IDEAS AND METHODS IN FIXED POINT THEORY FOR
PROBABILISTIC CONTRACTIONS

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Abstract. The notion of B-contraction mapping in probabilistic metric spaces is due to V. M. Sehgal (1966), who proved that any B-contraction on a complete Menger space \((X, F, \text{Min})\) has a unique fixed point. Important contributions are due to Sherwood (1970) and Istrățescu & Săcătu (1971). A fundamental step was made by O. Hadzić in 1978, who introduced a class of continuous t-norms, essentially weaker than \(\text{Min}\), for which the above result of Sehgal still holds.

Our aim is to present some comments and results related to the following statements concerning a triangular norm \(T\):

(B1) \(T\) is of Hadzić type; that is the family of its iterates is equicontinuous at \(x = 1\).

(BII) \(T\) has the fixed point property; that is each B-contraction on every complete Menger space \((X, F, T)\) has a fixed point.

(BIII) \(\forall a \in (0, 1), \exists b \geq a\) such that \(T(b, b) = b < 1\)

which are seen to correspond to different kinds of classical deterministic fixed point theorems, together with the main tools used in fixed point theory for probabilistic contractions.

Keywords: Probabilistic metric space, probabilistic contraction, fixed point

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1. Preliminaries

1.1. Menger norms and triangular norms

Definition 1.1.1 \(^1\)

A mapping

\[ T : [0, 1] \times [0, 1] \rightarrow [0, 1] = I \]

is called a Menger- norm (shortly \(M\)-norm) if it satisfies the following conditions:

N1) \(T(a, b) = T(b, a), \forall a, b \in I\)

N2) \(a \leq c, b \leq d \Rightarrow T(a, b) \leq T(c, d)\)

N3) \(T(a, 1) = a, (\forall) a \in I.\)

2) A triangular norm (shortly \(t\)-norm) is an associative \(M\)-norm:

N4) \(T(a, T(b, c)) = T(T(a, b), c), \forall a, b, c \in I.\)
It is easy to see that if $T$ is an $M$-norm then $T(a, b) \leq \min(a, b)$ (\forall) a, b \in I$, and $T(a, 0) = T(0, a) = 0$, (\forall)\ a \in I. Among the most important examples of t-norms, we will use:

$$T_1(a, b) = W(a, b) = \max(a + b - 1, 0),$$

$$T_2(a, b) = T_{2P}(a, b) = \prod(a, b),$$

$$T_{\infty}(a, b) = T_{\infty}(a, b) = \min(a, b).$$

Given a t-norm $T$ and an element $x \in [0, 1]$, we can define the $T$-powers of $x$ by:

$$x^0 = 1, x^1 = x \text{ and } x^{n+1} = T(x^n, x), (\forall) n \geq 1.$$ Since $([0, 1], T)$ is a semigroup, $T(x^n, x^m) = x^{n+m}$, (\forall)n, m \in N.

**Definition 1.1.2** A t-norm $T$ is called *Archimedean* if, for each $a \in [0, 1)$, $\lim_{n \to \infty} a^n = 0$ or, equivalently,

$$\forall a, \varepsilon \in (0, 1), \exists m \in N : a^m < \varepsilon .$$

**Proposition 1.1.3** A continuous (in the product topology on I) t-norm is Archimedean if and only if

$$\delta(x) < x (\forall)x \in (0, 1),$$

where $\delta(x) := T(x, x) = x^2$.

There are known many proofs of the representation theorems for continuous and Archimedean t-norms, which have a simple structure.

Simple proofs can be given by using the following

**Lemma 1.1.4** Let $T$ be a continuous and Archimedean t-norm.

a) If $T$ does not have interior nilpotents, then the semigroup $([0, 1], T)$ is isomorphic with the semigroup $([0, 1], \prod)$ (Faucett, 1955)

b) If $T$ has interior nilpotents, then $([0, 1], T)$ is isomorphic with $([1/2, 1], o)$ where $x \circ y = \max\{1/2, xy\}$ [M-S, 1957].

Therefore, if $T$ is a continuous and Archimedean t-norm, then there exist $\alpha \in \{0, 1/2\}$ and $h : [0, 1] \to [\alpha, 1]$, a continuous bijection, such that

$$T(u, v) = h^{-1}(\max\{\alpha, h(u) \cdot h(v)\}), (\forall) u, v \in [0, 1].$$

c) Let $h^{(-1)} : [0, 1] \to [0, 1], h^{(-1)}(x) = h^{-1}(\max\{\alpha, x\}).$ Then

$$\textit{(Multi)} \quad T(u, v) = h^{(-1)}(h(u) \cdot h(v)), (\forall) u, v \in [0, 1].$$

d) Moreover, for $f : [0, 1] \to [0, -\log \alpha], f(x) = -\log h(x)$, $f$ is strictly decreasing and continuous, with $f(1) = 0$, and

$$\textit{(Addit)} \quad T(a, b) = f^{(-1)}(f(a) + f(b))(\forall) u, v \in [0, 1],$$

where $f^{(-1)}(x) = f^{-1}(\max\{x, f(0)\})$ is the pseudo-inverse of $f$.

**Proposition 1.1.5** (The structure theorem for continuous t-norms). Let $T$ be a continuous t-norm. Then there exists an at most countable family of closed intervals $I_k = [\alpha_k, \beta_k] \subset [0, 1]$, such that

i) $[0, 1] = (\cup I_k) \cup C(\cup I_k)$

ii) $(\alpha_k, \beta_k) \cap (\alpha_l, \beta_l) = \emptyset, (\forall) k \neq l$
iii) $T(b,b) = b$, $(\forall) b \notin \cup (\alpha_k, \beta_k)$

iv) $T(a,b) = \begin{cases} T_k(a,b) \in I_k, & \text{if } a,b \in I_k \\ \min(a,b), & \text{otherwise} \end{cases}$

v) $T(a,a) < a$, $(\forall) a \in I_k$ (thus $T_k = T/_{I_k \times I_k}$ is Archimedean)

The proof is based on the fact that the set $i = \{b | T(b,b) = b\}$ is a closed subset of $[0,1]$, and $C_i = (0,1) \setminus i$ is an at most countable union of open disjoint intervals. More details on M-norms and t-norms can be seen in [SCSK83]. A proof of the representation theorem can also be found in [MRD93].

1.2. The strong topology and the strong semiuniformity on Menger spaces

In what follows $\Delta_+^+$ denotes the set of distribution functions $F : [0, \infty] \rightarrow [0,1]$ with the properties:

a) $F(0) = 0$ and $F(\infty) = 1$;

b) $F$ is increasing;

c) $F$ is left continuous on $(0, \infty)$.

$D_+$ is the subset of $\Delta_+^+$ containing functions $F$ which also satisfy the condition

$$\lim_{x \to \infty} F(x) = 1.$$ If $a \geq 0$, then $\varepsilon_a$ is defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a \\ 1, & \text{if } x > a \end{cases}.$$

Let $X$ be a nonempty set and $F : X \times X \rightarrow D_+$ a given mapping $(F(x,y)$ will be denoted by $F_{xy})$. The pair $(X,F)$ is called a probabilistic semi-metric space (shortly PSM-space) if

I. $F_{xy} = \varepsilon_0$ if and only if $x = y$

II. $F_{xy} = F_{yx} \forall x,y \in X$.

One uses the generic term probabilistic metric space (PM-space) if some kind of ”triangle inequality” is verified. The weakest one was proposed in [SCSK60]:

$$IIISS. [F_{xy}(t) = 1, F_{yz}(t) = 1] \Rightarrow F_{xz}(t+s) = 1$$

If there exists a triangular norm $T$ such that

$$III_M. F_{xz}(t+s) \geq T(F_{xy}(t), F_{yz}(s))$$

then we say that $(X,F,T)$ is a Menger space. A more general form for $III_M$, defining $\sigma$-Menger spaces, was formulated by using some operations $\sigma$ on $[0,\infty)$, instead of the addition (see [RD94] for more details).

In [HISH84] is proposed the inequality $(\forall) \varepsilon > 0 \exists \delta > 0$ such that

$$III_1. |1 - F_{xy}(\delta)| < \delta, 1 - F_{yz}(\delta) < \delta \Rightarrow 1 - F_{xz}(\varepsilon) < \varepsilon,$$

which can be generalized: $(\forall) \varepsilon > 0 \exists \delta > 0$ such that

$$III_f. |f \circ F_{xy}(\delta)| < \delta, f \circ F_{yz}(\delta) < \delta \Rightarrow f \circ F_{xz}(\varepsilon) < \varepsilon,$$

by using additive generators $f$ (see e.g. [RD94] for more details).
For every PSM-space \((X, \mathcal{F})\) we can consider the sets of the form 
\[ U_{\varepsilon, \lambda} = \{(x, y) \in X \times X, \ F_{xy}(\varepsilon) > 1 - \lambda\}, \ \varepsilon > 0, \lambda \in (0, 1) \]
which generates a semiuniformity, denoted by \(U_{\mathcal{F}}\), and a topology, \(T_{\mathcal{F}}\), named also the \((\varepsilon, \lambda)\)-topology, the strong topology, or the \(\mathcal{F}\)-topology. Namely,
\[ \mathcal{O} \in T_{\mathcal{F}} \iff \forall x \in \mathcal{O} \exists \varepsilon > 0, \exists \lambda \in (0, 1) \text{ s.t. } U_{\varepsilon, \lambda}(x) \subset \mathcal{O} \]
Actually, \(U_{\mathcal{F}}\) can also be generated by the family of the sets \(V_{\delta} := U_{\delta, \delta}\).

