those for compact maps, (e.g. Krasnoselskii and Rabinowitz theorems).

When A is a (nonlinear) maximal monotone map and $R(\mu, x) := \mu C(x)$ with C a compact map, we can define a new coincidence degree for the pair (A, C) and establish some existence results.

Abstract setting

Let X be a reflexive uniformly convex Banach space and X^* be its dual Banach space and $J : X \to X^*$, the duality mapping that in our case is strictly monotone and uniformly continuous operator on bounded set of X.

Consider the eingenvalue problem

$$Jx + \mu Ax + R(\mu, x) = 0 \tag{1}$$

where $A : X \to X^*$ is a linear continuous operator and $R(\mu, \cdot) : X \to X^*$ is a nonlinear perturbation such that $R(\mu, 0) = 0,\ \forall \mu \in \mathbb{R}$. In this case $(\mu, 0)$ are solutions of (1) for all $\mu \in \mathbb{R}$-named *trivial solutions* and the set of all trivial solutions are denoted by $\mathcal{C}$.

A point $(\mu_0, 0) \in \mathcal{C}$ is said to be a *bifurcation point* for (1) provided that there exist solutions $(\mu, x_\mu), x_\mu \neq 0$ in each neighborhood of $(\mu_0, 0)$. Let us denote by S_0 the set all of these nontrivial solutions and let $S := \overline{S}_0$ be its adherence in $\mathbb{R} \times X$.

The key step in our extension is the Browder-Ton theorem concerning the compact imbedding property for separable Banach spaces (e.g. D. Pascali, S. Sburlan p[1, p.302]) namely:

Let X be a separable reflexive Banach space and let S be a countable subset of X. Then there exists a separable Hilbert space H and a compact one-to-one linear operator $\psi : H \to X$ such that $S \subset \psi(H)$ and $\psi(H)$ is dense in X.

Define now the adjoint operator $\phi : X^* \to H$ using the inner product of H denoted by $(\cdot,\cdot)$:

$$(2) \qquad < \phi(u), v >= (u, \psi(v)),\ \forall u \in H,\ v \in X^*.$$

Then we have the following scheme $H \overset{\psi}{\to} X \underset{R(\mu,\cdot)}{\overset{A}{\to}} X^* \overset{\phi}{\to} H$ and the operator $L := -\psi\phi A : X \to X$ is linear and compact as the composition of a continuous map with a compact one. Since the spectrum $\sigma(L)$ is a discrete we can choose $\delta > 0$ such that $\sigma(L) \cap (I_1 \cup I_2) = \emptyset$, where $I_1 := (\varepsilon\mu_0 - \delta, \varepsilon\mu_0)$ and $I_2 := (\varepsilon\mu_0, \varepsilon\mu_0 + \delta)$ and $\mu_0 \in \sigma(L)$.

Suppose that A is a bounded from below in the sense

$$\text{(i)} \qquad < Ax, x >\geq -\frac{\varepsilon}{\varepsilon\mu_0 + \delta}||x||^2,\ \forall x \in X$$

and the complementary part is a asimptotical zero, i.e.,

$$\text{(ii)} \qquad Jx + R(\mu, x) = o(||x||)$$

uniformly in μ on bounded sets.

Of course, when A is linear and monotone the condition (i) holds.

Reasoning by contradiction we can prove an analogous of Krasnoselskii theorem for monotone operators (see S. Sburlan [10]).

Proposition. *Let μ_0 be a characteristic value with odd algebraic multiplicity of the linear compact operator $L \in L(X)$.*

If there exist $\varepsilon, \delta > 0$ such that (i)-(ii) hold, then $(\varepsilon, \mu_0) \in \mathbb{R} \times X$ is a bifurcation point for (1).

Example: If consider $R(\mu, x) : \mu||x||^2 JX$, then $< R(\mu, x)x >=$ $= \mu||x||^2 < Jx, x >= \mu||x||^4$. The above results can be applied in Sobolev space $X := H^1(\Omega)$ for any linear elliptic operator, A, defined there. The corresponding nonlinear part $Ju + R(\mu, u)$ is $-\Delta u + \mu||u||^2\Delta u$.

This result still remain true in the general case $X := W^{1,p}(\Omega)$, but in this case J is nonlinear, namely the p-laplacian $J(u) := div(|\nabla u|^{p-2}\nabla u)$ (see e.g. D. Pascali, S. Sburlan [9, p. 127]).

Let us denote

$$i_- := d_S(J + \mu A, B, 0) = d_{LS}(I + \frac{1}{\varepsilon}\mu\psi\phi A, B, 0),\ \mu \in I_1,$$

$$i_+ := d_S(J + \mu A, B, 0) = d_{LS}(I + \frac{1}{\varepsilon}\mu\psi\phi A, B, 0),\ \mu \in I_2,$$

and observe that these degrees are constant in $\mu_1 \in I_1$ and $\mu_2 \in I_2$.

