Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 73-98
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

IDEAS AND METHODS IN FIXED POINT THEORY FOR PROBABILISTIC CONTRACTIONS

VIOREL RADU

West University of Timişoara
Faculty of Mathematics
Bv. Vasile Parvan 4
1900 Timişoara, Romania
e-mail address radu@hilbert.math.uvt.ro

Abstract. The notion of B-contraction mapping in probabilistic metric spaces is due to **V. M. Sehgal(1966),** who proved that *any B-contraction on a complete Menger space* $(X, \mathcal{F}, Min)$ *has a unique fixed point.* Important contributions are due to *Sherwood(1970)* and *Istrăţescu & Săcuiu(1971).* A fundamental step was made by *O. Hadžić* in 1978, who introduced a class of continuous t-norms, *essentially weaker* than Min, for which the above result of Sehgal still holds.

Our aim is to present some comments and results related to the following *statements* concerning a triangular norm T:

($\mathbf{B_I}$*) T is of Hadžić type;* that is the family of its iterates is equicontinuous at $x = 1$.

($\mathbf{B_{II}}$*) T has the fixed point property;* that is each B-contraction on every complete Menger space $(X, \mathcal{F}, T)$ has a fixed point.

($\mathbf{B_{III}}$*)* $\forall a \in (0,1), \ \exists b \geq a$ *such that* $T(b,b) = b < 1$

which are seen to correspond to different kinds of classical deterministic fixed point theorems, together with the main tools used in fixed point theory for probabilistic contractions.

Keywords: Probabilistic metric space, probabilistic contraction, fixed point

AMS Subject Classification: 54E 70, 47H 10.

1. Preliminaries

1.1. Menger norms and triangular norms

Definition *1.1.1* 1^0 *A mapping*

$$T : [0,1] \times [0,1] \to [0,1] = I$$

*is called a Menger- norm (shortly **M-norm**) if it satisfies the following conditions:*

$$N1) \qquad T(a,b) = T(b,a) \ , \forall a,b \in I$$
$$N2) \quad a \leq c, \ b \leq d \Rightarrow T(a,b) \leq T(c,d)$$
$$N3) \qquad T(a,1) = a, \ (\forall) a \in I.$$

2^0 *A triangular norm (shortly **t-norm**) is an associative M-norm:*

$$N4) \ T(a, T(b,c) = T(T(a,b),c), \forall a,b,c \in I.$$

It is easy to see that if T is an M-norm then $T(a,b) \leq Min(a,b)$ $(\forall)a,b \in I$, and $T(a,0) = T(0,a) = 0$, $(\forall)a \in I$. Among the most important examples of t-norms, we will use:

$$\begin{aligned} T_1(a,b) &= W(a,b) = Max(a+b-1,0), \\ T_*(a,b) &= T_P(a,b) = Prod(a,b), \\ T_\infty(a,b) &= T_M(a,b) = Min(a,b). \end{aligned}$$

Given a t-norm T and an element $x \in [0,1]$, we can define the T-powers of x by:

$$x^0 = 1, x^1 = x \text{ and } x^{n+1} = T(x^n, x), (\forall)n \geq 1.$$

Since $([0,1],T)$ is a semigroup, $T(x^n, x^m) = x^{n+m}$, $(\forall)n,m \in \mathbf{N}$.

Definition 1.1.2 *A t-norm T is called **Archimedean** if, for each $a \in [0,1)$, $\lim_{n\to\infty} a^n = 0$ or, equivalently,*

$$\forall a, \varepsilon \in (0,1), \exists m \in N : a^m < \varepsilon \,.$$

Proposition 1.1.3 *A continuous (in the product topology on I) t-norm is Archimedean if and only if*

$$\delta(x) < x \ (\forall)x \in (0,1),$$

where $\delta(x) := T(x,x) = x^2$.

There are known many proofs of the representation theorems for continuous and Archimedean t-norms, which have a simple structure.

Simple proofs can be given by using the following

Lemma 1.1.4 *Let T be a continuous and Archimedean t-norm.*

a) *If T does not have interior nilpotents, then the semigroup$([0,1],T)$ is iseomorphic with the semigroup* $([0,1], Prod)$ (Faucett, 1955)

b) *If T has interior nilpotents, then* $([0,1],T)$ is iseomorphic with $([1/2,1], o)$ *where $x \circ y = Max\{1/2, xy\}$* [M-S, 1957].

Therefore, if T is a continuous and Archimedean t-norm, then there exist $\alpha \in \{0, 1/2\}$ and $h : [0,1] \to [\alpha, 1]$, a continuous bijection, such that

$$T(u,v) = h^{-1}(Max\,\{\alpha, h(u)\cdot h(v)\}),\ (\forall)u,v \in [0,1].$$

c) Let $h^{(-1)} : [0,1] \to [0,1]$, $h^{(-1)}(x) = h^{-1}(Max\,\{\alpha, x\})$. Then

$$(Mult) \quad T(u,v) = h^{(-1)}(h(u)\cdot h(v)),\ (\forall)u,v \in [0,1].$$

d) Moreover, for $f : [0,1] \to [0, -log\alpha]$, $f(x) = -logh(x)$, f is strictly decreasing and continuous, with $f(1) = 0$, and

$$(Addit) \quad T(a,b) = f^{(-1)}(f(a) + f(b))(\forall)u,v \in [0,1],$$

where $f^{(-1)}(x) = f^{-1}(Min\,\{x, f(0)\})$ is the pseudo-inverse of f.

Proposition 1.1.5 (The structure theorem for continuous t-norms). *Let T be a continuous t-norm. Then there exists an at most countable family of closed intervals $I_k = [\alpha_k, \beta_k] \subset [0,1]$, such that*

i) $[0,1] = (\cup I_k) \bigcup \mathcal{C}(\cup I_k)$

ii) $(\alpha_k, \beta_k) \bigcap (\alpha_l, \beta_l) = \emptyset$, $(\forall)k \neq l$

iii) $T(b,b) = b,\ (\forall) b \notin \cup(\alpha_k, \beta_k)$

iv) $T(a,b) = \begin{cases} T_k(a,b) \in I_k, & \textit{if } a,b \in I_k \\ Min(a,b), & \textit{otherwise} \end{cases}$

v) $T(a,a) < a,\ (\forall) a \in I_k$ (*thus* $T_k = T_{\ /I_k \times I_k}$ **is Archimedean**)

The *proof* is based on the fact that the set $i = \{b |\ T(b,b) = b\}$ is a closed subset of $[0,1]$, and $C_i = (0,1) \diagdown i$ is an at most countable union of open disjoint intervals. More details on M-norms and t-norms can be seen in [SCSK83]**.** A proof of the representation theorem can also be found in [MRD93].

1.2. The strong topology and the strong semiuniformity on Menger spaces

In what follows Δ_+ denotes the set of *distribution functions* $F : [0,\infty] \to [0,1]$ with the properties:

a) $F(0) = 0$ and $F(\infty) = 1$;

b) F is increasing ;

c) F is left continuous on $(0,\infty)$.

D_+ is the subset of Δ_+ containing functions F which also satisfy the condition $\lim\limits_{x\to\infty} F(x) = 1$. If $a \geq 0$, then ε_a is defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if} \quad x \leq a \\ 1, & \text{if} \quad x > a \end{cases} .$$

Let X be a nonempty set and $\mathcal{F} : X \times X \longrightarrow \mathcal{D}_+$ a given mapping ($\mathcal{F}(x,y)$ will be denoted by F_{xy}). The pair $(X, \mathcal{F})$ is called a **probabilistic semi-metric space** (shortly PSM-space) if

$I.$ $F_{xy} = \varepsilon_0$ if and only if $x = y$

$II.$ $F_{xy} = F_{yx}\ \forall x,y \in X$.

One uses the generic term **probabilistic metric space** (PM-space) if some kind of "triangle inequality" is verified. The weakest one was proposed in [SCSK60]:

$$III_{SS}.\ [F_{xy}(t) = 1, F_{yz}(t) = 1] \Rightarrow F_{xz}(t+s) = 1$$

If there exists a triangular norm T such that

$$III_M.\ F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$$

then we say that $(X, \mathcal{F}, T)$ is a **Menger space**. A more general form for III_M, defining **σ-Menger spaces,** was formulated by using some operations σ on $[0,\infty)$, instead of the addition (see [RD94] for more details).

In [HISH84] is proposed the inequality $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$III_1.[1 - F_{xy}(\delta) < \delta, 1 - F_{yz}(\delta) < \delta] \Rightarrow 1 - F_{xz}(\varepsilon) < \varepsilon,$$

which can be generalized : $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$III_f.[f \circ F_{xy}(\delta) < \delta, f \circ F_{yz}(\delta) < \delta] \Rightarrow f \circ F_{xz}(\varepsilon) < \varepsilon,$$

by using additive generators f (see e.g. [RD94] for more details).

For every PSM-space $(X, \mathcal{F})$ we can consider the sets of the form

$$U_{\varepsilon,\lambda} = \{(x, y) \in X \times X,\ F_{xy}(\varepsilon) > 1 - \lambda\},\ \varepsilon > 0, \lambda \in (0, 1)$$

which generates a semiuniformity, denoted by $\mathcal{U}_{\mathcal{F}}$, and a topology, $\mathcal{T}_{\mathcal{F}}$, named also the $(\varepsilon, \lambda)-$topology, the strong topology, or the $\mathcal{F}$-topology. Namely,

$$\mathcal{O} \in \mathcal{T}_{\mathcal{F}} \text{ iff } \forall x \in \mathcal{O}\ \exists \varepsilon > 0,\ \exists \lambda \in (0, 1) \text{ s.t. } U_{\varepsilon,\lambda}(x) \subset \mathcal{O}$$

Actually, $\mathcal{U}_{\mathcal{F}}$ can also be generated by the family of the sets $V_\delta := U_{\delta,\delta}$.In [SHR71] one can find details concerning the completion of Menger spaces under left-continuous t-norms(lc-t-norms).

2. B-Contractions on Menger spaces

The notion of contraction map in probabilistic metric spaces was introduced by V. M. Sehgal in [SHG66](cf. [SBHR72]).

Definition 2.1 *Let $(X, \mathcal{F})$ be a probabilistic metric space and $A : X \to X$. The mapping A is called a probabilistic contraction or **B-contraction** if there exists an $L \in (0, 1)$ such that, for all points $p, q \in X$ and all $u \geq 0$, the following inequality holds:*

$$(B) \quad F_{Ap,Aq}(Lu) \geq F_{p,q}(u).$$

In the paper[SBHR72] it is shown that *any B-contraction on a complete Menger space $(X, \mathcal{F}, Min)$ has a unique fixed point.*

Immediate contributions are due to H. Sherwood, who obtained a simple characterization for the existence of fixed points and proved that for a very large class of triangular norms it is possible to construct complete Menger spaces together with fixed points free contraction maps and to V.I. Istrăţescu-I. Săcuiu[ISS73].