2. B-Contractions on Menger spaces

The notion of contraction map in probabilistic metric spaces was introduced by V. M. Sehgal in [SHG66](cf. [SBHR72]).

**Definition 2.1** Let \((X, \mathcal{F})\) be a probabilistic metric space and \(A : X \to X\). The mapping \(A\) is called a probabilistic contraction or **B-contraction** if there exists an \(L \in (0, 1)\) such that, for all points \(p, q \in X\) and all \(u \geq 0\), the following inequality holds:
\[ (B) \quad F_{A_p, A_q}(L u) \geq F_{p, q}(u). \]

In the paper [SBHR72] it is shown that any B-contraction on a complete Menger space \((X, \mathcal{F}, \text{Min})\) has a unique fixed point.

Immediate contributions are due to H. Sherwood, who obtained a simple characterization for the existence of fixed points and proved that for a very large class of triangular norms it is possible to construct complete Menger spaces together with fixed points free contraction maps and to V.I. Istrătescu-I. Săcuiu [ISS73].

A fundamental step is made by O. Hadžić in 1978, who introduces a class of continuous t-norms, essentially weaker than Min, for which the above result of Sehgal still holds.

If \(T\) is a given t-norm, then \(T^m\) is defined on \(I^m\) by
\[ T^1(x) = x, T^{m+1}(x_1, \ldots, x_{m+1}) = T(T^m(x_1, \ldots, x_m), x_{m+1}). \]

**Definition 2.2** [RD83b] We say that \(T\) is an **h-t-norm** (of Hadžić type or of h-type), if the family of mappings \(H_T = \{T_m\}_{m \in \mathbb{N}}\), defined on \(I\) by
\[ T_m(x) = T^m(x, x, \ldots, x), \]
is equicontinuous at \(x = 1\).

There are nontrivial examples of h-t-norms, due also to Olga Hadžić.

**Definition 2.3** [RD84, 87, 99] We say that the t-norm \(T\) has the **fixed point property** (shortly f.p.p.) if each B-contraction on every complete Menger space \((X, \mathcal{F}, T)\) has a fixed point (which clearly is unique and globally attractive).

In this section we will present some comments and results related to the following three statements concerning a triangular norm:
\[ (B_1) \quad T \text{ is of Hadžić type}; \]
\[ (B_2) \quad T \text{ has the fixed point property}; \]
Moreover, it suffices to consider contractions with the Lipschitz constant

\( A \).

A unique extension

\( A^* \) of \( A \) such that \( A^* \) is a \( B \)-contraction on \( X \), with the same

Lipschitz constant.

The converse holds for lc-t-norms. \((\text{space not have the f.p.p.})\)

Since \( T \) is of h-type and \( T \) is a \( B \)-contraction on \( X \), then there exists a completion \((X^*, F^*, T)\) of \((X, F, T)\) and a

unique extension \( A^* \) of \( A \) such that \( A^* \) is a \( B \)-contraction on \( X \), with the same Lipschitz constant.

Lemma 1. If \((X, F, T)\) is a Menger space, \( T \) is an lc-t-norm and \( A \) is a \( B \)-

contraction on \( X \), then there exists a completion \((X^*, F^*, T)\) of \((X, F, T)\) and a

unique extension \( A^* \) of \( A \) such that \( A^* \) is a \( B \)-contraction on \( X \), with the same

Lipschitz constant.

Lemma 2. A t-norm \( T \) has the f.p.p. if for every \( B \)-contraction \( A \) on a Menger

space \((X, F, T)\) and for each fixed \( p_0 \) in \( X \), the sequence \( p_n = A^n p_0 \) is \( F \)-Cauchy.

Moreover, it suffices to consider contractions with the Lipschitz constant in \((0, \frac{1}{2}]\).

The converse holds for lc-t-norms.

2.1. B-Contractions and the t-norms of Hadžić-type

Remark 2.1.1 Olga Hadžić proved in [HAD80] that each continuous t-norm of h-

type has the f.p.p. The following theorem shows that the continuity is not necessary.

Theorem 2.1.2 [RD83]. Every t-norm of h-type has the fixed point property.

Proof. Let \((X, F, T)\) be a Menger space such that \( T \) is of h-type and consider a

mapping \( A : X \to X \) which verifies (B) with \( L \in (0, \frac{1}{2}] \).

Let \( p_0 \in X \) and \( x \in (0, \infty) \) be fixed. If \( m \) is a positive integer, then

\[
F_{p_0 A^m p_0}(2x) \geq T(F_{p_0 A^m p_0}(x), F_{p_0 A^m p_0}(x)) \\
\geq T(F_{p_0 A^m p_0}(x), F_{p_0 A^m p_0}(2x))
\]

and, therefore,

\[
F_{p_0 A^m p_0}(2x) \geq T_m(F_{p_0 A^m p_0}(x)), \forall m \geq 1.
\]

Thus we obtain that for any positive integers \( n, m \),

\[
F_{A^n p_0}(A^{m+n} p_0)(2x) \geq T_m(F_{A^n p_0}(x L^{-n})).
\]

Since \( T \) is of h-type and \( F_{p_0 A^m p_0} \in D_+ \), then it follows that

\[
\lim_{n \to \infty} F_{A^n p_0 A^{m+n} p_0}(2x) = 1,
\]

uniformly in \( m \), for each \( x \in (0, \infty) \). By definition, (4) means that \( \{A^n p_0\} \) is \( F \)-Cauchy

and the theorem follows from Lemma 2.

Lemma 2.1.3[RD84] Let \( T \) be an lc-t-norm and fix an \( F \) in \( D_+ \). Let \( X =
\{1, 2, \ldots\} \) and define a probabilistic metric on \( X \) by

\[
F_{n+m}(x) = T^n[F(2^n+1 x), F(2^n+2 x), \ldots, F(2^{n+m} x)], m \neq 0
\]

and \( F_{n+m} = H_0, m = 0 \). Then \((X, F, T)\) is a Menger space and the mapping

\( n \to n + 1 \) is a contraction with the Lipschitz constant \( \frac{1}{2} \).

A partial converse to Theorem 2.1.2 is the following.

Theorem 2.1.4[RD84] If \( T \) is an lc-t-norm which is not of h-type, then \( T \) does
not have the f.p.p.
Proof. If $T$ is not of $h$-type, then there exists $a \in (0, 1)$ such that for each $b > a$ there is $m_b \geq 1$ for which $T_{m_b}(b) < a$. Let $b_n \in (a, 1)$ be increasing to 1, and $m_n \geq 1$, strictly increasing and such that

\[(6) \quad T_{m_n}(b_n) < a, \quad n = 1, 2, \ldots\]

Let $F \in D_+$ be defined by

\[(7) \quad F(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ b_1 & \text{if } x \in (1, 2^{2+m_1}] \\ b_{n+1} & \text{if } x \in (2^{2n+m_n}, 2^{2n+m_{n+1}}], \quad n \geq 1 \end{cases}\]

If we consider the Menger space from Lemma 2.1.3, then we have successively:

\[
\begin{align*}
F_{nn+m_n}(1) & \leq F_{nn+m_n}(2^n) \\
& = T_{m_n}[F(2^{2n+1}), F(2^{2n+2}), \ldots, F(2^{2n+m_n})] \\
& \leq T_{m_n}[F(2^{2n+m_n}), \ldots, F(2^{2n+m_n})] \\
& \leq T_{m_n}(b_n) < a
\end{align*}
\]

Therefore the sequence $\{A^n1\}$ is not $\mathcal{F}$-Cauchy. From Lemma 2 it follows that $T$ does not have the fixed point property and the theorem is proved.

**Lemma 2.1.5.** [RD83] Let $T$ be a right continuous t-norm of Hadzic type. Then

\[
\forall a \in (0, 1), \exists b \geq a \text{ such that } T(b, b) = b < 1,
\]

that is $(B_I \Rightarrow B_{III})$ in this case.

**Proof.** Suppose that $(B_I)$ holds, and let $a > 0$ be fixed. Then there exists $c > a$ such that $T_m(x) > a, \forall x \geq c, \forall m \leq 1$. Since clearly $\{T_m(c)\}$ is nonincreasing, then it is convergent to some limit $b \geq a$. As

\[
T_{2m}(c) = T(T_m(c), T_m(c))
\]

then $b = T(b, b)$ and we obtain that $(B_I)$ implies $(B_{III})$.