For any fixed $r > 0$ define the mapping $H_r : \mathbb{R}_+ \times X \to \mathbb{R}_+ \times X^*$ as follows

$$H_r(\mu, x) := \left(||x||^2 - r^2, Jx + \mu Ax + R(\mu, x)\right),\ \forall(\mu, x) \in \mathbb{R} \times X.$$

Since we can prove a formula similar to Ize's formula

$$(3) \qquad d_S(H_r, \mathcal{B}, 0) = i_- - i_+,$$

where $\mathcal{B} = \{(\nu, x) \in \mathbb{R} \times X \mid {}^2 + ||x||^2 < \delta^2 + r^2\}$ (see Sburlan [10], a global results concerning the bifurcation under monotonocity condition similar to those under compactness condition proved by Rabinowitz is true:

Theorem. *If $\mathcal{E}$ is a connected component containing the bifurcation point $(\varepsilon\mu_0, 0) \in S$, then we have one of the following two possibilities:*

(j) $\mathcal{E}$ is unbounded in $\mathbb{R}\times X$.

(jj) $\mathcal{E}$ contains a finite number of bifurcation points $(\varepsilon\mu_j, 0)$ where $\frac{1}{\mu_j} \in \sigma(L)$. Moreover, the number of these points, including $(\varepsilon\mu_0, 0)$, is even.

Suppose now that the operator $A : D(A) \subset X \to X^*$ is maximal monotone and $C : \bar{D} \subset X \to X^*$ is a compact one, both of them nonlinear ones. Consider an eingenvalue problem of the form

$$\lambda Jx + Ax - Cx = 0, \; \lambda > 0, \tag{4}$$

with $x \in X$. Since A is maximal monotone, there exists $(\lambda J + A)^{-1}$ and it is continuous, for every $\lambda > 0$. Equation (4) can be written as

$$(\lambda J + A)\, x = Cx \Leftrightarrow x - (\lambda J + A)^{-1} Cx = 0.$$

For each $\lambda > 0$, we set $M_\lambda = (\lambda J + A)^{-1}C$. It is easy to see that $M_\lambda : X \to X$ is compact, as the product of a continuous operator with a compact one. Therefore $I - M_\lambda : X \to X$ is a compact perturbation of the identity, so the Leray-Schauder topological degree can be considered. From the equivalence:

$$\lambda Jx + Ax - Cx = 0 \Leftrightarrow (I - M_\lambda)x = 0$$

it follows the next natural definition of the coincidence degree of the pair of nonlinear operators (A, C) :

Assume that the operator $A : D(A) \subset X \to X^*$ is maximal monotone and $C : \bar{D} \subset X \to X^*$ is compact, both of them nonlinear. If $0 \notin (\lambda J + A - C)(D(A) \cap \partial D)$, then define the coincidence degree of the pair (A, C) with respect to D by the formula:

$$d_\lambda((a, C), D) = d_{LS}(I - M_\lambda, D, 0),$$

where d_{LS} stands for the Leray-Schauder degree.

The next properties of the coincidence degree follow easily from the properties of the Leray-Schauder topological degree.

(a) **Solution property:** If $d_\lambda((A, C), D) \neq 0$, then $0 \in (\lambda J + A)(D(A) \cap D)$.

(b) **Additivity with respect to the domain:** If $D_1, D_2 \subset D$, $D_1 \cap D_2 = \emptyset$ and $0 \notin (\lambda J\_A - C)\ (D(A) \cap (D \backslash D_1 \cup D_2))$,then

$$d_\lambda((AC), D) = d_\lambda((A, C), D_1) + d_\lambda(A, C), D_2).$$

(c) **The invariance to homotopy:** Let $C_t : \bar{D} \subset X \to X^*,\ 0 \leq t \leq 1$ be compact and $A_t : D(A_t) \subset X \to X^*,\ 0 \leq t \leq 1$ be maximal monotone such that $\bigcap_{0 \leq t \leq 1} D(A_t) \neq \varnothing$.

If $0 \notin (\lambda J + a_t - C_t)(D(A_t) \cap \partial D) <$ for all $0 \leq t \leq 1$, then the coincidence degree $d_\lambda((A_t, C_t), D)$ is independent on $t \in [0,1]$.

Application

Let X be a real, reflexive Banach space. Assume, without loss of generality, that X and X^* are locally uniform convex, according to a result due to Trojanski (e.g. [3]).