A fundamental step is made by O. Hadžić in 1978, who introduces a class of continuous t-norms, *essentially weaker* than Min, for which the above result of Sehgal still holds.

If T is a given t-norm, then T^m is defined on I^m by

$$(1) \quad T^1(x) = x, T^{m+1}(x_1, ..., x_{m+1}) = T(T^m(x_1, ..., x_m), x_{m+1}).$$

Definition 2.2[RD83b] *We say that T is an **h-t-norm** (of Hadžić type or of h-type), if the family of mappings $\mathcal{H}_T = \{T_m\}_{m\in\ \mathbb{N}}$, defined on I by*

$$(2) \quad T_m(x) = T^m(x, x, ..., x),$$

is equicontinuous at $x = 1$.

There are nontrivial examples of h-t-norms, due also to Olga Hadžić.

Definition 2.3[RD84,87,99] *We say that the t-norm T has the **fixed point property** (shortly f.p.p.) if **each** B-contraction on **every** complete Menger space (X $\mathcal{F}, T$) has a fixed point (which clearly is **unique** and **globally attractive**).*

In this section we will present some comments and results related to the following three *statements* concerning a triangular norm:

*(***B_I***)* T is of Hadžić type*;*

*(***B_II***)* T has the fixed point property*;*

($\mathbf{B_{III}}$) $\forall a \in (0,1)$, $\exists b \geq a$ such that $T(b,b) = b < 1$,
which will be seen to correspond to different kinds of classical deterministic fixed point theorems.

We note for references, the following lemmas which are immediate consequences of Sherwood's results, and their proofs are easy to reproduce:

Lemma 1. *If $(X, \mathcal{F}, T)$ is a Menger space, T is an lc-t-norm and A is a B-contraction on X, then there exists a completion $(X^*, \mathcal{F}^*, T)$ of $(X, \mathcal{F}, T)$ and a unique extension A^* of A such that A^* is a B-contraction on X, with the same Lipschitz constant.*

Lemma 2. *A t-norm T has the f.p.p. if for every B-contraction A on a Menger space $(X\mathcal{F}, T)$ and for each fixed p_0 in X, the sequence $p_n = A^n p_0$ is $\mathcal{F}$-Cauchy. Moreover, it suffices to consider contractions with the Lipschitz constant* in $(o, \frac{1}{2}]$. *The converse holds for lc-t-norms.*

2.1. B-Contractions and the t-norms of Hadžić -type

Remark 2.1.1 Olga Hadžič proved in [HAD80] that each *continuous* t-norm of h-type has the f.p.p.. The following theorem shows that the continuity is not necessary.

Theorem 2.1.2 [RD83]. *Every t-norm of h-type has the fixed point property.*

Proof. Let $(X, \mathcal{F}, T)$ be a Menger space such that T is of h-type and consider a mapping $A : X \to X$ which verifies (B) with $L \in (0, \frac{1}{2}]$.

Let $p_0 \in X$ and $x \in (0, \infty)$ be fixed. If m is a positive integer, then

$$\begin{aligned} F_{p_0 A^{m+1} p_0}(2x) &\geq T(F_{p_0 A p_0}(x), F_{A p_0 A^{m+1} p_0}(x)) \\ &\geq T(F_{p_0 A p_0}(x), F_{p_0 A^m p_0}(2x)) \end{aligned}$$

and, therefore,

$$F_{p_0 A^m p_0}(2x) \geq T_m(F_{p_0 A p_0}(x)), \forall m \geq 1.$$

Thus we obtain that for any positive integers n, m,

$$(3) \quad F_{A^n p_0 A^{n+m} p_0}(2x) \geq T_m(F_{p_0 A p_0}(x L^{-n})).$$

Since T is of h-type and $F_{p_0 A p_0} \in D_+$, then it follows that

$$(4) \qquad \lim_{n \longrightarrow \infty} F_{A^n p_0 A^{n+m} p_0}(2x) = 1,$$

uniformly in m, for each $x \in (0, \infty)$. By definition, (4) means that $\{A^n p_0\}$ is $\mathcal{F}$-Cauchy and the theorem follows from Lemma 2.

Lemma 2.1.3[RD84] *Let T be an lc-t-norm and fix an F in D_+. Let $X = \{1, 2, ...\}$ and define a probabilistic metric on X by*

$$(5) \; F_{n n+m}(x) = T^m[F(2^{n+1}x), F(2^{n+2}x), ..., F(2^{n+m}x)], m \neq 0$$

and $F_{n n+m} = H_0, m = 0$. *Then $(X, \mathcal{F}, T)$ is a Menger space and the mapping $n \xrightarrow{A} n+1$ is a contraction with the Lipschitz constant $\frac{1}{2}$.*

A partial converse to Theorem 2.1.2 is the following.

Theorem 2.1.4[RD84] *If T is an lc-t-norm which is not of h-type, then T does not have the f.p.p.*

Proof. If T is not of h-type, then there exists $a \in (0,1)$ such that for each $b > a$ there is $m_b \geq 1$ for which $T_{m_b}(b) < a$. Let $b_n \in (a,1)$ be increasing to 1, and $m_n \geq 1$, strictly increasing and such that

$$(6) \qquad T_{m_n}(b_n) < a, \quad n = 1, 2, ...$$

Let $F \in D_+$ be defined by

$$(7) \quad F(x) = \begin{cases} 0 \; if \;\; x \leq 1 \\ b_1 \; if \;\; x \in (1, 2^{2+m_1}] \\ b_{n+1} \; if \;\; x \in (2^{2n+m_n}, 2^{2n+m_{n+1}}], \;\; n \geq 1 \end{cases}$$

If we consider the Menger space from Lemma 2.1.3, then we have successively:

$$\begin{aligned} F_{nn+m_n}(1) &\leq F_{nn+m_n}(2^n) \\ &= T^{m_n}[F(2^{2n+1}), F(2^{2n+2}), ..., F(2^{2n+m_n})] \\ &\leq T^{m_n}[F(2^{2n+m_n}), ..., F(2^{2n+m_n})] \\ &\leq T_{m_n}(b_n) < a \end{aligned}$$

Therefore the sequence $\{A^n 1\}$ is not $\mathcal{F}$-Cauchy. From Lemma 2 it follows that T does not have the fixed point property and the theorem is proved.

Lemma 2.1.5.[RD83] Let T *be a right continuous t-norm of Hadžić type. Then*

$$\forall a \in (0,1), \;\; \exists b \geq a \;\; such \; that \; T(b,b) = b < 1,$$

that is ($\mathbf{B_I}$*)⇒(*$\mathbf{B_{III}}$*) in this case.*

Proof. Suppose that *(*$\mathbf{B_I}$*)* holds, and let $a > 0$ be fixed. Then there exists $c > a$ such that $T_m(x) > a, \forall x \geq c, \forall m \leq 1$. Since clearly $\{T_m(c)\}$ is nonincreasing, then it is convergent to some limit $b \geq a$. As

$$T_{2m}(c) = T(T_m(c), T_m(c))$$

then $b = T(b,b)$ and we obtain that *(*$\mathbf{B_I}$*) implies (*$\mathrm{B_{III}}$*).*

By combining the above results we obtain the following

Theorem 2.1.6 *[RD84,87] Let* T *be a continuous t-norm. Then the statements (*$\mathbf{B_I}$*), (*$\mathbf{B_{II}}$*) and (*$\mathrm{B_{III}}$*) are equivalent.*

Lemma 2.1.7 [RD84b,99] *Let* T *be a continuous t-norm. Then*

1^0 $T \notin \mathcal{H}$ *iff there exists* $a \in [0,1)$ *such that*

$$T(a,a) = a, \; and \; T(x,x) < x, \forall x \in (a, 1).$$

$\mathbf{2^0}$ $T \notin \mathcal{H}$ *iff there exist* $a_T \in [0,1)$ *and an increasing bijection* $h_T : [a_T, 1] \to [0,1]$ *such that*

$$(8) : T(\alpha,\beta) = h_T^{-1}[\widetilde{T}(h_T(\alpha), h_T(\beta))], \forall \alpha, \beta \geq a_T$$

where $\widetilde{T}= T_1$ *or* $\widetilde{T}= Prod$ *(*$\widetilde{T}$ *depends only on* T*).*

The following result is easy to reproduce:

Proposition 2.1.8 [RD84b,87,99] *Let* T *be a continuous t-norm,* $T \notin \mathcal{H}$*.*

(i) If $(X, \mathcal{F}, T_1)$ *is a Menger space, then* $(X, e^{\mathcal{F}-1}, Prod)$ *is a Menger space with the same* (ε, λ)*-uniformity;*

(ii) If $(X,\mathcal{F},\widetilde{T})$ is a Menger space, then $(X,h_T^{-1}\circ\mathcal{F},T)$ is a Menger space with the same (ε,λ)-uniformity;

(iii) If $(X,\mathcal{F},T)$ is a Menger space, then $(X,h_T\circ\mathcal{F},\widetilde{T})$ is a Menger space with the same (ε,λ)-uniformity.

Corollary 2.1.9 [RD84b,87,99] *Let T be an arbitrary but fixed t-norm such that $T\notin\mathcal{H}$. Then the following are equivalent*

(i) T does not have the f.p.p.;

(ii) Prod does not have the f.p.p.;

(iii) T_1 does not have the f.p.p.

Focus on the statement ($\mathbf{B_{III}}$) and a family of pseudo-metrics

The following two lemmas are well-known :

Lemma 2.2.1 *If $a\le b\le c$, $T(b,b)=b$ and T is continuous, then*

1^0 *T(a,b)=a;*

2^0 *T(b,c)=b;*

3^0 *T(a,c)=a.*

Lemma 2.2.2[RD92] *Let $\{b_n\}\subset[0,1)$ be a strictly increasing sequence such that $b_n\to 1$. Then, for every distribution function $G\in D_+$, the function G^* defined by:*

$$4^0\ G^*(t)=\begin{cases} 0 & iff \quad G(t)\le b_0\\ b_n & iff \quad G(t)\in(b_n,b_{n+1}]\\ 1 & iff \quad G(t)=1\end{cases}$$

is also in D_+. Moreover, $G^*\le G$.