By combining the above results we obtain the following

**Theorem 2.1.6.** [RD84, 87] Let $T$ be a continuous t-norm. Then the statements $(B_I), (B_{II})$ and $(B_{III})$ are equivalent.

**Lemma 2.1.7.** [RD84b, 99] Let $T$ be a continuous t-norm. Then

1° $T \not\in \mathcal{H}$ iff there exists $a \in [0, 1)$ such that

\[
T(a, a) = a, \text{ and } T(x, x) < x, \forall x \in (a, 1).
\]

2° $T \not\in \mathcal{H}$ iff there exist $a_T \in [0, 1)$ and an increasing bijection $h_T : [a_T, 1] \rightarrow [0, 1]$ such that

\[(8) : T(\alpha, \beta) = h_T^{-1}[\tilde{T}(h_T(\alpha), h_T(\beta))], \forall \alpha, \beta \geq a_T
\]

where $\tilde{T} = T_1$ or $\tilde{T} = \text{Prod} (\tilde{T}$ depends only on $T$).

The following result is easy to reproduce:

**Proposition 2.1.8.** [RD84b, 87, 99] Let $T$ be a continuous t-norm, $T \not\in \mathcal{H}$.

(i) If $(X, \mathcal{F}, T_1)$ is a Menger space, then $(X, e^{\mathcal{F}^{-1}}, \text{Prod})$ is a Menger space with the same $(\varepsilon, \lambda)$-uniformity;
(ii) If \((X, \mathcal{F}, \tilde{T})\) is a Menger space, then \((X, h_T^{-1} \circ \mathcal{F}, T)\) is a Menger space with the same \((\varepsilon, \lambda)\)-uniformity;

(iii) If \((X, \mathcal{F}, T)\) is a Menger space, then \((X, h_T \circ \mathcal{F}, \tilde{T})\) is a Menger space with the same \((\varepsilon, \lambda)\)-uniformity.

**Corollary 2.1.9** [RD84b, 87, 99] Let \(T\) be an arbitrary but fixed t-norm such that \(T \notin H\). Then the following are equivalent

(i) \(T\) does not have the f.p.p.;

(ii) Prod does not have the f.p.p.;

(iii) \(T_1\) does not have the f.p.p.

**Focus on the statement (BIII) and a family of pseudo-metrics**

The following two lemmas are well-known:

**Lemma 2.2.1** If \(a \leq b \leq c\), \(T(b, b) = b\) and \(T\) is continuous, then

\[
\begin{align*}
1^0 & \quad T(a,b) = a; \\
2^0 & \quad T(b,c) = b; \\
3^0 & \quad T(a,c) = a.
\end{align*}
\]

**Lemma 2.2.2** [RD92] Let \(\{b_n\} \subset [0,1)\) be a strictly increasing sequence such that \(b_n \to 1\). Then, for every distribution function \(G \in D_+\), the function \(G^*\) defined by:

\[
4^0 G^*(t) = \begin{cases} 
0 & \text{iff } G(t) \leq b_0 \\
b_n & \text{iff } G(t) \in (b_n, b_{n+1}] \\
1 & \text{iff } G(t) = 1
\end{cases}
\]

is also in \(D_+\). Moreover, \(G^* \leq G\).

Using these results we can obtain the following

**Theorem 2.2.3** [RD92] Let \(\{b_n\} \subset [0,1)\) be a strictly increasing sequence. If we suppose that \(b_n \to 1\), \(T(b_n, b_n) = b_n\) and

\[
5^0 \quad a, b > b_n \Rightarrow T(a, b) > b_n,
\]

then, for every Menger space \((X, \mathcal{F}, T)\), we have that \((X, \mathcal{F}^*, \text{Min})\) is a Menger space with the same \((\varepsilon, \lambda)\)-topology. Moreover, if \((X, \mathcal{F})\) is complete, then \((X, \mathcal{F}^*)\) is complete.

**Definition 2.2.4** [RD92] The continuous t-norm \(T\) is of type Hadžić-Budinčević if the family \(\{T_n\}\) is equicontinuous at \(b = 1\) and each \(T_k\) is strict.

We know that a t-norm of type Hadžić-Budinčević verifies the condition \(5^0\), for some sequence \(\{b_n\}\).

If we have that \(I_k \cap I_{k'} = \emptyset\) for \(k \neq k'\) and \(\lim b_k = 1\), then \(T\) is of Hadžić type (that is \(\{T^n\}\) is equicontinuous at \(1\)) and we can choose a sequence \(\{b'_n\}\) for which \(5^0\) holds and \(T\) is not necessarily of Hadžić-Budinčević type.

The following results are very clear:

**Theorem 2.2.5 (of Hadžić-Budinčević)** [HBD78, 79] If \((X, \mathcal{F}, T)\) is a complete Menger space, \(T\) is continuous, \(\{T^n\}\) is equicontinuous at \(1\) and each \(T_k\) is strict, then every B-contraction on \(X\) has a unique fixed point.

**Theorem 2.2.6** (see e.g. [CKS75, KAS]). Let \((X, \mathcal{F}, T)\) be a complete Menger space, where \(T\) verifies the condition \(5^0\) from Theorem 1.3. Then every probabilistic contraction on \(X\) has a unique fixed point.

**Theorem 2.2.7** [RD92] The theorems 2.2.5 and 2.2.6 are equivalent.
of G.L.Cain can be applied. In fact, for statement that the mapping obtain a countable family (2.2.6, then, by Theorem 2.2.3., (X, F*, Min) is a complete Menger space. It is easy to see that any probabilistic contraction in (X, F, T) is a probabilistic contraction in (X, F*, Min) and, by 2.2.5., it has a unique fixed point.

Remark 2.2.8 [RD92] The result of Theorem 2.2.5 is proven by using deterministic semi-metrics of the form (9) ρₜ(x, y) = sup{t/Fₓₜ(t) ≤ b} In [RD83b, RD84] we used a different method, by using a generalized metric, in order to prove a result more general. Actually formula (9) can be slightly modified:

(9') dₜ(x, y) = inf{t/Fₓₜ(t) ≥ b}

and if b = T(b, b) then dₜ is a pseudo-metric. Moreover, if T is of Hadžić type, then we obtain a countable family {dₜ} which generate the (ε, λ)-uniformity and the method of G.L.Cain can be applied. In fact, for t-norms of Hadžić-Budincević, formula (9) can be used for F* and one obtains generally (9').

The pseudo-metrics of type (9') have been successfully used by D. Miheţ to prove fixed point theorems for more general contraction-type mappings.

Definition 2.2.9 [MIH93,97]. Let (bₙ) ∈ b, that is strictly increasing to 1. We say that the PSM space (X, F) is 1° a (bₙ)-probabilistic metric structure or a (bₙ)-strict structure if the following triangle inequality takes place:

(PM₁(bₙ)) (Fₚₗ(s) > bₙ, Fₚᵣ(t) > bₙ) ⇒ Fₚᵣ(s + t) > bₙ

2° a (bₙ) - probabilistic metric structure if the following relation takes place:

(PM₂(bₙ)) (Fₚₗ(s) ≥ bₙ, Fₚᵣ(t) ≥ bₙ) ⇒ Fₚᵣ(s + t) ≥ bₙ.

Note that every Menger space relative to a (bₙ)-M norm T of Hadžić-Budincević type is a (bₙ)-probabilistic metric structure:

(Fₚₗ(s) > bₙ, Fₚᵣ(t) > bₙ, Fₚᵣ(s + t) ≥ T(Fₚₗ(s), Fₚᵣ(t))) ⇒ Fₚᵣ(s + t) > bₙ

and every Menger space relative to a (bₙ)-M norm of Hadžić type is a (bₙ)-probabilistic metric structure.

Definition 2.2.10 [MIH97]. Let (X, F) be a PSM space and let (bₙ) ∈ b. We say that the mapping f : X → X is a strict (bₙ)-probabilistic contraction (shortly a s-(bₙ) contraction) if:

(∀)n ∈ N (∃)k ∈ (0, 1) : Fₚₗ(t) > bₙ ⇒ Fₚᵣₕₗₙ(t) > bₙ.

Obviously, every B-contraction is a s-(bₙ) contraction for every (bₙ) ∈ b.

Theorem 2.2.11 [MIH93,97]. If (X, F) is a complete s-(bₙ) probabilistic metric structure and f : X → X is an s-(bₙ)-contraction, then f has a unique fixed point.

Corollary 2.2.12 [HDB78]. Let T be a (bₙ)- t-norm of Hadžić-Budincević type. If (X, F, T) is a complete Menger space and f : X → X is a mapping with the property that for every n ∈ N there exists kₙ ∈ (0, 1) such that

Fₚₗ(t) > bₙ ⇒ Fₚᵣₕₗₙ(t) ≥ Fₚₗ(t),

then f has a unique fixed point.
Definition 2.2.13 [MIH97]. Let \((X, \mathcal{F})\) be a PSM space and let \((b_n) \in b\). We say that \(f : X \to X\) is a \((b_n)\)-probabilistic contraction if
\[
(\forall) n \in \mathbb{N} \ (\exists) k_n \in (0, 1) : F_{pq}(t) \geq b_n \Rightarrow F_{fpfq}(k_n t) \geq b_n.
\]

Theorem 2.2.14 [MIH97]. If \((X, \mathcal{F})\) is a complete \((b_n)\)-probabilistic metric structure and \(f : X \to X\) is a \((b_n)\)-probabilistic contraction, then \(f\) has a unique fixed point which can be obtained by the successive approximation method, starting from an arbitrary point of \(X\).