In the sequel, we use the above coincidence degree to establish an existence result for the operator equations of the form:

$$Ax + Tx + Cx = y,\ y \in X^*, \tag{5}$$

where $A : D(A) \subset X \to X^*, 0 \in D(A)$ and satisfy the following hypotesis:

(i) A is bounded demicontinuous and strongly monotone with $A(0) = 0$;

(ii) T is linear, compact;

(iii) C is completely continuous;

(iv) there exists $p > 0$ and $g : \overline{B(0,1)} \subset X \to [0, \infty)$ a completely continuous function with $g(u) = 0 \Leftrightarrow u = 0$, such that

$$< Cu, u > \geq g\left(\frac{u}{||u||}\right) ||u||^{2+p}, \forall u \in X \backslash \{0\}.$$

Theorem 1. *Under the assumption (i)-(iv), for every $y \in X^*$ the equation (5) has solutions in $D(A)$.*

Proof. Let $c > 0$ be such that $> Au - Av, u - v > \geq c||u - v||^2$, for all $u \in D(A)$. Then the operator $A' : D(A) \subset X \to X^*$, defined by
$A'x = Ax - cJx,\ x \in D(A)$, is *maximal monotone*. The equation (5) can be written as: $cJx + A'x + Cx = y$.

First, we will prove that the solution set of the equation

$$cJx + A'x + tCx - ty = 0 \tag{6}$$

is uniformly bounded in $t \in [0,1]$. Indeed, let us suppose on the contrary that there exists $(x_n)_{n \in \mathbb{N}} \in X$ with $||x_n|| \to \infty$, and $t_n \in [0,1]$ such that $cJx_n + A'x_n + t_nTx_n + t_nCx_n - t_ny = 0$.

Now, we can find $\varepsilon > 0$ such that

$$g\left(\frac{x_n}{||x_n||}\right) \geq \varepsilon,\ \forall n \in \mathbb{N}. \tag{7}$$

If assume on the contrary that $g\left(\frac{x_n}{||x_n||}\right) \to 0$, then $\frac{x_n}{||x_n||} \to u_0$, eventually on a subsequence, according with Minty theorem (e.g., D. Pascali, S. Sburlan [9, p. 2]). In this case, using the fact that g is completely continuous, we obtain $g(u_0) = 0$ and thus $u_0 = 0$. Further,

$$c < Jx_nx_n > = - < A'x_n, x_n > -t_n < Tx_n, x_n > -t_n < Cx_n, x_n > +$$

$+t_n < y, x_n > \leq ||Tx_n|| \, ||x_n|| + ||y|| \;\; ||x_n||$, so $c \leq \left|\left| T\left(\dfrac{x_n}{||x_n||} \right) \right|\right| +$

$+\dfrac{||f||}{||x_n||} \to ||T(0)|| = 0$ is a contradiction. Hence (7) holds true. Now,

$c < Jx_n, x_n > +¡A'x_n, x_n > +t_n < Tx_n, x_n > +t_n < Cx_n, x_n > -$
$-t_n < y, x_n >=0$.

But A' is monotone and $A'(0) = 0$, so

$$0 \geq < tx_n, x_n > + < Cx_n, x_n > - < y, x_n > \geq g\left(\frac{x_n}{||x_n||}\right) ||x_n||^{2+p} -$$

$$-||T|| \, ||x_n||^2 - ||y|| \, ||x_n|| \geq \varepsilon ||x_n||^{2+p} - ||T|| \, ||x_n||^2 - ||y|| \, ||x_n|| \overset{n\to\infty}{\to} \infty$$

is a contradiction. In fact, we proved that there exists $R > 0$ such that the equation (6) has no solutions on $\partial B(0, R)$. Finally, we will use the invariance to homotopy $C_t(x) = tTx + tCx - ty, \; 0 \leq t \leq 1$, of the coincidence degree. As we have proved, $0 \notin (cJ + A' + C_t)(D(A) \cap \partial B(0, R))$ and consequently, the coincidence degree $d_\lambda(cJ + A' + C_t - y)(D(A) \cap B(0, R), 0)$ is independent on $t \in [0, 1]$. According to the solution property (a) and Minty theorem, the equation (3.1) has solutions. ■

By imposing usual conditions of monotonocity it can be easily obtained the unique solvability of (5).

The above methods can be applied to study:

The simplest Model Problems:

Let $g : I \times \mathbb{R}^2 \to \mathbb{R}^2$ be a bounded continuous function

$$|g(t, \xi, \eta)| \; \leq M, \; \forall t \in I, \; (\xi, \eta) \in \mathbb{R}^2$$

and consider the eingenvalue problem

$$\begin{cases} u''(t) + \lambda u(t) + g(t, u(t), u'(t)) = f(t), \; t \in I \\ B(u, t)u'(t) = 0, \; t \in \partial I, \end{cases}$$

where $I := [0, \pi] \subset \mathbb{R}$ and B denotes either Dirichlet boundary conditions $u(0) = u(\pi) = 0$ or Neumann boundary conditions $u'(0) = u'(\pi) = 0$ or periodic boundary conditions $u(0) = u(\pi), u'(0) = u'(\pi)$.

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