Using these results we can obtain the following

Theorem. 2.2.3[RD92] *Let $\{b_n\}\subset[0,1)$ be a strictly increasing sequence. If we suppose that $b_n\to 1$, $T(b_n,b_n)=b_n$ and*

5^0 $a,b>b_n\Rightarrow T(a,b)>b_n$,

then, for every Menger space $(X\mathcal{F},T)$, we have that $(X,\mathcal{F}^*,Min)$ is a Menger space with the same (ε,λ)-topology. Moreover, if $(X,\mathcal{F})$ is complete, then $(X,\mathcal{F}^*)$ is complete.

Definition. 2.2.4 [RD92] *The continuous t-norm T is of type **Hadžić-Budinčević** if the family $\{T^n\}$ is equicontinuous at $b=1$ and each T_k is strict.*

We know that a *t-norm of type **Hadžić-Budinčević*** verifies the condition 5^0, for some sequence $\{b_n\}$.

If we have that $I_k\bigcap I_{k'}=\emptyset$ for $k\ne k'$ and $\lim b_k=1$, then T is of Hadžić type (that is $\{T^n\}$ is equicontinuous at 1) and we can choose a sequence $\{b'_n\}$ for which 5^0 holds and T is not necessarily of Hadžić-Budinčević type.

The following results are very clear:

Theorem 2.2.5 (of Hadžić-Budinčević) [HBD78,79] *If $(X,\mathcal{F},T)$ is a complete Menger space, T is continuous, $\{T^n\}$ is equicontinuous at 1 and each T_k is strict, then every B- contraction on X has a unique fixed point.*

Theorem 2.2.6 (see e.g. [CKS75,KAS]). *Let $(X,\mathcal{F},T)$ be a complete Menger space, where T verifies the condition 5^0 from Theorem 1.3. Then every probabilistic contraction on X has a unique fixed point.*

Theorem 2.2.7 [RD92] *The theorems 2.2.5 and 2.2.6 are equivalent*

Proof. Clearly 2.2.5 is a particular case of 2.2.6. Now if $(X, \mathcal{F}, T)$ is as in Theorem 2.2.6, then, by Theorem 2.2.3., $(X, \mathcal{F}^*, Min)$ is a complete Menger space. It is easy to see that any probabilistic contraction in $(X, \mathcal{F}, T)$ is a probabilistic contraction in $(X, \mathcal{F}^*, Min)$ and, by 2.2.5., it has a unique fixed point.

Remark 2.2.8 [RD92] The result of Theorem 2.2.5 is proven by using deterministic semi-metrics of the form

(9) $\rho_b(x, y) = \sup\{t / F_{xy}(t) \leq b\}$

In [RD83b, RD84] we used a different method, by using a generalized metric, in order to prove a result more general . Actually formula (9) can be slightly modified:

(9′) $d_b(x, y) = \inf\{t, \mathcal{F}_{x,y}(t) \geq b\}$

and if $b = T(b, b)$ then d_b is a pseudo-metric. Moreover, if T is of Hadžić type, then we obtain a countable family $\{d_{b_n}\}$ which generate the (ε, λ)-uniformity and the method of G.L.Cain can be applied. In fact, for t-norms of Hadžić-Budinčević, formula (9) can be used for F^* and one obtains generally (9′).

The pseudo-metrics of type (9′) have been successfully used by D. Miheţ to prove fixed point theorems for more general contraction-type mappings.

Definition 2.2.9 [MIH93,97]. *Let $(b_n) \in b$, that is strictly increasing to 1. We say that the PSM space $(X, \mathcal{F})$ is*

1^0 *a* (b_n)**- probabilistic metric structure** *or a (b_n)-strict structure if the following triangle inequality takes place* :

$$(PM3_{b_n}) \ \ (F_{pq}(s) > b_n, F_{qr}(t) > b_n) \Rightarrow F_{pr}(s + t) > b_n$$

2^0 *a* $(\overline{b_n})$ *- probabilistic metric structure if the following relation takes place* :

$$(PM3_{(\overline{b_n})}) \ \ (F_{pq}(s) \geq b_n, F_{qr}(t) \geq b_n) \Rightarrow F_{pr}(s + t) \geq b_n.$$

Note that every Menger space relative to a (b_n)-M norm T of *Hadžić-Budincević* type is a (b_n)-probabilistic metric structure:

$$\begin{aligned} (F_{pq}(s) &> b_n, F_{qr}(t) > b_n, F_{pr}(s + t) \\ &\geq T(F_{pq}(s), F_{qr}(t))) \Rightarrow F_{pr}(s + t) > b_n \end{aligned}$$

and every Menger space relative to a (b_n)-M norm of *Hadžić* type is a $(\overline{b_n})$-probabilistic metric structure.

Definition 2.2.10 [MIH97]. *Let $(X, \mathcal{F})$ be a PSM space and let $(b_n) \in b$. We say that the mapping $f : X \to X$ is a* **strict** (b_n)**-probabilistic contraction** *(shortly a $s - (b_n)$ contraction) if* :

$$(\forall) n \in \mathbf{N} \ (\exists) k = k_n \in (0, 1) \ : \ F_{pq}(t) > b_n \Rightarrow F_{fpfq}(k_n t) > b_n.$$

Obviously, every B-contraction is a $s - (b_n)$ contraction for every $(b_n) \in b$.

Theorem 2.2.11 [MIH93,97]. *If $(X, \mathcal{F})$ is a complete s-(b_n) probabilistic metric structure and $f : X \to X$ is an $s-(b_n)$-contraction, then f has a unique fixed point.*

Corollary 2.2.12 [HDB78]. *Let T be a $(b_n) - t$-norm of Hadžić-Budincević type. If $(X, \mathcal{F}, T)$ is a complete Menger space and $f : X \to X$ is a mapping with the property that for every $n \in N$ there exists $k_n \in (0, 1)$ such that*

$$F_{pq}(t) > b_n \Rightarrow F_{fpfq}(k_n t) \geq F_{pq}(t),$$

then f has a unique fixed point.

Definition 2.2.13 [MIH97]. *Let $(X, \mathcal{F})$ be a PSM space and let $(b_n) \in b$. We say that $f : X \to X$ is a* (b_n)**-probabilistic contraction** *if*

$$(\forall)n \in \mathbf{N}\ (\exists)k_n \in (0,1)\ :\ F_{pq}(t) \geq b_n \Rightarrow F_{fpfq}(k_n t) \geq b_n.$$

Theorem 2.2.14 [MIH97] *If $(X, \mathcal{F})$ is a complete (b_n)-probabilistic metric structure and $f : X \to X$ is a (b_n)- probabilistic contraction, then f has a unique fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of X.*

2.3. B-Contractions and generalized metrics

Let E be an element of D_+ and consider a Menger space $(X, \mathcal{F}, T)$, where T is an lc-t-norm.

Theorem 2.3.1[RD83a,83b] *The function $d_E : X \times X \to [0, \infty]$, defined by*

$$d_E(p,q) = \inf\{a > 0, F_{pq}(ax) \geq E(x)\ ,\ \forall x \in \mathbb{R}\}$$

has the following properties:

1^0 $d_E(p,q) = 0 \Leftrightarrow p = q$;

2^0 $d_E(p,q) = d_E(q,p)$;

3^0 $d_{\pi_T(E,G)}(p,q) \leq d_E(p,r) + d_G(r,q)$;

4^0 *If $d_E \neq \infty$, then the semiuniformity and the topology generated by d_E are stronger than those generated by $\mathcal{F}$*;

5^0 If $\pi_T(E,E) = E$, *then d_E is a* **generalized metric** *on X .*

6^0 *Every B-contraction on (X, F, T) is a strict contraction on (X, d_E) for each d_E, with the same Lipschitz constant.*

Remark 2.3.2 The left continuity of E is not used above.

Example 2.3.3 If

$$E(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases},$$

then

$$d_E(p,q) = \inf\{x, F_{pq}(x) = 1\},$$

which shows that d_E needs not be nontrivial. Moreover , if (X, d) is a metric space, considered as a Menger space with $F_{pq}(x) = H(x - d(p,q))$, then $d_E(p,q) = d(p,q)$.

Corollary 2.3.4 [RD83]. *Let $(X, \mathcal{F}, Min)$ be a complete Menger space and suppose that $A : X \to X$ is a B-contraction . Then A has a unique fixed point p_* , and for each $p \in X$, $p_* = \lim_{n\to\infty} A^n(p)$, in the (ε, λ) topology.*

Proof. Let $E = F_{pAp}$. Then d_E is a generalized metric space, $d_E(p, A(p)) \leq 1$, and it is easy to see that $(A^n(p))$ is d_E-Cauchy. Therefore it is $\mathcal{F}$-Cauchy, and the theorem follows.

The following lemma shows how to construct generalized metrics on a Menger space under an h-t-norm:

Lemma 2.3.5 [RD83a,83b] *Let T be an h-t-norm. For*

$$0 < a_1 < a_2 < \ldots,\ a_n \to \infty$$

and

$$0 < b_1 < b_2 < \ldots,\ b_n \to 1,$$

such that $T(b_n, b_n) = b_n$, *let us set*

$$F(x) = \begin{cases} 0 & if \quad x \leq a_1 \\ b_n & if \quad x \in (a_n, a_{n+1}],\ n = 1, 2, \ldots \end{cases}$$

Consider a Menger space $(X, \mathcal{F}, T)$ *and define*

$$d(p, q) = inf\{a > 0,\ F_{pq}(ax) \geq F(x), \forall x \in R\}$$

Then
(i) d is a generalized metric on X;
(ii) If X is $\mathcal{F}$-complete, then X is d-complete;
(iii) The d-uniformity is ***stronger*** *than the $\mathcal{F}$-uniformity.*
Proof. (i) We prove only the triangle inequality. If

$$d(p, q) < a' < a, :: d(q, r) < b' < b$$

and $x \in (a_n, a_{n+1}]$, then

$$\begin{aligned} F_{pr}(a'x + b'x) &\geq T(F_{pq}(a'x), F_{qr}(b'x)) \geq T_1(F(x)) \\ &= T(b_n, b_n) = b_n = F(x). \end{aligned}$$

Therefore $d(p, q) \leq a' + b' < a + b$, and we obtain the triangle inequality.

(ii) and (iii) : Let $\{p_n\}$ be a d-Cauchy sequence. and fix $a > 0$.