2.3. B-Contractions and generalized metrics

Let \(E\) be an element of \(D_+\) and consider a Menger space \((X, \mathcal{F}, T)\), where \(T\) is an lc-t-norm.

Theorem 2.3.1 [RD83a,83b]. The function \(d_E : X \times X \to [0, \infty]\), defined by
\[
d_E(p, q) = \inf \{a > 0, F_{pq}(ax) \geq E(x), \forall x \in \mathbb{R}\}
\]
has the following properties:
1\(^{st}\) \(d_E(p, q) = 0 \iff p = q;\)
2\(^{nd}\) \(d_E(p, q) = d_E(q, p);\)
3\(^{rd}\) \(d_T(E, G)(p, q) \leq d_E(p, r) + d_G(r, q);\)
4\(^{th}\) If \(d_E \neq \infty\), then the semiuniformity and the topology generated by \(d_E\) are stronger than those generated by \(F;\)
5\(^{th}\) If \(\pi_T(E, E) = E\), then \(d_E\) is a generalized metric on \(X\).
6\(^{th}\) Every B-contraction on \((X, F, T)\) is a strict contraction on \((X, d_E)\) for each \(d_E\), with the same Lipschitz constant.

Remark 2.3.2. The left continuity of \(E\) is not used above.

Example 2.3.3 If
\[
E(x) = \begin{cases} 0 & x \leq 1 \\ 1 & x > 1 \end{cases},
\]
then
\[
d_E(p, q) = \inf \{x, F_{pq}(x) = 1\},
\]
which shows that \(d_E\) need not be nontrivial. Moreover, if \((X, d)\) is a metric space, considered as a Menger space with \(F_{pq}(x) = H(x - d(p, q))\), then \(d_E(p, q) = d(p, q).

Corollary 2.3.4 [RD83a]. Let \((X, F, Min)\) be a complete Menger space and suppose that \(A : X \to X\) is a B-contraction. Then \(A\) has a unique fixed point \(p^*\), and for each \(p \in X\), \(p^* = \lim_{n \to \infty} A^n(p)\), in the \((\varepsilon, \lambda)\) topology.

Proof. Let \(E = F_{pA}\). Then \(d_E\) is a generalized metric space, \(d_E(p, A(p)) \leq 1\), and it is easy to see that \((A^n(p))\) is \(d_E\)-Cauchy. Therefore it is \(\mathcal{F}\)-Cauchy, and the theorem follows.

The following lemma shows how to construct generalized metrics on a Menger space under an h-t-norm:

Lemma 2.3.5 [RD83a,83b]. Let \(T\) be an h-t-norm. For
\[
0 < a_1 < a_2 < \ldots, a_n \to \infty
\]
2. It is easy to see that

\[ 2.3.5 \text{ and such that} \]

result still holds, with a new proof.

\[ F(i) \text{ We prove only the triangle inequality. If} \]

\[ \text{Then} \]

\[ d(p, q) = \inf \{a > 0, \ F_{pq}(ax) \geq F(x), \forall x \in R \} \]

Consider a Menger space \((X, \mathcal{F}, T)\) and define

\[ d(p, q) = \inf \{a > 0, \ F_{pq}(ax) \geq F(x), \forall x \in R \} \]

Then

(i) \(d\) is a generalized metric on \(X\);

(ii) If \(X\) is \(\mathcal{F}\)-complete, then \(X\) is \(d\)-complete;

(iii) The \(d\)-uniformity is stronger than the \(\mathcal{F}\)-uniformity.

Proof. (i) We prove only the triangle inequality. If

\[ d(p, q) < a' < a, \forall \langle a, q, r \rangle < b' < b \]

and \(x \in (a_n, a_{n+1}]\), then

\[ F_{pq}(a'x + b'x) \geq T(F_{pq}(a'x), F_{pq}(b'x)) \geq T_1(F(x)) \]

Therefore \(d(p, q) \leq a' + b' < a + b\), and we obtain the triangle inequality.

(ii) and (iii): Let \(\{p_n\}\) be a \(d\)-Cauchy sequence. and fix \(a > 0\).

For \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), let \(x_0 \in R\) such that \(F(x_0) > 1 - \lambda\) and let \(a > 0\) such that \(ax_0 < \varepsilon\). By the definition of \(d\), there exists \(n_0 \geq 1\) such that

\[ F_{p_n}(ax) \geq F(x), \forall : n \geq n_0, \forall m \geq 1, \forall x \in R. \]

If \(n \geq n_0, m \geq 1\), then \(F_{p_n}(ax) > 1 - \lambda\), which shows that \(\{p_n\}\) is \(\mathcal{F}\)-Cauchy.

If we suppose that \(X\) is \(\mathcal{F}\)-complete, then \(\{p_n\}\) is convergent to some limit \(p\).

Therefore

\[ F(x) \leq \lim_{m \to \infty} \inf_{n \geq n_0} F_{p_n}(ax) = F_{p_n}(ax) \]

for each \(x\) and all \(n \geq n_0\), that is \(d(p, q) \leq a, \forall n \geq n_0\). Thus \(p_n\) is \(d\)-convergent and the lemma is proved.

**Remark 2.3.6** For given \(p, q\) in \(X\), we can take \(a_n\) in the lemma such that \(F_{pq}(a_n) \geq b_n\). Therefore the metric \(d\) is nontrivial and we obtain that the above result still holds, with a new proof.

**Remark 2.3.7** As it is well known the Banach contraction principle is a consequence of the above Corollary. Actually we can modify the above proof in order to see that the Banach fixed point principle implies Corollary 2.3.4: Let \((X, \mathcal{F}, T)\) and \(A\) be as in Corollary 2.3.4. If \(p_0\) is given in \(X\), then let \(a_n\) and \(b_n\) be as in Lemma 2.3.5 and such that \(F_{p_0}(a_n) \geq b_n\). Consider the generalized metric \(d\) as in Lemma 2. It is easy to see that \(d(p_0, A) < \infty\) and therefore \(X_0 := \{q_0 \in X, d(p_0, q_0) < \infty\}\)
is a complete metric space and $A$ is a strict contraction on $X_0$. Therefore $p_n = A^n p_0$ will $d$-converge to the (evidently unique) fixed point of $A$.

### 2.3.1. A generalized metric on probabilistic $f$-metric structures and the fixed point alternative

Let $f : [0, 1] \rightarrow [0, \infty)$ be a continuous function which is strictly decreasing and vanishes at 1.

**Definition 2.3.1.1** ([RD85]). The pair $(X, F)$ which has the properties $(PM0) - (PM2)$ is called a **probabilistic $f$-metric structure** iff $\forall \epsilon > 0, \exists s > 0$ such that

$$III f_0 - (PM2) \Rightarrow f_0$$

**Remark 2.3.1.2** If $(X, F)$ is a probabilistic $f$-metric structure then the family

$$W_f := \{W_f s \in (0, f(0)): W_f \in \{x, y \in F_s(x) > f^{-1}(s)\},$$

is a uniformity base which generates the uniformity $U_f$.

**Lemma 2.3.1.3** ([PRD97]) Consider a Menger space $(X, F, T)$, where $T \geq T_f$. For each $k > 0$ let us define

$$d_k(x, y) := \sup_{s>0} s^{k} \int s^{k} f_0 \frac{F_s(x)}{t} dt$$

and

$$\rho_k(x, y) := (d_k(x, y))^{\frac{1}{k+1}}.$$ 

Then $\rho_k$ is a generalized metric on $X$.

**Lemma 2.3.1.4** Let $(X, F, T)$ be a Menger space with $T \geq T_f$. Then $U_f \subset U_{\rho_k}$.

**Lemma 2.3.1.5** If $(X, F)$ is a probabilistic $f$-metric structure and $A$ is a $B$-contraction then $A$ is, for each $k > 0$, a strict contraction in $(X, \rho_k)$.

**Theorem 2.3.1.6** ([PRD97]) Let $(X, F, T)$ be a complete Menger space with $T \geq T_f$. If there exists some $k > 0$ such that for every pair $(x, y) \in X$ one has

$$\sup_{s>0} s^{k} \int s^{k} f_0 \frac{F_s(x)}{t} dt < \infty$$

then every $B$-contraction on $X$ has a unique fixed point.

**Corollary 2.3.1.7** ([cf. SHR71]) Let $(X, F, T)$ be a complete Menger space under $T \geq T_f$, where $f(0) < \infty$ and suppose that for each pair $(x, y) \in X^2$ there exists $t_{xy}$ for which $F_{xy}(t_{xy}) = 1$. Then every $B$-contraction on $X$ has a unique fixed point.

**Corollary 2.3.1.8** ([TAR92]) Let $(X, F, T)$ be a complete Menger space under $T \geq T_1$ and suppose that there exists $k > 0$ such that every $F_{xy}$ has a finite $k$-moment. Then every $B$-contraction on $X$ has a unique fixed point.

Generally, from the fixed point alternative ([DMG79]) we obtain the following

**Theorem 2.3.1.9** Let $(X, F, T)$ be a complete Menger space under $T \geq T_f$ and $A$ an $B$-contraction. Then for each $x \in X$ either,

$(A_1)$ there is some $k > 0$ such that $(A^k x)$ is $\rho_k$-convergent to the unique fixed point of $A$, or
for all $k > 0$, for all $n \in \mathbb{N}$ and for all $M > 0$ there exists $s := s(k, n, M)$ such that
\[ s^k \int_s^\infty \frac{f \circ F_{A_n, x, A_n+1, y}(t)}{t} \, dt > M. \]

2.4. Using a single metric

The results of this section are related to the following well known classical result.

**Lemma 2.4** Let $(X, d)$ be a complete metric space, and $A$ a continuous self-mapping of $X$. Then the following two statements are equivalent:

i) $A$ has a fixed point;

ii) There exists $p \in X$ such that $\sum_{n \geq 0} d(A^n p, A^{n+1} p) < \infty$.

2.4.1. A metric for the strong uniformity in some Menger spaces

**Proposition 2.4.1.1** [RD98] The two-place function $\rho_0$, defined by
\[ \rho_0(p, q) = \int_0^1 (1 - F_{pq}(x)) \, dx, \quad \forall (p, q) \in X \times X \]

is a semi-metric, on $X$, which generates $U_F$.

**Remark 2.4.1.2** Let $K$ be the semi-metric of type Ky Fan, defined by
\[ K(p, q) := \sup \{ t \mid t \leq 1 - F_{pq}(t) \}. \]

Then $\rho_0$ and $K$ are related by the inequalities
\[ K^2 \leq \rho_0 \leq 2K - K^2. \]

**Theorem 2.4.1.3** Let $(X, F, T)$ be a Menger space relatively to the $t$-norm $T \geq W$. Then the mapping $R_0 : X \times X \to \mathbb{R}$, given by:
\[ R_0(p, q) = \left\{ \int_0^1 |1 - F_{pq}(x)| \, dx \right\}^{\frac{1}{2}}, \quad \forall p, q \in X \times X \]

is a metric, on $X$, which generates the strong uniformity $U_F$. Moreover,
\[ K(p, q) \leq R_0(p, q) \leq \sqrt{2K(p, q)} \quad \forall p, q \in X \]

so that $(X, F, T)$ is complete iff $(X, R_0)$ is complete.

**Remark 2.4.1.4** If $(X, F, W)$ is either an $E$-space or a non-Archimedean Menger space, then $\rho_0$ itself is seen to be a metric.

**Example 2.4.1.5** Consider the random variables (on $(0, 1)$ with Lebesgue measure) $a, b, c$ defined by:
\[
 a(t) = t^2, \quad b(t) = t + t^2, \quad c(t) = t.
\]

If $F_{pq}(x) = \lambda(|p - q| < x)$, then $(X, (p, q) \to F_{pq}, W)$ is a Menger space. In this case $\rho_0(a, b) = \frac{1}{2}$, $\rho_0(b, c) = \frac{1}{2}$ and $\rho_0(a, c) = \frac{1}{2}$; thus $\rho_0$ is a metric.

Now if we take $F_{ab}$ and $F_{bc}$ as above, and $F_{ac} = \tau_W(F_{ab}, F_{bc})$, then one obtains $\rho_0(a, c) = \frac{31}{32} > \frac{1}{2} + \frac{1}{4} = \rho_0(a, b) + \rho_0(b, c)$, so that $\rho_0$ is not a metric. Clearly $R_0$ is a metric.
2.4.2. A family of metrics which generate the $\mathcal{F}$ - uniformity
The above idea can be easily extended. Let $\lambda$ be fixed in $[0, 1]$ and define

$$(12) \quad R_{\lambda}(p, q) := \left( \int_0^1 \frac{1 - F_{pq}(x)}{(1 + x)^{\lambda}} \, dx \right)^{\frac{1}{2}}, \quad \forall p, q \in X.$$ 

Then we have the following theorem:

**Theorem 2.4.2.1** [RD99] If $(X, \mathcal{F}, T)$ is a Menger space and $T \geq W$, then

(i) $R_{\lambda}$ is a metric, for each $\lambda \in [0, 1]$.

(ii) For $0 \leq \lambda < \mu \leq 1$ one has

$$\frac{1}{\sqrt{2}} R_0(p, q) \leq R_{\lambda}(p, q) \leq R_{\mu}(p, q) \leq \frac{1}{\sqrt{2}} R_1(p, q), \forall p, q.$$ 

(iii) $R_{\lambda}$ generates the strong $\mathcal{F}$-uniformity on $X$.

(iv) $(X, \mathcal{F}, T)$ is complete iff $(X, R_{\lambda})$ is complete for some $\lambda \in [0, 1]$.

**Remarks 2.4.2.2**

(a) It is easy to see that

$$2^{-\frac{1}{2}} K(p, q) \leq R_{\lambda}(p, q) \leq \sqrt{2} K(p, q).$$

(b) For $E$-spaces or non-Archimedean Menger spaces $\rho_{\lambda} := R_{\lambda}^2$ is a metric.

2.4.3. A fixed point principle

Let $(X, \mathcal{F}, T)$ be a complete Menger space and consider a B-contraction $A$ on $X$. The following lemma is obvious.

**Lemma 2.4.3.1** For every $\lambda \in [0, 1]$ one has

$$(12') \quad R_{\lambda}(Ap, Aq) \leq L \int_0^{L^{-\lambda}} \left( \int_0^1 \frac{1 - F_{pq}(x)}{(1 + x)^{\lambda}} \, dx \right)^{\frac{1}{2}}.$$ 

We are in position to give a characterization of probabilistic B-contractions with fixed points.

**Theorem 2.4.3.2** [RD99] If $T \geq W$, then the following statements are equivalent

1) $A$ has a fixed point.

2) There exist $p \in X$ and $\lambda \in [0, 1)$ such that

$$(13) \quad E^{(\lambda)}_{pAp} := \int_0^\infty \frac{1 - F_{pAp}(x)}{(1 + x)^{\lambda}} \, dx < \infty.$$

**Proof.** The implication $1^0 \Rightarrow 2^0$ is obvious : $p = Ap \Rightarrow F_{pAp}(x) = 1, \forall x > 0 \Rightarrow \lim_{x \to \infty} F_{pAp}(x) = 0, \forall \lambda < 1$. Let us prove the implication $2^0 \Rightarrow 1^0$.

Thus we suppose that there exists $p \in X$ which verifies $(13)$ for some $\lambda < 1$. Since

$$\lambda < \mu \Rightarrow \frac{1}{(1 + x)^{\lambda}} \leq \frac{1}{(1 + x)^{\lambda}}, \quad \forall x \geq 0,$$

then it is clear that $(13) \Rightarrow (13_{\mu})$ for $\lambda < \mu$. So it suffices to consider the case $\lambda \in [0, 1)$. 

Since $F_{Ap^{n+1}}(x) \geq F_{pAp}(\frac{x}{L_n})$, $\forall x \geq 0, \forall n \geq 0$, where $A^n$ is the $n$-iterate of $A$, then from formula (12$\lambda$) we obtain the inequality

$$R_\lambda (A^n p, A^{n+1} p) \leq \left(L^{\frac{1}{1-\lambda}}\right)^n \left\{ \int_0^\infty \frac{1 - F_{pAp}(x)}{(1 + x)^\lambda} dx \right\}^{\frac{1}{\lambda}}$$

which implies that

$$(14\lambda) \quad R_\lambda (A^n p, A^{n+1} p) \leq \left(L^{\frac{1}{1-\lambda}}\right)^n \left\{ E_{pAp}^{(\lambda)} \right\}^{\frac{1}{\lambda}}.$$

From (14$\lambda$) and (13$\lambda$) it results that $\sum_{n=0}^{\infty} R_\lambda (A^n p, A^{n+1} p) < \infty$

Therefore $(A^n p)$ is a Cauchy sequence in the complete metric space $(X, R_\lambda)$, thus it converges to some element $p_\ast \in X$.

From the continuity of $A$, one obtains that $p_\ast$ is a fixed point for $A$, which is necessarily unique. The theorem is completely proved.