For $\varepsilon > 0$ and $\lambda \in (0, 1)$,let $x_0 \in R$ such that $F(x_0) > 1 - \lambda$ and let $a > 0$ such that $ax_0 < \epsilon$. By the definition of d, there exists $n_a \geq 1$ such that

$$F_{p_n p_m}(ax) \geq F(x),\ \forall : n \geq n_a,\ \forall m \geq 1, \forall x \in R.$$

If $n \geq n_a$, $m \geq 1$, then $F_{p_n p_{n+m}}(\epsilon) > 1 - \lambda$, which shows that $\{p_n\}$ is $\mathcal{F}$-Cauchy.

If we suppose that X is $\mathcal{F}$-complete, then $\{p_n\}$ is convergent to some limit p. Therefore

$$F(x) \leq \lim \inf_{m \to \infty} F_{p_n p_{n+m}}(ax) = F_{p_n p}(ax)$$

for each x and all $n \geq n_a$, that is $d(p, q) \leq a, \forall n \geq n_0$. Thus p_n is d-convergent and the lemma is proved.

Remark 2.3.6 For given p, q in X, we can take a_n in the lemma such that $F_{pq}(a_n) \geq b_n$. Therefore the metric d is nontrivial and we obtain that the above result still holds, with a new proof.

Remark 2.3.7 As it is well known the Banach contraction principle is a consequence of the above Corollary. Actually we can modify the above proof in order to see that *the Banach* fixed point principle implies Corollary 2.3.4 : Let $(X, \mathcal{F}, T)$ and A be as in Corollary 2.3.4. If p_0 is given in X, then let a_n and b_n be as in Lemma 2.3.5 and such that $F_{p_0 A p_0}(a_n) \geq b_n$. Consider the generalized metric d as in Lemma 2. It is easy to see that $d(p_0, Ap_0) < \infty$ and therefore $X_0 := \{q_0 \in X, d(p_0, q_0) < \infty\}$

is a **complete metric space** and A is a strict contraction on X_0. Therefore $p_n = A^n p_0$ will d-converge to the (evidently unique) fixed point of A.

2.3.1. A generalized metric on probabilistic f-metric structures and the fixed point alternative

Let $f : [0,1] \longrightarrow [0,\infty]$ be a continuous function which is strictly decreasing and vanishes at 1.

Definition 2.3.1.1 ([RD85]). *The pair* $(X,\mathcal{F})$ *which has the properties* $(PM0)-(PM2)$ *is called a* **probabilistic f-metric structure** *iff* $\forall t > 0\,, \exists s > 0$ *such that*

$$III_f[f \circ F_{xz}(s) < s,\ f \circ F_{zy}(s) < s] \Rightarrow f \circ F_{xy}(t) < t.$$

Remark 2.3.1.2 If $(X,\mathcal{F})$ is a probabilistic f-metric structure then the family $\mathcal{W_F}^f := \{W_\epsilon^f\}_{\epsilon \in (0,f(0))}$, where $W_\epsilon^f := \{(x,y) | F_{xy}(\epsilon) > f^{-1}(\epsilon)\}$, is a uniformity base which generates the uniformity $\mathcal{U_F}$.

Lemma 2.3.1.3[PRD97] Consider a Menger space $(X,\mathcal{F},T)$, where $T \geq T_f$. For each $k > 0$ let us define

$$d_k(x,y) := \sup_{s>0} s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t} dt$$

and

$$\rho_k(x,y) := (d_k(x,y))^{\frac{1}{k+1}}.$$

Then ρ_k is a generalized metric on X .

Lemma 2.3.1.4 Let $(X,\mathcal{F},T)$ be a Menger space with $T \geq T_f$. Then $\mathcal{U_F} \subset \mathcal{U}_{\rho_k}$.

Lemma 2.3.1.5 If $(X,\mathcal{F})$is a probabilistic f-metric structure and A is a B-contraction then A is, for each $k > 0$, a strict contraction in (X,ρ_k)

Theorem 2.3.1.6 [PRD97] Let $(X,\mathcal{F},T)$ be a complete Menger space with $T \geq T_f$. If there exists some $k > 0$ such that for every pair $(x,y) \in X$ one has

$$(10) \qquad \sup_{s>0} s^k \int_s^\infty \frac{f \circ F_{xy}(t)}{t} dt < \infty$$

then every B-contraction on X has a unique fixed point.

Corollary.2.3.1.7 ([cf. SHR71]) *Let* (X,F,T) *be a complete Menger space under* $T \geq T_f$, *where* $f(0) < \infty$ *and suppose that for each pair* $(x,y) \in X^2$ *there exists* t_{xy} *for which* $F_{xy}(t_{xy}) = 1$. *Then every* B*-contraction on* X *has a unique fixed point.*

Corollary 2.3.1.8 [TAR92] *Let* (X,F,T) *be a complete Menger space under* $T \geq T_1$ *and suppose that there exists* $k > 0$ *such that every* F_{xy} *has a finite* k*-moment. Then every* B*-contraction on* X *has a unique fixed point.*

Generally, from the fixed point alternative ([DMG79]) we obtain the following

Theorem 2.3.1.9 *Let* (X,F,T) *be a complete Menger space under* $T \geq T_f$ *and* A *an* B*-contraction. Then for each* $x \in X$ *either,*

(A_1) *there is some* $k > 0$ *such that* $(A^i x)$ *is* ρ_k*-convergent to the unique fixed point of* A, *or*

(A_2) for all $k>0$, for all $n\in N$ and for all $M>0$ there exists $s:=s(k,n,M)$ such that

$$s^k\int\limits_s^\infty \frac{f\circ F_{A^nxA^{n+1}x}(t)}{t}dt > M.$$

2.4. Using a single metric

The results of this section are related to the following well known classical result.

Lemma 2.4 *Let (X,d) be a complete metric space, and A a **continuous** self-mapping of X. Then the following two statements are equivalent:*

i) A has a fixed point;

ii) There exists $p\in X$ such that $\sum_{n\geq 0} d(A^np,A^{n+1}p)<\infty$

2.4.1. A metric for the strong uniformity in some Menger spaces

Proposition 2.4.1.1 [RD98] *The two-place function ρ_0, defined by*

$$(11)\quad \rho_0(p,q)=\int_0^1 (1-F_{pq}(x))dx, \quad \forall (p,q)\in X\times X$$

is a semi-metric, on X, which generates $\mathcal{U}_{\mathcal{F}}$.

Remark 2.4.1.2 Let $\mathbf{K}$ be the semi-metric of type Ky Fan, defined by

$$\mathbf{K}(p,q):=sup\{t \mid t\leq 1-F_{pq}(t)\}.$$

Then ρ_0 and $\mathbf{K}$ are related by the inequalities

$$\mathbf{K}^2\leq \rho_0\leq 2\mathbf{K}-\mathbf{K}^2$$

Theorem 2.4.1.3 *Let $(X,\mathcal{F},T)$ be a Menger space relatively to the t-norm $T\geq W$. Then the mapping $R_0: X\times X\longrightarrow \mathbf{R}$, given by:*

$$(12)\ R_0(p,q)=\left\{\int_0^1 [1-F_{pq}(x)]\,dx\right\}^{\frac{1}{2}} \qquad \forall p,q\in X\times X$$

is a metric, on X, which generates the strong uniformity $\mathcal{U}_{\mathcal{F}}$. Moreover,

$$\mathbf{K}(p,q)\leq R_0(p,q)\leq \sqrt{2\,\mathbf{K}(p,q)}\quad \forall p,q\in X$$

so that $(X,\mathcal{F},T)$ is complete iff (X,R_o) is complete.

Remark 2.4.1.4 If $(X,\mathcal{F},W)$is either an E-space or a non-Archimedean Menger space, then ρ_0 *itself is seen to be a metric.*

Example 2.4.1.5 Consider the random variables (on $(0,1)$ with Lebesgue measure) a,b,c defined by:

$$a(t)=t^2,\ b(t)=t+t^2\ ,\ c(t)=t.$$

If $F_{pq}(x)=\lambda(|p-q|<x)$,then $(X,(p,q)\longrightarrow F_{pq},W)$is a Menger space. In this case $\rho_0(a,b)=\frac{1}{2},\ \rho_0(b,c)=\frac{1}{3}$ and $\rho_0(a,c)=\frac{1}{6}$; thus ρ_0 is a metric.

Now if we take F_{ab} and F_{bc} as above, and $F_{ac}=\tau_W(F_{ab},F_{bc})$, then one obtains $\rho_0(a,c)=\frac{31}{32}>\frac{1}{2}+\frac{1}{3}=\rho_0(a,b)+\rho_0(b,c)$, so that ρ_0 is *not* a metric. Clearly R_0 is a metric.

2.4.2. A family of metrics which generate the $\mathcal{F}$ - uniformity

The above idea can be easily extended. Let λ be fixed in $[0, 1]$ and define

$$(12_\lambda) \quad R_\lambda(p, q) := \left(\int_0^1 \frac{1 - F_{pq}(x)}{(1 + x)^\lambda} dx \right)^{\frac{1}{2}}, \qquad \forall p, q \in X.$$

Then we have the following theorem:

Theorem 2.4.2.1 [RD99] *If* $(X, \mathcal{F}, T)$ *is a Menger space and* $T \geq W$, *then*
(i) R_λ *is a metric, for each* $\lambda \in [0, 1]$.
(ii) *For* $0 \leq \lambda < \mu \leq 1$ *one has*

$$\frac{1}{\sqrt{2}} R_0(p, q) \leq R_1(p, q) \leq R_\mu(p, q) \leq R_\lambda(p, q) \leq R_0(p, q), \forall p, q.$$

(iii) R_λ *generates the strong* $\mathcal{F}$*-uniformity on* X.
(iv) $(X, \mathcal{F}, T)$ *is complete iff* (X, R_λ) *is complete for some* $\lambda \in [0, 1]$.
Remarks 2.4.2.2 (a) It is easy to see that

$$2^{-\frac{\lambda}{2}} \mathbf{K}(p, q) \leq R_\lambda(p, q) \leq \sqrt{2\mathbf{K}(p, q)}.$$

(b) For E-spaces or nonArchimedean Menger spaces $\rho_\lambda := R_\lambda^2$ is a *metric*.

2.4.3. A fixed point principle

Let $(X, \mathcal{F}, T)$ be a *complete Menger space* and consider a B-contraction A on X. The following lemma is obvious.

Lemma 2.4.3.1 *For every* $\lambda \in [0, 1]$ *one has*

$$(12') \quad R_\lambda(Ap, Aq) \leq L^{\frac{1-\lambda}{2}} \left\{ \int_0^{\frac{1}{L}} \frac{1 - F_{pq}(x)}{(1 + x)^\lambda} dx \right\}^{\frac{1}{2}}.$$

We are in position to give a characterization of probabilistic B-contractions with fixed points.