**Remark 2.4.3.3** Since every Archimedean t-norm $T$ has the representation

$$(8') T (a, b) = h^{-1} \left( \tilde{T} (h(a), h(b)) \right), \quad \forall a, b \in [0, 1]$$

where $h : [0, 1] \rightarrow [0, 1]$, an increasing homeomorphism, and $\tilde{T} \in \{ W, Prod \}$ are precisely determined by $T$, then it is easy to see that, for every Menger space $(X, F, T)$, the probabilistic metric $h \circ F$ verifies the triangle inequality with $W$. Therefore Theorem 2.4.3.2 can be applied:

**Corollary 2.4.3.4** Let $(X, F, T)$ be a complete Menger space such that the Archimedean t-norm $T$ has the representation $(8')$. Then a given $B$-contraction $A$ on $X$ has a fixed point if and only if there exist $p \in X$ and $\lambda < 1$ such that

$$(13\lambda, h) \quad \int_0^\infty \frac{1 - h \circ F_{pAp}(x)}{(1 + x)^\lambda} dx < \infty.$$

**Remark 2.4.3.5** Our results are clearly applicable in the case of E-spaces, which are Menger spaces under $W$. The condition (13$\lambda$) says that the random variable $dist (p, Ap)$ is in the Lebesgue space $L_{1-\lambda}$ (that is it has a finite moment of order $1 - \lambda$) for one element $p$ and some value $\lambda < 1$, a condition which appears to be reasonable strong and easy to verify in concrete applications.

### 2.4.4. A family of semi-metrics on $PM$-spaces

In the following lemma we introduce a family of nonnegative functions which measure the distance between $e_0$ and the elements of $D_+$. Let $k$ be a (fixed) positive real number.

**Lemma 2.4.4.1** [RD98] The one-place mapping $\delta_k : D_+ \rightarrow R_+$, given by

$$(15) \quad \delta_k (F) := \sup_{x>0} \{ x^k [1 - F(x)]e^{-x} \},$$

has the following properties:

(i) $\delta_k (F) = 0 \Leftrightarrow F = e_0$;
and given probabilistic contraction

An an homeomorphism

(19) holds, Q.E.D.

Proof

slightly extends our above results

if (19) holds, Q.E.D.

Proof

Remark 2.4.4.6

Let $(X, \mathcal{F})$ be a probabilistic metric space and define

Then

1° $e_k$ is a semi-metric which generates the strong $\mathcal{F}$-topology;

2° $e_k$ generates the $\mathcal{F}$-uniformity, if this exists;

3° If $(X, \mathcal{F}, W)$ is a Menger space, then

(17) $(p, q) \rightarrow \theta_k(p, q) := \{e_k(p, q)\}^{\mathcal{F}}$

gives a metric on $X$. Moreover, $(X, \mathcal{F})$ is complete if and only if $(X, \theta_k)$ is complete.

Theorem 2.4.4.3 [RD98] Let $(X, \mathcal{F}, T)$ be a complete Menger space such that $T \geq W$. If $A : X \rightarrow X$ is a $B$-contraction, then the following statements are equivalent:

(1) $A$ has a fixed point;

(2) There exist $p \in X$ and $k \in (0, \infty)$ such that

(18) $E_k(p) := \sup_{x \geq 0} \{x^k[1 - F_{Ap}(x)]\} < \infty$.

Remarks 2.4.4.4

a) Simple examples show that $A$ is generally not contractive relatively to $\theta_k$ (or $e_k$).

b) The supremum in (18) may be infinite for some different values of $k$ or for different points in $X$.

c) Our condition is verified if there exists an element $p$ such that $F_{Ap}(t_p) = 1$ for some $t_p > 0$ (Note that H. Sherwood in [SHER71, Corollary] imposed this condition for all $F_{pq}$).

d) The condition (18) is verified if $F_{Ap}$ has a finite $k$ moment. Thus Theorem 2.1. slightly extends our above results.

Corollary 2.4.4.5 If $T \geq W$ and $(X, \mathcal{F}, T)$ is a complete Menger space, then a given probabilistic contraction $A$ on $X$ has a fixed point if and only if there exist $k > 0$ and $p \in X$ such that

(19) $\int_0^{+\infty} x^k dF_{Ap}(x) < +\infty$.

Proof. It is well known and easy to see that

(20) $\lim_{x \rightarrow +\infty} x^k(1 - F_{Ap}(x)) = 0$,

if (19) holds, Q.E.D.

Remark 2.4.4.6 A $t$-norm is Archimedean if and only if there exists an increasing homeomorphism $h : [0, 1] \rightarrow [0, 1]$ such that

(8') $T(a, b) = h^{-1}(T_+(h(a), h(b))) := T_*(a, b)$
where $T_\ast = W$ or $T_\ast = \text{Prod}$ (see Theorem 0.4). Since $ab \geq a+b-1$ for all $a, b \in [0,1]$, then we obtain the following.

**Theorem 2.4.4.7 [RD98]** Let $(X, \mathcal{F}, T)$ be a complete Menger space such that $T \geq T_h$ for some increasing homeomorphism $h : [0,1] \to [0,1]$. Then a probabilistic contraction $A$ of $X$ has a fixed point if and only if there exist $k > 0$ and $p \in X$ such that

$$
(21) \quad \sup_{x>0} x^k [1 - h \circ F_p A_p(x)] < +\infty.
$$

The proof follows from the fact that $(X, h \circ \mathcal{F}, W)$ is seen to be a complete Menger space, Q.E.D.

3. C-Contractions

A second type of contractions in Menger spaces was introduced by T.L. Hicks [HIC83] who also proved that for the t-norm $\text{Min}$ the Banach principle is valid (and is essentially equivalent to the classical one).

We improved [RD87] the above result by showing that it remains true in any complete Menger space $(X, \mathcal{F}, T)$ for which $\sup_{a<1} T(a, a) = 1$. It is to be noted that this condition on $T$ is the weakest one which ensures the existence of the $(\varepsilon, \lambda)$-uniformity ([MNG78]).

Using the method of the so called T-conjugate transforms, in [MOYS79] is introduced a (deterministic) metric which generates the $(\varepsilon, \lambda)$-topology in a Menger spaces under an Archimedean t-norm. This metric is given in terms of a multiplicative generator of the t-norm $T$.

Considering a modified form of this metric, in terms of an additive generator of the t-norm, we have given a direct simpler proof of the above result. Our formulas can be considered as direct generalizations of the Fréchet metrics for the convergence in probability. In [MOYS79] the following is proved.

**Theorem 3.1** For any multiplicative generator $h$ of $T$ and for any positive real number $z$, the mapping $d_a$ defined on $X \times X$ by

$$
(22) \quad d_a(p, q) = -\sup_{x>0} e^{ax} h F_{pq}(x)
$$

is a metric on $X$, which generates the $(\varepsilon, \lambda)$-topology. Moreover, the metrics $d_a$ are uniformly equivalent.

This is proved by using the properties of the so called T-conjugate transform.

We restated the above Theorem in terms of additive generators and so we have given a direct simpler proof. Namely

**Theorem 3.2** [RD82]. Let $f$ be an additive generator of $T$ and define the mapping

$$
(23) \quad \rho_f(p, q) = \inf_{t>0} \{t + f \circ F_{pq}(t)\}, p, q \in X
$$

Then

(i) $\rho_f$ is a metric on $X$;

(ii) The uniformity generated by $\rho_f$ is the $(\varepsilon, \lambda)$-uniformity;
(iii) If $a$ is a positive real number, then $\rho_a^f$ defined by

\begin{equation}
(23a) \quad \rho_a^f(p, q) = \inf_{t>0} \{at + f \circ F_{pq}(t)\}
\end{equation}

has the properties (i)-(ii);

(iv) For each $a \in (0, 1]$ one has

$$a\rho_f \leq \rho_a f \leq \rho_f$$

and so all $\rho_f$ are uniformly equivalent.

Our proof is based on the well known inequality

$$f \circ F_{pq}(x + y) \leq f \circ F_{pr}(x) + f \circ F_{qr}(y), \forall p, q, r, x, y.$$

**Corollary 3.3** If $(X, F, T)$ is a Menger space under an Archimedean t-norm $T$, then there exists an increasing bijection $h : [0, 1] \to [0, 1]$ such that the two-place function $k_h$ defined by

\begin{equation}
(24) \quad k_h(p, q) = \inf_{t>0} \{t + 1 - h \circ F_{pq}(t)\}
\end{equation}

is a metric on $X$, which metricizes the $(\varepsilon, \lambda)$-uniformity.

The main result of Hicks reads as follows:

**Theorem 3.4** Every C-contraction on a complete Menger space $(X, F, \text{Min})$ has a unique fixed point, which is the limit of the successive approximations.

The proof of the above result is obtained from the deterministic Banach principle, by constructing a metric on $X$ which generates the $(\varepsilon, \lambda)$-uniformity and is such that $f$ is a contraction with respect to that metric.

As a matter of fact, the same proof is valid for a larger class of t-norms. This is due to the fact that the two-place function $d$ constructed in [HIC83] is a metric in any Menger space $(X, F, T)$ if $T \geq T_1$. We proved this fact using a slightly modified form of $d$.

**Proposition 3.5.** Let $(X, F)$ be a PSM-space and define the two place mapping

\begin{equation}
(25) \quad K(x, y) = \sup \{t \geq 0 : t \leq 1 - F_{xy}(t)\}
\end{equation}

Then $K$ is semi-metric on $(X, T_F)$ and

\begin{equation}
(26) \quad K(x, y) < \delta \Leftrightarrow F_{xy}(\delta) > 1 - \delta \forall \delta > 0,
\end{equation}

which shows that $K$ generates the semi-uniformity $\mathcal{U}_F$.

**Examples 3.6** (i) If $d$ is a semi-metric on $X$ and we set $F_{xy} := \varepsilon_{d(x, y)}$ then $(X, \varepsilon_{d(\cdot, \cdot)})$ is a PSM-space and $K(x, y) = \min(d(x, y), 1)$.

(ii) Let $X$ be the family of all classes of real random variables on a probability measure space $(\Omega, \mathcal{K}, P)$. If we set $F(x, y) = F_{|x-y|}$, the distribution function of $|x - y|$, then $(X, F, W)$ is a Menger space and $K$ is the Ky Fan metric of the convergence in probability.

It is to be noted that, generally, $K$ need not be a metric. But we have proven the following.
**Theorem 3.7** [RD85,87] Let \((X, \mathcal{F}, T_1)\) be a Menger space and define

\[
(27) \quad d(p, q) = \sup \{ t \mid F_{pq}(t) \leq 1 - t \}
\]

Then

(i) \(d\) is a metric on \(X\), which generates the \((\varepsilon, \lambda)\) - uniformity;
(ii) \(X\) is \(\mathcal{F}\)-complete iff \(X\) is \(d\)-complete;
(iii) \(f : X \to X\) is a \(C\)-contraction iff \(f\) is \(d\)-contraction.

In order to ensure the verification of the triangle inequality for \(K\), T. L. Hicks [HIC96] proposed the following form of the triangle inequality for \((X, \mathcal{F})\):

\[III^1. \quad [F_{xy}(t) > 1 - t, F_{yz}(s) > 1 - s] \Rightarrow F_{xz}(t + s) > 1 - (t + s)\]

and he observed that the property \(III^1\) holds for every Menger space \((X, \mathcal{F}, T)\) for which \(T \geq W\).

As a matter of fact one has the following

**Proposition 3.8** Let \(T\) be a \(t\)-norm such that \((III^1)\) holds for every Menger space \((X, \mathcal{F}, T)\). Then \(T \geq W\).

**Proof.** Let \(X = \{x, y, z\}\), \(F_{xy} = F_{yx}, F_{yz} = F_{zy}, F_{xz} = F_{zx}\) where

\[
F_{xy}(t) = \begin{cases} 0 & t \leq 0 \\ a & t \in (0, 1] \\ 1 & t > 1 \end{cases}, \quad F_{yz}(t) = \begin{cases} 0 & t \leq 0 \\ b & t \in (0, 1] \\ 1 & t > 1 \end{cases},
\]

and \(F_{xz} = F_{yy} = F_{zz} = \varepsilon_0\). Then \((X, \mathcal{F}, T)\) is a Menger space (for which \(T\) is the best \(t\)-norm) and \(K(x, y) = 1 - a, K(y, z) = 1 - b, \) while \(K(x, z) = 1 - T(a, b)\). Thus we see that \(K(x, z) \leq K(x, y) + K(y, z) \Leftrightarrow T(a, b) \geq a + b - 1\).

**Remark 3.9** Let \((X, \mathcal{F}, T)\) as in the proof of the theorem and suppose that \(T(a, b) < a + b - 1\). Therefore \(0 < a, b < 1\) and there exists \(p > 1\) such that \(((1 - a) \hat{p} + (1 - b) \hat{p})^p > 1 - T(a, b)\). Thus \((1 - a) \hat{p} + (1 - b) \hat{p} > (1 - T(a, b)) \hat{p}\) and we see that \(K_p\), given by

\[
K_p(u, v) = \sup \{ t^p \leq 1 - F_{uv}(t) \},
\]

is verifying the triangle inequality. This shows that the general formulas proposed in [RD86b] can give metrics in more general situations.

Let \(\mathcal{M}\) denote the family of all mappings \(\mu : [0, \infty] \to [0, \infty]\) which are such that

a) \(\mu(t) + \mu(s) \leq \mu(t + s), \forall t, s \geq 0\)

b) \(\mu(t) = 0 \Leftrightarrow t = 0\)

c) \(\mu\) is continuous.
It is easy to see that every \( \mu \in \mathcal{M} \) is finite and strictly increasing on a uniquely determined interval \([0, b_\mu]\) and \( \mu(b_\mu) = \infty \).

If we set, for any PSM-space \((X, \mathcal{F})\),
\[
(1_\mu) \ K_\mu(x, y) = \sup\{t | t \geq 0, \mu(t) \leq 1 - F_{xy}(t)\}
\]
then \( K_\mu \) is a semi-metric. Moreover,
\[
(2_\mu) \ K_\mu(x, y) < \delta \iff F_{xy}(\delta) > 1 - \mu(\delta),
\]
from which it follows that \( K_\mu \) generates \( \mathcal{U}_\mathcal{F} \).

### 3.1. Probabilistic metric spaces of type \( \mathcal{M} \)

The above remarks suggest the following definition, which extends \((III)\):

**Definition 3.1.1.** [PRD99] A PSM-space \((X, \mathcal{F})\) for which takes place the following triangle inequality

\[
III^\mu. \ [F_{xy}(t) > 1 - \mu(t), F_{yz}(s) > 1 - \mu(s)] \Rightarrow \ \Rightarrow \ F_{xz}(t + s) > 1 - \mu(t + s)
\]
is called **PM-space of type \( \mathcal{M} \).**

**Remark 3.1.2** The triangle inequality \((III^\mu)\) can be useful and appropriate in many cases. For example, if \((X, \mathcal{F})\) verifies \((III)\) — which is \((III^\mu)\) for \( \mu(t) = t \) — then it is easy to see that \( \mathcal{F} \) defined by \( F_{xy} \circ \mu \), is a probabilistic semi-metric and

\[
\tilde{F}_{xy}(\delta) > 1 - \mu(\delta) \iff F_{xy}(\mu(\delta)) > 1 - \mu(\delta).
\]
The formula \((1_\mu)\) leads to \( K_\mu(x, y) = \mu^{-1}(K(x, y)) \), a very convenient one, for \( \mu^{-1} \) is clearly sub-additive. In particular, for spaces of random variables (see Example 1.3 (ii)), \( \tilde{F}_{xy}(t) = P(|x - y| < \mu(t)) = P(\mu^{-1}(|x - y|) < t) \) and \( \mu^{-1} \circ K \) is a metric which gives the convergence in probability, too. Generally, we can prove the following

**Theorem 3.1.3** [PRD99] Let \((X, \mathcal{F})\) be a PM-space of type \( \mathcal{M} \), that is the triangle inequality \((III^\mu)\) holds. Then the two-place function \( K_\mu \), defined by formula \((1_\mu)\), is a metric on \( X \) which generates \( \mathcal{T}_\mathcal{F} \) and \( \mathcal{U}_\mathcal{F} \).

**Proof.** We only have to prove the triangle inequality for \( K_\mu \). From \((2_\mu)\) we see that \( [K_\mu(x, y) < \delta \text{ and } K_\mu(y, z) < \varepsilon] \Rightarrow [F_{xy}(\delta) > 1 - \mu(\delta) \text{ and } F_{yz}(\varepsilon) > 1 - \mu(\varepsilon)] \).

From \((III^\mu)\) it follows that \( F_{xz}(\delta + \varepsilon) > 1 - \mu(\delta + \varepsilon) \), which shows that \( K_\mu(x, z) < \delta + \varepsilon \) and we obtain the triangle inequality for \( K_\mu \).

**Corollary 3.1.4** Let \( T \) be a \( t \)-norm such that

\[
T(a, b) \geq T_\mu(a, b) := \max\{1 - \mu[\mu^{-1}(1 - a) + \mu^{-1}(1 - b)], 0\}.
\]

Then \( K_\mu \) is a metric for every Menger space \((X, \mathcal{F}, T)\).

**Proof.** Since \( T(a, b) \geq T_\mu(a, b) \) then \( T(a, b) \geq 1 - \mu[\mu^{-1}(1 - a) + \mu^{-1}(1 - b)] \).

From the inequality \( III_M \) it follows that

\[
F_{xz}(t + s) \geq 1 - \mu[\mu^{-1}(1 - F_{xy}(t)) + \mu^{-1}(1 - F_{yz}(s))].
\]

Now, if we suppose that \( F_{xy}(t) > 1 - \mu(t), F_{yz}(s) > 1 - \mu(s) \) and \( \mu(t + s) < 1 \), then \( \mu(t), \mu(s) < 1 \) and

\[
F_{xz}(t + s) \geq 1 - \mu(t + s).
\]
Therefore $\mathcal{F}$ verifies III$^\mu$.