Theorem 2.4.3.2 [RD99] *If* $T \geq W$, *then the following statements are equivalent*
1^0 *A has a fixed point.*
2^0 *There exist* $p \in X$ *and* $\lambda \in [0, 1)$ *such that*

$$(13_\lambda) \quad E_{pAp}^{(\lambda)} := \int_0^\infty \frac{1 - F_{pAp}(x)}{(1 + x)^\lambda} dx < \infty.$$

Proof. The implication $1^0 \Rightarrow 2^0$ is obvious : $p = Ap \Rightarrow F_{pAp}(x) = 1, \forall \dot{x} > 0 \Rightarrow E_{pAp}^{(\lambda)} = 0, \forall \lambda < 1$. Let us prove the implication $2^0 \Rightarrow 1^0$.

Thus we suppose that there exists $p \in X$ which verifies (13_λ) for some $\lambda < 1$. Since $\lambda < \mu \Rightarrow \frac{1}{(1+x)^\mu} \leq \frac{1}{(1+x)^\lambda}, \forall x \geq 0$, then it is clear that $(13_\lambda) \Rightarrow (13_\mu)$ for $\lambda < \mu$. So it suffices to consider the case $\lambda \in [0, 1)$.

Since $F_{Ap^n Ap^{n+1}}(x) \geq F_{pAp}\left(\frac{x}{L^n}\right) \quad, \forall x \geq 0, \forall n \geq 0$, where A^n is the n-iterate of A, then from formula $(12'_\lambda)$ we obtain the inequality

$$R_\lambda\left(A^n p, A^{n+1} p\right) \leq \left(L^{\frac{1-\lambda}{2}}\right)^n \left\{\int_0^{\frac{1}{L^n}} \frac{1-F_{pAp}(x)}{(1+x)^\lambda} dx\right\}^{\frac{1}{2}}$$

which implies that

$$(14_\lambda) \quad R_\lambda\left(A^n p, A^{n+1} p\right) \leq \left(L^{\frac{1-\lambda}{2}}\right)^n \left\{E_{pAp}^{(\lambda)}\right\}^{\frac{1}{2}}.$$

From (14_λ) and (13_λ) it results that $\sum\limits_{n=0}^{\infty} R_\lambda\left(A^n p, A^{n+1} p\right) < \infty$

Therefore $(A^n p)$ is a Cauchy sequence in the complete metric space (X, R_λ), thus it converges to some element $p_* \in X$.

From the continuity of A, one obtains that p_* is a fixed point for A, which is necessarily unique. The theorem is completely proved.

Remark 2.4.3.3 Since every Archimedean t-norm T has the representation

$$(8') \ T(a,b) = h^{-1}\left(\widetilde{T}(h(a), h(b))\right), \quad \forall a, b \in [0,1]$$

where $h : [0,1] \longrightarrow [0,1]$, an increasing homeomorphism, and $\widetilde{T} \in \{W, Prod\}$ are precisely determined by T, then it is easy to see that, for every Menger space $(X, \mathcal{F}, T)$, the probabilistic metric $h \circ \mathcal{F}$ verifies the triangle inequality with W. Therefore Theorem 2.4.3.2 can be applied:

Corollary 2.4.3.4 *Let $(X, \mathcal{F}, T)$ be a complete Menger space such that the Archimedean t-norm T has the representation $(8')$. Then a given B-contraction A on X has a fixed point if and only if there exist $p \in X$ and $\lambda < 1$ such that*

$$(13_{\lambda,h}) \quad \int_0^\infty \frac{1 - h \circ F_{pAp}(x)}{(1+x)^\lambda} dx < \infty.$$

Remark 2.4.3.5 Our results are clearly applicable in the case of E-spaces, which are Menger spaces under W. The condition (13_λ) says that the random variable *dist* (p, Ap) is in the Lebesgue space $L_{1-\lambda}$ (that is it has a finite moment of order $1-\lambda$) for *one element* p *and some value* λ <1, a condition which appears to be reasonable strong and easy to verify in concrete applications.

2.4.4. A family of semi-metrics on PM-spaces

In the following lemma we introduce a family of nonnegative functions which measure the distance between ϵ_0 and the elements of D_+ Let k be a (fixed) positive real number.

Lemma 2.4.4.1 [RD98] *The one-place mapping δ_k: $D_+ \to R_+$, given by*

$$(15) \quad \delta_k(F) := \sup_{x>0}\{x^k[1-F(x)]e^{-x}\},$$

has the following properties:

(i) $\delta_k(F) = 0 \Leftrightarrow F = \epsilon_0$;

(ii) *If* $F_1 \le F_2$, then $\delta_k(F_1) \ge \delta_k(F_2)$;

(iii) $\delta_k(\lambda \circ F) \le \lambda^k \delta_k(F), \forall \lambda \ge 1$;

(iv) $\delta^{k+1} e^{-\delta} \le \delta_k(F) \le \max\{\delta^k, \delta k^k e^{-k}\}$,

where $\delta = \delta(F) := \sup\{t | t \le 1 - F(t)\}$ *is the écart of Ky Fan.*

(v) $\delta_k(F_n) \to 0 \Leftrightarrow F_n(x) \to 1$, for each $x > 0$.

Proposition 2.4.4.2 [RD98] *Let* $(X, \mathcal{F})$ *be a probabilistic metric space and define*

$$(16) \quad e_k(p,q) := \delta_k(F_{pq}) = \sup_{x>0} x^k [1 - F_{pq}(x)] e^{-x}, \ \forall p, q \in X.$$

Then

1^o e_k *is* **a semi-metric** *which generates the strong* $\mathcal{F}$*-topology;*

2^o e_k *generates the* $\mathcal{F}$*-uniformity, if this exists;*

3^o *If* $(X, \mathcal{F}, W)$ *is a Menger space, then*

$$(17) \quad (p,q) \to \theta_k(p,q) := \{e_k(p,q)\}^{\frac{1}{k+1}}$$

gi*ves* **a metric** *on X. Moreover,* $(X, \mathcal{F})$ *is complete if and only if* (X, θ_k) *is complete.*

Theorem 2.4.4.3 [RD98] *Let* $(X, \mathcal{F}, T)$ *be a complete Menger space such that* $T \ge W$*. If* $A : X \to X$ *is a B-contraction, then* **the following statements are equivalent:**

(1) A has a fixed point;

(2) There exist $p \in X$ *and* $k \in (0, \infty)$ *such that*

$$(18) \qquad E_k(p) := \sup_{x>0} \{x^k [1 - F_{pAp}(x)]\} < \infty.$$

Remarks 2.4.4.4 a) Simple examples show that A is generally not contractive relatively to θ_k(or e_k).

b) The suppremum in (18) may be infinite for some different values of k or for different points in X.

c) Our condition is verified if there exists an element p such that $F_{pAp}(t_p) = 1$ for some $t_p > 0$ (Note that H. Sherwood in [SHER71, Corollary] imposed this condition **for all** F_{pq})

d) The condition (18) is verified if F_{pAp} has a finite k moment. Thus Theorem 2.1. slightly extends our above results

Corollary 2.4.4.5 *If* $T \ge W$ *and* $(X, \mathcal{F}, T)$ *is a complete Menger space, then a given probabilistic contraction A on X has a fixed point if and only if there exist* $k > 0$ *and* $p \in X$ *such that*

$$(19) \quad \int_0^\infty x^k dF_{pAp}(x) < +\infty.$$

Proof . It is well known and easy to see that

$$(20) \lim_{x \to \infty} x^k (1 - F_{pAp}(x)) = 0,$$

if (19) holds, Q.E.D.

Remark 2.4.4.6 A t-norm T is Archimedean if and only if there exists an increasing homeomorphism $h : [0,1] \to [0,1]$ such that

$$(8') \quad T(a,b) = h^{-1}(T_*(h(a), h(b))) =: T_*(a,b)$$

where $T_* = W$ or $T_* = Prod$ (see Theorem 0.4). Since $ab \geq a+b-1$ for all $a, b \in [0,1]$, then we obtain the following.

Theorem 2.4.4.7 [RD98] *Let $(X, \mathcal{F}, T)$ be a complete Menger space such that $T \geq T_h$ for some increasing homeomorphism $h : [0,1] \to [0,1]$. Then a probabilistic contraction A of X has a fixed point if and only if there exist $k > 0$ and $p \in X$ such that*

$$\sup_{x>0} x^k [1 - h \circ F_{pAp}(x)] < +\infty. \tag{21}$$

The *proof* follows from the fact that $(X, h \circ \mathcal{F}, W)$ is seen to be a complete Menger space, Q.E.D.

3. C-Contractions

A second type of contractions in Menger spaces was introduced by T.L. Hicks [HIC83] who also proved that for the t-norm Min the Banach principle is valid (and is essentially equivalent to the classical one).

We improved [RD87]the above result by showing that it remains true in any complete Menger space $(X, \mathcal{F}, T)$ for which $\sup_{a<1} T(a,a) = 1$. It is to be noted that this condition on T is the weakest one which ensures the existence of the (ε, λ) - uniformity ([MNG78]).

Using the method of the so called T-conjugate transforms, in [MOYS79] is introduced a (deterministic) metric which generates the (ε, λ) - topology in a Menger spaces under an Archimedean t-norm. This metric is given in terms of a multiplicative generator of the t-norm T.

Considering a modified form of this metric, in terms of an additive generator of the t-norm, we have given a direct simpler proof of the above result. Our formulas can be considered as direct generalizations of the Fréchet metrics for the convergence in probability. In [MOYS79] the following is proved.

Theorem 3.1 For any multiplicative generator h of T and for any positive real number z, the mapping d_a defined on $X \times X$ by

$$d_a(p,q) = -\sup_{x>0} e^{ax} hF_{pq}(x) \tag{22}$$

is a metric on X, which generates the (ε, λ)-topology. Moreover, the metrics d_a are uniformly equivalent.

This is proved by using the properties of the so called T-conjugate transform.

We restated the above Theorem in terms of additive generators and so we have given a direct simpler proof. Namely

Theorem 3.2 [RD82]. Let f be an additive generator of T and define the mapping

$$\rho_f(p,q) = \inf_{t>0} \{t + f \circ F_{pq}(t)\}, p, q \in X \tag{23}$$

Then

(i) ρ_f is a metric on X;

(ii) The uniformity generated by ρ_f is the (ε, λ)-uniformity;

(iii) If a is a positive real number, then ρ_f^a defined by

$$(23a) \qquad \rho_f^a(p,q) = \inf_{t>0}\{at + f \circ F_{pq}(t)\}$$

has the properties (i)-(ii);

(iv) For each $a \in (0,1]$ one has

$$a\rho_f \leq \rho_{af} \leq \rho_f$$

and so all ρ_f are uniformly equivalent.