**Remark 3.1.5** Since $\mu$ is super-additive, then $\mu(\mu^{-1}(1-a)+\mu^{-1}(1-b)) \geq 2-a-b$, which shows that $T_\mu(a,b) \leq \max(a+b-1,0) = W(a,b)$. Therefore our Theorem 3.1.3 and Corollary 3.1.4 essentially extend the corresponding Theorem 2 and Corollary 2 of [HIC96]. Actually if we take an increasing sequence $(\mu_n) \subseteq \mathcal{M}$, it is clear that $(T_{\mu_n})$ is decreasing and so the class of Menger spaces, for which formula (1$\mu$) gives us a metric, is increasing. For example, if $\mu_n^{-1}(t) \rightarrow 1$ for $t \in (0,1]$, then we see that $T_{\mu_n}(a,b) \rightarrow T_\mu(a,b)$, the weakest $t$-norm (see also the example in Remark 3.1.2).

Consider an Archimedean $t$-norm $T_\mu$ with the additive generator $f$, and let $\mu_1$, $\mu_2 \in \mathcal{M}$ be fixed. Then we have the following

**Theorem A** [RD88] For every Menger space $(X, \mathcal{F}, T)$ with $T \succeq T_\mu$, the mapping $d$ given by

$$
\quad (28) \quad d(p,q) = \sup \{t, \mu_1(t) \leq f \circ F_{pq}(\mu_2(t))\}
$$

is a metric on $X$. Moreover,

$$
\quad (29) \quad d(p,q) < t \Leftrightarrow f \circ F_{pq}(\mu_2(t)) < \mu_1(t)
$$

and so $d$ and $\mathcal{F}$ generate the same uniformity.

1. It suffices to consider the case $\mu_2(t) = t$, for $(X, \mathcal{F} \circ \mu_2, T)$ is a Menger space for every $(X, \mathcal{F}, T)$.

2. The formula (28) gives a metric on every Menger space $(X, \mathcal{F}, Min)$ and any $f : [0,1] \rightarrow [0,\infty]$ which is continuous, strictly decreasing and such that $f(1) = 0$. The case $f(t) = 1-t$, $m_1(t) = \mu_2(t) = t$ and $T = Min$, was considered in [HIC83] in a different formulation.

In [RD84] we observed that the method used in [HIC83] can be applied for a larger class of $t$-norms, namely for $T \succeq T_1$.

In [CNS85] this case $T \succeq T_1$ was considered for a larger class of mappings: Let $\mathcal{L}$ be the family of functions $L : [0,\infty) \rightarrow [0,\infty]$ with the following three properties:

- $(L_1)$ $L$ is strictly increasing;
- $(L_2)$ $L$ is right continuous;
- $(L_3)$ $\lim_{n \rightarrow \infty} L^n(t) = 0$, $\forall t \geq 0$.

A mapping $A$ is called $L$-probabilistic contractions iff

$$(C_L) \quad t > 0, F_{pq}(t) > 1-t \Rightarrow F_{A_\mu A_n}(L(t)) > 1-L(t)$$

We considered (RD88) a slightly more general case, suggested by the following remark. If we set $f(s) = 1-s$ then $(C_L)$ can be formulated as

$$(C_L^f) \quad f \circ F_{pq}(t) < t \Rightarrow f \circ F_{A_\mu A_n}(L(t)) < L(t)$$

As a matter of facts, a fixed point theorem holds in more general conditions:

**Theorem 3.1.6** [RD88] Let $(X, \mathcal{F}, T)$ be a complete Menger space such that $T \succeq T_\mu$. Then every mapping $A : X \rightarrow X$ which satisfies the condition ($\mu$ is fixed)

$$
\quad (30) \quad f \circ F_{pq}(t) < \mu(t) \Rightarrow f \circ F_{A_\mu A_n}(L(t)) < \mu(L(t))
$$

has a unique fixed point which is the limit of successive approximations.

### 3.2. A special case: $\mathcal{L} - \mathcal{M}$ contractions
Definition 3.2.1 [PRD99] We say that $A : X \to X$ is an $L - M$ probabilistic contraction if there exist $L \in L$ and $\mu \in M$ such that

$$(L\mu - c) \quad [F_{xy}(t) > 1 - \mu(t)] \Rightarrow [F_{Ax, Ay}(L(t)) > 1 - \mu \circ L(t)]$$

For a concrete pair $L - \mu$ we use the term $L - \mu$ probabilistic contraction.

Example 3.2.2. Suppose that $A$ is a contraction of Hicks type – that is $(L\mu - c)$ holds for $\mu(t) = t$ (the case of Hicks) and consider the probabilistic semi-metric $\tilde{F}$ defined by $\tilde{F}_{xy} = F_{xy} \circ \mu$, where

$$F_{xy} \circ \mu(t) = \begin{cases} 0, & t \leq 0 \\ F_{xy}(\mu(t)), & t > 0 \end{cases}$$

and $t_\mu = \infty$. If we set $\tilde{L} = \mu^{-1} \circ L \circ \mu$, then it is easy to see that $\tilde{L} \in L$ and

$$\tilde{F}_{xy}(t) > 1 - \mu(t) \iff F_{xy}(\mu(t)) > 1 - \mu(t)$$

$$\Rightarrow F_{Ax, Ay}(L \circ \mu(t)) > 1 - \mu \circ L(t)$$

$$\iff F_{Ax, Ay}(\mu[\mu^{-1} \circ L \circ \mu(t)]) > 1 - \mu \circ \mu \circ L(t)$$

$$\Rightarrow \tilde{F}_{Ax, Ay}(\tilde{L}(t)) > 1 - \mu \circ \tilde{L}(t).$$

But this says that $A$ verifies $(\tilde{L}\mu - c)$ for every $\mu$.

Theorem 3.2.3 [PRD99]. Let $(X, \mathcal{F})$ be a complete PM-space of type $M$, for which the triangle inequality $(III^\mu)$ holds. Then every $L - \mu$ probabilistic contraction has a unique fixed point which can be obtained by successive approximations.

Corollary 3.2.4 Let $(X, \mathcal{F}, T)$ be a complete Menger space, for which $T \geq T_\mu$. Then every $L - \mu$ probabilistic contraction on $X$ has a unique fixed point.

Remark 3.2.5 (i) For $\mu(t) = t$ or from Example 3.2.2 we obtain the Theorem 3 of [HIC96]. Actually we can extend to PM-spaces of type $M$ all the results of [HIC96] and that obtained by the present author.

(ii) It is clear that our Corollary 3.2.4 is applicable for Menger spaces in a class essentially larger than that from [HIC83,96].

4. Generalized $C$-contractions on Menger spaces

Let $(X, \mathcal{F})$ be a given PSM-space and $A : X \to X$ a fixed mapping.

Definition 4.1 We say that $A$ is a generalized $C$- contraction if for each pair of real numbers $(a, b)$, with $0 < a < b$, there exists $L = L_{ab} \in (0, 1)$ such that if

$$a \leq 1 - F_{pq}(a) \quad \text{and} \quad 1 - F_{pq}(b+) \leq b,$$

then the following implication holds:

$$(C_{ab}) : \quad F_{pq}(x) > 1 - x \Rightarrow F_{Ax, Ay}(L_{ab}x) > 1 - L_{ab}x$$

We can prove the following.

Theorem 4.2 Every generalized $C$- contraction on a complete Menger space $(X, \mathcal{F}, T)$, where $T \geq W$, has a unique fixed point, which is globally attractive.

Proof. Let us first note the following simple useful...
**Lemma 4.3** In every PSM-space \((X, \mathcal{F}), K(p, q) = \sup\{t, t \leq 1 - F_{pq}(t)\}\) is the only nonnegative real number \(k\) with the property
\[
1 - F_{pq}(k+) \leq k \leq 1 - F_{pq}(k).
\]

Now let us suppose that
\[
K(p, q) = k = (1 - t)a + tb
\]
where \(0 < a < b\) and \(t \in [0, 1]\) are fixed.

1°. Since \(a \leq k\), then \(a \leq 1 - F_{pq}(k) \leq 1 - F_{pq}(a)\), and we see that
\[
a \leq 1 - F_{pq}(a).
\]

2°. Since \(k \leq b\), then
\[
1 - F_{pq}(b+) \leq 1 - F_{pq}(k+) \leq k \leq b,
\]
which says that
\[
1 - F_{pq}(b+) \leq b.
\]

2°. Since \(A\) is a generalized \(C\) contraction, then \((C_{ab})\) holds. But, for any \(d > 0\), we have \(1 - F_{pq}(k + d) \leq 1 - F_{pq}(k+) \leq k < k + d\), which implies
\[
F_{AxAy}(L_{ab}(k + d)) > 1 - L_{ab}(k + d),
\]
so that
\[
K(Ap, Aq) \leq L_{ab}(k + d), \forall d > 0
\]
and we see that
\[
K(Ap, Aq) \leq L_{ab}K(p, q), \text{ if } K(p, q) \in [a, b].
\]

Therefore \(A\) is a Krasnoselski contraction [KREM69] in the complete metric space \((X, K)\), which proves the theorem.

We used this type of methods in a recent joint paper with Olga Hadžić and Endre Pap [HPR2001].

**References**


IDEAS AND METHODS IN FIXED POINT THEORY FOR PROBABILISTIC CONTRACTIONS


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