Our proof is based on the well known inequality

$$f \circ F_{pq}(x+y) \leq f \circ F_{pr}(x) + f \circ F_{pr}(y), \forall p,q,r,x,y.$$

Corollary 3.3 If $(X,\mathcal{F},T)$ is a Menger space under an Archimedean t-norm T, then there exists an increasing bijection $h:[0,1]\to[0,1]$ such that the two-place function k_h defined by

$$(24) \qquad k_h(p,q) = \inf_{t>0}\{t + 1 - h \circ F_{pq}(t)\}$$

is a metric on X, which metricizes the (ε,λ)-uniformity.

The main result of Hicks reads as follows:

Theorem 3.4 *Every C-contraction on a complete Menger space $(X,\mathcal{F},Min)$ has a unique fixed point , which is the limit of the successive approximations.*

The proof of the above result is obtained from the deterministic Banach principle , by constructing a metric on X which generates the (ε,λ)- uniformity and is such that f is a contraction with respect to that metric.

As a matter of fact , the same proof is valid for a larger class of t-norms. This is due to the fact that the two-place function d constructed in [HIC83] is a metric in any Menger space $(X,\mathcal{F},T)$ if $T \geq T_1$. We proved this fact using a slightly modified form of d.

Proposition 3.5. *Let $(X,\mathcal{F})$ be a PSM-space and define the two place mapping*

$$(25) \qquad \mathbf{K}(x,y) = \sup\{t\ ,\ t \leq 1 - F_{xy}(t)\}.$$

Then K is semi-metric on $(X,\mathcal{T}_{\mathcal{F}})$ and

$$(26) \qquad \mathbf{K}(x,y) < \delta \Leftrightarrow F_{xy}(\delta) > 1-\delta\ \forall\delta>0,$$

which shows that K generates the semi-uniformity $\mathcal{U}_F$.

Examples 3.6 (i) If d is a semi-metric on X and we set $F_{xy} := \varepsilon_{d(x,y)}$ then $(X,\varepsilon_{d(.,.)})$ is a PSM-space and $\mathbf{K}(x,y) = \min(d(x,y),1)$.

(ii) Let X be the family of all classes of real random variables on a probability measure space $(\Omega,\mathcal{K},P)$. If we set $\mathcal{F}(x,y) = F_{|x-y|}$, the distribution function of $|x-y|$, then $(X,\mathcal{F},W)$ is a Menger space and $\mathbf{K}$ is the Ky Fan metric of the convergence in probability.

It is to be noted that, generally, **K need not be a metric**. But we have proven the following.

Theorem 3.7 [RD85,87] *Let* $(X, \mathcal{F}, T_1)$ *be a Menger space and define*

$$(27) \quad d(p,q) = sup\{t,\ F_{pq}(t) \le 1-t\}$$

Then
(i) d *is a metric on* X *, which generates the* $(\varepsilon, \lambda) - uniformity$;
(ii) X *if* $\mathcal{F}$*-complete iff* X *is* d*-complete;*
(iii) $f : X \to X$ *is a C-contraction iff* f *is* d*-contraction*

In order to ensure the verification of the triangle inequality for **K**, T. L. Hicks [HIC96] proposed the following form of the triangle inequality for $(X, \mathcal{F})$:

$$\begin{aligned} III^1.\ [F_{xy}(t) &> 1-t, F_{yz}(s) > 1-s] \\ &\Rightarrow F_{xz}(t+s) > 1-(t+s) \end{aligned}$$

and he observed that the property III^1 holds for every Menger space $(X, \mathcal{F}, T)$ for which $T \ge W$.
As a matter of fact one has the following
Proposition 3.8 Let T be a t-norm such that (III^1) holds for every Menger space $(X, \mathcal{F}, T)$. Then $T \ge W$.
Proof. Let $X = \{x, y, z\}, F_{xy} = F_{yx}, F_{yz} = F_{zy}, F_{xz} = F_{zx}$ where

$$F_{xy}(t) = \begin{cases} 0 & t \le 0 \\ a & t \in (0,1] \\ 1 & t > 1 \end{cases}, \quad F_{yz}(t) = \begin{cases} 0 & t \le 0 \\ b & t \in (0,1] \\ 1 & t > 1 \end{cases},$$

$$F_{zx}(t) = \begin{cases} 0 & t \le 0 \\ T(a,b) & t \in (0,1] \\ 1 & t > 1 \end{cases}$$

and $F_{xx} = F_{yy} = F_{zz} = \varepsilon_0$. Then $(X, \mathcal{F}, T)$ is a Menger space (for which T is the best t-norm) and $\mathbf{K}(x,y) = 1-a, \mathbf{K}(y,z) = 1-b$, while $\mathbf{K}(x,z) = 1 - T(a,b)$. Thus we see that $\mathbf{K}(x,z) \le \mathbf{K}(x,y) + \mathbf{K}(y,z) \Leftrightarrow T(a,b) \ge a+b-1$.
Remark 3.9 Let $(X, \mathcal{F}, T)$ as in the proof of the theorem and suppose that $T(a,b) < a+b-1$. Therefore $0 < a,\ b < 1$ and there exists $p > 1$ such that $((1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}})^p > 1 - T(a,b)$. Thus $(1-a)^{\frac{1}{p}} + (1-b)^{\frac{1}{p}} > (1-T(a,b))^{\frac{1}{p}}$ and we see that $\mathbf{K}_p$, given by

$$\mathbf{K}_p(u,v) = \sup\{t | t^p \le 1 - F_{uv}(t)\},$$

is verifying the triangle inequality. This shows that the general formulas proposed in [RD86b] can give metrics in more general situations.
Let $\mathcal{M}$ denote the family of all mappings $\mu : [0,\infty] \to [0,\infty]$ which are such that
a) $\mu(t) + \mu(s) \le \mu(t+s)$, $\forall\ t, s \ge 0$
b) $\mu(t) = 0 \Leftrightarrow t = 0$
and
c) μ is continuous.

It is easy to see that every $\mu \in \mathcal{M}$ is finite and strictly increasing on a uniquely determined interval $[0, b_\mu]$ and $\mu(b_\mu) = \infty$.

If we set, for any PSM-space $(X, \mathcal{F})$,

(1_μ) $\mathbf{K}_\mu(x, y) = \sup\{t | t \geq 0, \mu(t) \leq 1 - F_{xy}(t)\}$

then $\mathbf{K}_\mu$ is a semi-metric. Moreover,

(2_μ) $\mathbf{K}_\mu(x, y) < \delta \Leftrightarrow F_{xy}(\delta) > 1 - \mu(\delta)$,

from which it follows that $\mathbf{K}_\mu$ generates $\mathcal{U}_\mathcal{F}$.

3.1. Probabilistic metric spaces of type $\mathcal{M}$

The above remarks suggest the following definition, which extends (III^1) :

Definition 3.1.1. [PRD99] A PSM-space $(X, \mathcal{F})$ for which takes place the following triangle inequality

$$\begin{aligned} III^\mu.\ [F_{xy}(t) &> 1 - \mu(t), F_{yz}(s) > 1 - \mu(s)] \Rightarrow \\ &\Rightarrow F_{xz}(t + s) > 1 - \mu(t + s) \end{aligned}$$

is called **PM-space of type $\mathcal{M}$**.

Remark 3.1.2 The triangle inequality (III^μ) can be useful and appropriate in many cases. For example, if $(X, \mathcal{F})$ verifies (III^1) – which is (III^μ) for $\mu(t) = t$ – then it is easy to see that $\widetilde{\mathcal{F}}$ defined by $F_{xy} \circ \mu$, is a probabilistic semi-metric and

$$\widetilde{F}_{xy}(\delta) > 1 - \mu(\delta) \Leftrightarrow F_{xy}(\mu(\delta)) > 1 - \mu(\delta).$$

The formula (1_μ) leads to $\mathbf{K}_\mu(x, y) = \mu^{-1}(\mathbf{K}(x, y))$, a very convenient one, for μ^{-1} is clearly sub-additive. In particular, for spaces of random variables (see Example 1.3 (ii)), $\widetilde{F}_{xy}(t) = P(|x - y| < \mu(t)) = P(\mu^{-1}(|x - y|) < t)$ and $\mu^{-1} \circ \mathbf{K}$ is a metric which gives the convergence in probability, too. Generally, we can prove the following

Theorem 3.1.3 [PRD99] *Let $(X, \mathcal{F})$ be a PM-space of type $\mathcal{M}$, that is the triangle inequality (III^μ) holds. Then the two-place function K_μ, defined by formula (1_μ), is a metric on X which generates $T_\mathcal{F}$ and $U_\mathcal{F}$.*

Proof. We have only to prove the triangle inequality for $\mathbf{K}_\mu$. From (2_μ) we see that $[\mathbf{K}_\mu(x, y) < \delta$ and $\mathbf{K}_\mu(y, z) < \varepsilon] \Rightarrow [F_{xy}(\delta) > 1 - \mu(\delta)$ and $F_{yz}(\varepsilon) > 1 - \mu(\varepsilon)]$. From (III^μ) it follows that $F_{xz}(\delta + \varepsilon) > 1 - \mu(\delta + \varepsilon)$, which shows that $\mathbf{K}_\mu(x, z) < \delta + \varepsilon$ and we obtain the triangle inequality for $\mathbf{K}_\mu$.

Corollary 3.1.4 *Let T be a t-norm such that*

$$T(a, b) \geq T_\mu(a, b) := \max\{1 - \mu[\mu^{-1}(1 - a) + \mu^{-1}(1 - b)], 0\}.$$

Then K_μ is a metric for every Menger space $(X, \mathcal{F}, T)$.

Proof. Since $T(a, b) \geq T_\mu(a, b)$ then $T(a, b) \geq 1 - \mu[\mu^{-1}(1 - a) + \mu^{-1}(1 - b)]$. From the inequality III_M it follows that

$$F_{xz}(t + s) \geq 1 - \mu[\mu^{-1}(1 - F_{xy}(t)) + \mu^{-1}(1 - F_{yz}(s))].$$

Now, if we suppose that $F_{xy}(t) > 1 - \mu(t), F_{yz}(s) > 1 - \mu(s)$ and $\mu(t + s) < 1$, then $\mu(t), \mu(s) < 1$ and

$$F_{xz}(t + s) \geq 1 - \mu(t + s).$$

Therefore $\mathcal{F}$ verifies III^{μ}.

Remark 3.1.5 Since μ is super-additive, then $\mu(\mu^{-1}(1-a)+\mu^{-1}(1-b)) \geq 2-a-b$, which shows that $T_\mu(a,b) \leq \max(a+b-1,0) = W(a,b)$. Therefore our Theorem 3.1.3 and Corollary 3.1.4 essentially extend the corresponding Theorem 2 and Corollary 2 of [HIC96]. Actually if we take an increasing sequence $(\mu_n) \subset \mathcal{M}$, it is clear that (T_{μ_n}) is decreasing and so the class of Menger spaces, for which formula (1_μ) gives us a metric, is increasing. For example, if $\mu_n^{-1}(t) \to 1$ for $t \in (0,1]$, then we see that $T_{\mu_n}(a,b) \to T_w(a,b)$, the weakest t-norm (see also the example in Remark 3.1.2).

Consider an Archimedean t-norm T_f with the additive generator f, and let μ_1, $\mu_2 \in M$ be fixed. Then we have the following

Theorem A [RD88] For every Menger space $(X,\mathcal{F},T)$ with $T \geq T_f$, the mapping d given by

$$(28) \qquad d(p,q) = \sup\{t, \mu_1(t) \leq f \circ F_{pq}(\mu_2(t))\}$$

is a metric on X. Moreover,

$$(29) \qquad d(p,q) < t \Leftrightarrow f \circ F_{pq}(\mu_2(t)) < \mu_1(t)$$

and so d and $\mathcal{F}$ generate the same uniformity.

1. It suffices to consider the case $\mu_2(t) = t$, for $(X, \mathcal{F} \circ \mu_2, T)$ is a Menger space for every $(X, \mathcal{F}, T)$.

2. The formula (28) gives a metric on every Menger space $(X,\mathcal{F},Min)$ and any $f:[0,1] \to [0,\infty]$ which is continuous, strictly decreasing and such that $f(1) = 0$. The case $f(1) = 1-t$, $m_1(t) = \mu_2(t) = t$ and $T = Min$, was considered in [HIC83] in a different formulation.

In [RD84] we observed that the method used in [HIC83] can be applied for a larger class of t-norms, namely for $T \geq T_1$.

In [CNS85] this case $T \geq T_1$ was considered for a larger class of mappings: Let $\mathcal{L}$ be the family of functions $L:[0,\infty) \longrightarrow [0,\infty)$ with the following three properties:

(L_1) L is strictly increasing;
(L_2) L is right continuous;
(L_3) $\lim_{n\to\infty} L^n(t) = 0,\ \forall t \geq 0.$

A mapping A is called L-probabilistic contractions iff

$$(C_L) \qquad t > 0, F_{pq}(t) > 1-t \Rightarrow F_{A_pA_q}(L(t)) > 1-L(t)$$

We considered (RD88) a slightly more general case, suggested by the following remark. If we set $f(s) = 1-s$ then (C_L) can be formulated as

$$(C_L^f) \qquad f \circ F_{pq}(t) < t \Rightarrow f \circ F_{A_pA_q}(L(t)) < L(t)$$

As a matter of facts, a fixed point theorem holds in more general conditions:

Theorem 3.1.6 [RD88] Let $(X,\mathcal{F},T)$ be a complete Menger space such that$T \geq T_f$. Then every mapping $A: X \to X$ which satisfies the condition (μ is fixed)

$$(30) \qquad f \circ F_{pq}(t) < \mu(t) \Rightarrow f \circ F_{A_pA_q}(L(t)) < \mu(L(t))$$

has a unique fixed point which is the limit of successive approximations.

3.2. A special case: $\mathcal{L} - \mathcal{M}$ contractions

Definition 3.2.1[PRD99] We say that $A : X \to X$ is an $\mathcal{L}-\mathcal{M}$ **probabilistic contraction** if there exist $L \in \mathcal{L}$ and $\mu \in \mathcal{M}$ such that

$$(L\mu - c) \quad [F_{xy}(t) > 1-\mu(t)] \Rightarrow [F_{AxAy}(L(t)) > 1-\mu \circ L(t)]$$

For a concrete pair $L-\mu$ we use the term $L-\mu$ **probabilistic contraction**.

Example 3.2.2. Suppose that A is a contraction of Hicks type – that is $(L\mu - c)$ holds for $\mu(t) = t$ (the case of Hicks) and consider the probabilistic semi-metric $\widetilde{\mathcal{F}}$ defined by $\widetilde{F}_{xy} = F_{xy} \circ \mu$, where

$$F_{xy} \circ \mu(t) = \begin{cases} 0, & t \leq 0 \\ F_{xy}(\mu(t)), & t > 0 \end{cases},$$

and $t_\mu = \infty$. If we set $\widetilde{L} = \mu^{-1} \circ L \circ \mu$, then it is easy to see that $\widetilde{L} \in \mathcal{L}$ and

$$\begin{aligned} \widetilde{F}_{xy}(t) &> 1-\mu(t) \Leftrightarrow F_{xy}(\mu(t)) > 1-\mu(t) \\ &\Rightarrow F_{AxAy}(L \circ \mu(t)) > 1 - L \circ \mu(t) \\ &\Leftrightarrow F_{AxAy}\{\mu[\mu^{-1} \circ L \circ \mu(t)]\} > 1-\mu[\mu^{-1} \circ L \circ \mu(t)] \\ &\Leftrightarrow \widetilde{F}_{AxAy}(\widetilde{L}(t)) > 1-\mu \circ \widetilde{L}(t). \end{aligned}$$

But this says that A verifies $(\widetilde{L}\mu - c)$ for every μ.

Theorem 3.2.3 [PRD99]**.** Let $(X, \mathcal{F})$ be a complete PM-space of type $\mathcal{M}$, for which the triangle inequality (III^μ) holds. Then every $L-\mu$ probabilistic contraction has a unique fixed point which can be obtained by successive approximations.

Corollary 3.2.4 Let $(X, \mathcal{F}, T)$ be a complete Menger space, for which $T \geq T_\mu$. Then every $L-\mu$ probabilistic contraction on X has a unique fixed point.

Remark 3.2.5 (i) For $\mu(t) = t$ or from Example 3.2.2 we obtain the Theorem 3 of [HIC96]. Actually we can extend to PM-spaces of type $\mathcal{M}$ all the results of [HIC96] and that obtained by the present author.

(ii) It is clear that our Corollary 3.2.4 is applicable for Menger spaces in a class essentially larger than that from [HIC83,96].

4. Generalized C–contractions on Menger spaces

Let $(X, \mathcal{F})$ be a given PSM-space and $A : X \to X$ a fixed mapping.

Definition 4.1 *We say that A is a generalized C– contraction if for each pair of real numbers (a, b), with $0 < a < b$, there exists $L = L_{ab} \in (0, 1)$ such that if*

$$a \leq 1 - F_{pq}(a) \text{ and } 1 - F_{pq}(b+) \leq b,$$

then the following implication holds:

$$(C_{ab}) : \quad F_{pq}(x) > 1 - x \Rightarrow F_{AxAy}(L_{ab}x) > 1 - L_{ab}x$$

We can prove the following.

Theorem 4.2 Every generalized C– contraction on a complete Menger space $(X, \mathcal{F}, T)$, where $T \geq W$, has a unique fixed point, which is globally attractive.

Proof. Let us first note the following simple useful

Lemma 4.3 *In every PSM-space* $(X,\mathcal{F}),\mathbf{K}(p,q)=\sup\{t\ ,\ t\le 1-F_{pq}(t)\}$ *is the only nonnegative real number* k *with the property*

$$1-F_{pq}(k+)\le k\le 1-F_{pq}(k).$$

Now let us suppose that

$$\mathbf{K}(p,q)=k=(1-t)a+tb$$

where $0<a<b$ and $t\in[0,1]$ are fixed.

1^0. Since $a\le k$, then $a\le 1-F_{pq}(k)\le 1-F_{pq}(a)$, and we see that

$$a\le 1-F_{pq}(a).$$

2^0. Since $k\le b$, then

$$1-F_{pq}(b+)\le 1-F_{pq}(k+)\le k\le b,$$

which says that

$$1-F_{pq}(b+)\le b.$$

2^0. Since A is a generalized C– contraction, then (C_{ab}) holds. But, for any $d>0$, we have $1-F_{pq}(k+d)\le 1-F_{pq}(k+)\le k<k+d$, which implies

$$F_{AxAy}(L_{ab}(k+d)>1-L_{ab}(k+d),$$

so that

$$\mathbf{K}(Ap,Aq)\le L_{ab}(k+d),\forall d>0$$

and we see that

$$\mathbf{K}(Ap,Aq)\le L_{ab}\mathbf{K}(p,q),\ \text{if}\ \mathbf{K}(p,q)\in[a,b].$$

Therefore A is a *Krasnoselski contraction* [KREM69]in the *complete metric space* $(X,\mathbf{K})$, which proves the theorem.

We used this type of methods in a recent joint paper with Olga Hadžić and Endre Pap[HPR2001].

References

[BHR76] A. T. Bharucha-Reid: Fixed point theorems in probabilistic analysis. Bull . Amer. Math. Soc. ,82 (1976), 641-657.

[BCN73] Gh. Bocsan, Gh. Constantin: The Kuratowski function and applications to the probabilistic metric spaces. Atti Accad. Naz. Lincei LV (1973), 236-240.

[CAIN85] G. L. Cain: Fixed points of maps on uniform spaces with a contractive iterate at a point. Math. Japonica 30(1985) 691-99.

[CCK95] S. S. Chang, Y. J. Cho, S.M. Kang: Probabilistic metric spaces and nonlinear operator theory. Sichuan University Press, 1995

[CKS75-76] G. L. Cain, Jr. and R. H. Kasriel: Fixed and periodic points of local contractions on PM-spaces. Math. Systems Theory, vol. 9, No. 4 (1975-76) 289-297.

[CIS81] Gh. Constantin , I. Istrăţescu: *Elemente de Analiză Probabilistică şi Aplicaţii.* Ed. Acad., RSR, Bucuresti 1981.

[CNS85] Gh. Constantin: On some classes of contraction mappings in Menger Spaces. In: Seminarul de Teoria Probalităţilor si Aplicaţii. Univ.Timişoara, Nr.76 (1985).

[CIS89] Gh. Constantin and I. Istrăţescu: *Elements of Probabilistic Analysis with Applications.* Ed. Accad. and Kluwer Academic Publ., 1989.

[DMRG68] J. B. Diaz, B. Margolis: A Fixed Point Theorem of the Alternative for Contractions on a Generalized Complete Metric Space. Bull. Amer. Math. Soc. 74, 305-309, 1968.

[HAD78] O. Hadžić: On the (ε, λ)-topology of locally convex spaces. Glasnik Mat.13(33) (1978) 61-66.

[HAD80] O. Hadžić: A generalization of the contraction principle in PM-spaces. Review of Research (Zb. Radova) Prir. Mat. Fac. 10 (1980) 13-21.

[HAD84] O.Hadžić: *Fixed point Theory in topological vector spaces.* Novi Sad, 1984.

[HAD85] O. Hadžić: Some fixed point theorems in PM-spaces. Review of Research, Faculty of Science, Univ. of Novi Sad, 15, 23-35 (1985).

[HAD92] O. Hadžić: On (n, f, g) - locally contractions in PM-spaces. Review of Research, Faculty of Science, Univ. of Novi Sad, 22, 1, 1-10 (1992).

[HAD95] O. Hadžić: *Fixed Point Theory in* PM-*Spaces.* Novi Sad, 1995.

[HDB78] O. Hadžić., M. Budinčević: A fixed point theorem in PM-spaces. Colloq. Math. Soc. J. Bolyai, 23, Topology, 569-579 (1978).

[HDB79] O. Hadžić., M. Budinčević: A class of t-norms in the fixed point theory on PM-spaces. Rev. Research Univ. Novi Sad, 9, 37-41 (1979).

[HDPRD2001] O. Hadžić, E. Pap, V. Radu:Some generalized contraction mapping principles in probabilistic metric spaces (to appear)

[HIC83] T. L. Hicks:Fixed point theory in probabilistic metric spaces. Review of Research, Faculty of Science, Univ. of Novi Sad, 13, 63-72 (1983).

[HIC96] T. L. Hicks: Fixed point theory in probabilistic metric spaces II. Math. Japonica, 44, No. 3, 487-493 (1996).

[HISH84] T. L. Hicks, P. L. Sharma: Probabilistic metric structures. Topological classification. Review of Research Faculty of Science, Univ. of Novi Sad 14,1 43-50 (1984).

[IMT66] M. Iosifescu, G. Mihoc, R. Theodorescu: *Teoria Probabilităţilor şi Statistică Matematică.* Ed. Tehnică, Bucureşti, 1966.

[II81] I.Istrăţescu: A fixed point theorem for mappings with a probabilistic contractive iterate. Rev. Roum. Math. Pures et Appl. (1981), 431-435

[ISSC73] V.I.Istrăţescu, I.Săcuiu: Fixed point theorems for contractions mappings on PM-spaces. Rev. Roum. Math. Pures Appl. 18(1973) 1375-1380.

[IV74] V. I. Istrăţescu: *Introducere in teoria spaţiilor metrice probabiliste şi aplicaţii.* Ed. Tehnica, Bucuresti, 1974.

[KIN64] J. F. C. Kingman: A metric for Wald Spaces. J. London Math.Soc.39 (1964) 129-130.

[KREM69] M. A. Krasnoselski et al: *Approximate solutions of operator equations.* Moscow, 1969.

[MRG68] K. Margolis: On some fixed point theorems in generalized complete metric spaces. Bull. Amer. Math. Soc. 74 (1968) 275-282.

[MEN42] K. Menger: Statistical metrics. Proc. Nat. Academy. Sci. U.S.A., 28 (1942) 535-537.

[MIH93] D. Miheţ: A fixed point theorem for mappings with contractive iterate in H-spaces. Analele Univ. Timişoara, vol. XXXI, 2 (1993).

[MRD93] D. Miheţ, V. Radu: Representable and non-representable Menger norms and applications. Seminarul de Teoria Probabilităţilor şi Aplicaţii, No. 105 1993, Univ. Timişoara.

[MRD93] D. Miheţ, V. Radu: A fixed point theorem for mappings with contractive iterate in PM-spaces. Analele St. ale Univ. Al. I. Cuza, Iasi, t XLII, 1996, 311-314.

[MIH93] D. Miheţ: Probabilistic (b_n) contractions. Seminarul de Teoria Probabilităţilor şi Aplicaţii, No. 107 1993, Univ. Timişoara.

[MIH95] D. Miheţ: A type of contractions mappings in PM - spaces. In Seminarul de Teoria Probabilităţilor şi Aplicaţii, No. 114 1995, Univ. Timişoara.

[MIH97] D. Miheţ: The triangle inequality and fixed points in PM - spaces. Ph.D. Thesis, Universitatea de Vest din Timisoara, 1997.

[MNG78] B. Morrel, J. Nagata: Statistical metric spaces as related to topological spaces. Gen. Top. Appl. , 9 , 233-237 (1978).

[MOSH75] P. Mostert, A. Shields: On the structure of semigroups on a compact manifold with boundary. Annals of Math, 65(1957) 117-143

[MUSR66] D. H. Mushtari, A. N. Serstnev: On methods of introducing a topology in random metric spaces. Izv. vysh. Uch. Zav. Math. 6(55)(1966), 99-106.

[PHM96] E. Pap, O.Hadžić, R. Mesiar: A fixed point theorem in probabilistic metric spaces and applications. J. Math. Anal. Appl. 202 (1996) 431-440.

[PRD97] E. Părău and V. Radu: Some remarks on Tardiff's fixed point theorem on Menger Spaces. Portugaliae Math. 54(4) (1997) 431-440.

[PRD99] E. Părău and V. Radu:On the triangle inequality and contractive mappings in PM-spaces. Analele Univ. Bucureşti, Matematică, Anul XLVIII (1998), nr.1, pp.57-64.

[RD8284] V. Radu: On a family of metrics for the distribution functions. In: Proc.7-th Conf. Prob. Theory August 29-September 4, 1982, Brasov Romania. Ed. Acad. RSR, Bucuresti 1984, 487-492.

[RD83b] V. Radu: A remark on contractions in Menger spaces. Seminarul de Teoria Probabilităţilor şi Aplicaţii, No. 64 1983, Univ. Timişoara.

[RD83a] V. Radu: On the t-norms of Hadžić-type and fixed points in PM-spaces. Review of Research[Zb. Radova]. Prir. Mat. Fak. 13(1983) 81-85.

[RD84a] V. Radu: On the t-norms with the fixed point property, Seminarul de Teoria Probalităţilor si Aplicaţii. Univ.Timişoara 72, 1984.

[RD84b] V. Radu: On the contraction principle in Menger spaces. Analele Univ. Timişoara vol. XXII(1984), fasc. 1-2 83-88.

[RD85] V. Radu:On some contraction - type mappings in Menger spaces. Analele Univ. Timişoara, vol. XXIII, 23, 61-65 (1985).

[RD86a] V. Radu: On obtaining Levi's metric by probabilistic metric. Analele Univ. Timisoara XXIV, 60-65 (1986).

[RD86b] V. Radu: A family of deterministic metrics on Menger spaces. In: Seminarul de Teoria Probalităţilor si Aplicaţii. Univ.Timişoara, nr.78 (1986).

[RD86c] V. Radu: Some remarks on the triangle inequality in probabilistic metric spaces. In: Seminarul de Teoria Probalităţilor si Aplicaţii. Univ.Timişoara nr.80 (1986).

[RD87] V. Radu: Some fixed Point Theorems in Probabilistic Metric Spaces. In "Lectures Notes in Math.", Vol. 1223, pp. 125-133, Springer-Verlag, New York, 1987.

[RD88] V. Radu: Deterministic metrics on Menger spaces and applications to fixed point theorems. Babeş-Bolyai University, Faculty of Mathematics and Physics Research Seminar, Preprint nr.2, 163-166 (1988).

[RD92] V. Radu: Some remarks on the representation of triangular norms. Analele Univ. Timişoara, Seria Ştiinţe Matematice, Vol. XXX, fasc. 1, 1992, 119–130.

[RD94] V. Radu: *Lectures on PM-Spaces.* University of Timişoara, Surveys Lectures Notes and Monographs Series on Probability Statistics and Applied Mathematics, vol. 2, 1994.

[RD98] V. Radu: A fixed point principle in probabilistic metric spaces under Archimedean triangular norms. Analele Univ. Timişoara, Seria Matematică-Informatică, Vol. XXXVI, fasc. 1, 1998, 109–116.

[RD99] V. Radu: Triangular norms and Menger spaces with the fixed point property for probabilistic contractions. Analele Univ. Timişoara, Seria Matematică-Informatică, Vol. XXXVII, fasc. 1, 1999, 123–132.

[RDM94] V. Radu, D. Miheţ: A fixed point principle in σ-Menger spaces. Analele Univ. Timişoara, vol. XXXII, fasc. 2, 99-110 (1994).

[RUS79] I. Rus: *Principii şi Aplicaţii ale Teoriei Punctului Fix.* Ed. Dacia, 1979.

[SCSK60] B. Schweizer , A. Sklar: Statistical metric spaces. Pacific J. Math. 10(1960)133-334.

[SST88] B. Schweizer, H. Sherwood, R. Tardiff: Contractions in Probabilistic Metric Spaces:examples and counterexamples. Stochastica 12(1988) 5-17.

[SCSK83] B. Schweizer, A. Sklar: *Probabilistic Metric Spaces.* North-Holland, New York, 1983.

[SSTH60] B. Schweizer, A. Sklar, E. Thorp: The metrization of SM- spaces. Pacific I. Math. 10 (1960) 673-75.

[SHG66] V. M. Sehgal: Some fixed point theorems in functional analysis and probability. Ph. D. dissertation, Wayne State Univ. 1966.

[SBHR72] V. M. Sehgal, A.T.Bharucha-Reid: Fixed points of contraction mappings on PM-spaces. Math. Syst. Theory 6(1972)97-100.

[SHR71] H. Sherwood: Complete probabilistic metric spaces. Z. Wahr . verw. Geb. 20(1971) 117-128.

[SER63] A.N. Šerstnev: The notion of random normed space. Doklady Akad. Nauk SSSR, 149, 280-283 (1963) (Russian).

[SER64] A. N. Šerstnev: On the probabilistic generalization of metric spaces. Kazan Gos. Univ. Uch. Zap. 124(1964) 3-11.

[SRJ88] M. Stojaković: On some classes of contraction mappings. Math. Japonica 33, 2, 311-318 (1988).

[TAR92] R. M. Tardiff: Contraction Maps on Probabilistic Metric Spaces. Journal Math. Anal. Appl.,165 ,1992,517-523.

[TAY85] M. D. Taylor: New metrics for weak convergence of distribution functions. Stochastica Vol.I, 1985.

[WAL43] A. Wald.: On a statistical generalization of metric spaces. Proc. Nat. Acad. Sci. USA 29, 196-197 (1943).