FIXED POINT THEOREMS FOR ACYCLIC MULTIVALUED MAPS AND INCLUSIONS OF HAMMERSTEIN TYPE

RADU PRECUP

Departement of Applied Mathematics
"Babeș-Bolyai" University
3400 Cluj-Napoca, Romania
E-mail: r.precup@math.ubbcluj.ro

Abstract. The aim of this lecture is to present a new compactness method for operator inclusions in general, and for Hammerstein like inclusions, in particular. This method applies to acyclic multivalued maps which satisfy a generalized compactness condition of Mönch type.

Keywords: Multivalued map, acyclic map, Hammerstein operator, operator inclusion, compactness, fixed point.

1. The operator form of the initial and boundary value problems

STEP I: Consider the initial value problem (IVP) and the boundary value problem (BVP):

\[ \begin{cases}
  u' = f(t,u), & t \in I = [0,T] \\
  u(0) = 0;
\end{cases} \quad \begin{cases}
  u'' = f(t,u), & t \in I \\
  u \in B
\end{cases} \]

for a system of \( n \) differential equations. Here \( B \) stands for the boundary conditions. Under standard conditions, both problems (1) can be put under the operator form

\[ u = N(u), \quad u \in C(I;\mathbb{R}^n), \]

where \( N : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n) \) is the composite operator \( N = JSF \), of the Nemytskii operator \( F \),

\[ F : C(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n), \quad F(u)(t) = f(t,u(t)), \]

of a linear integral operator \( S \), of the form

\[ S : C(I;\mathbb{R}^n) \to C^1(I;\mathbb{R}^n), \quad S(u)(t) = \int_0^T k(t,s)u(s)\,ds \]

and of the imbedding map \( J \),

\[ J : C^1(I;\mathbb{R}^n) \to C(I;\mathbb{R}^n), \quad J(u) = u. \]

For the (IVP), the kernel \( k \) has the expression

\[ k(t,s) = \begin{cases}
  1, & s < t \\
  0, & t < s
\end{cases} \]
while for the (BVP), \(-k\) is the Green’s function corresponding to the boundary conditions \(\mathcal{B}\), assuming its existence. Assume \(F\) and \(S\) are bounded continuous operators. Then, since by the Ascoli-Arzela Theorem, the imbedding map \(J\) is completely continuous, we have that \(N\) is completely continuous and so, we may think to apply Schauder’s Fixed Point Theorem or the Leray-Schauder Principle (see [7]) in order to guarantee the existence of solutions to each of problems (1).

2. Equations in Banach spaces

**STEP II:** Consider the problems (1) in a Banach space \(E\).

The imbedding map \(J\) of \(C^1(I;E)\) into \(C(I;E)\) is not completely continuous when \(E\) is infinite dimensional. Consequently, to say something about the compactness of \(N\), for each bounded set \(C\) of \(C(I;E)\) we have to analyze the compactness of the section sets \(N(C)(t)\) for \(t \in I\), where

\[
N(C)(t) = \left\{ \int_0^T k(t,s)f(s,u(s))\, ds : u \in C \right\}.
\]

If \(C\) is countable, then the integral and the Kuratowski’s measure of noncompactness interchange as follows (see [3], Theorem 1.2.2):

\[
\alpha(N(C)(t)) \leq \int_0^T |k(t,s)| \alpha(f(s,C(s)))\, ds.
\]

Next we require the following compactness property holds for \(f\):

\[
\alpha(f(t,M)) \leq L(t) \alpha(M)
\]

for each bounded set \(M \subset E\). Then we obtain

\[
\alpha(N(C)(t)) \leq \int_0^T |k(t,s)| L(s) \alpha(C(s))\, ds.
\]

From, we would like to derive that

\[
\alpha(N(C)(t)) = 0, \ \text{for all} \ t \in I.
\]

This is not easy for general sets \(C\), but it is possible if \(C\) satisfies

\[
C \subset \text{conv}(\{u_0\} \cup N(C))
\]

for some \(u_0 \in C(I;E)\). Indeed, for such a set \(C\), we have

\[
\alpha(C(t)) \leq \alpha(N(C)(t)) \leq \int_0^T |k(t,s)| L(s) \alpha(C(s))\, ds.
\]

If we let \(\phi(t) = \alpha(C(t))\), then

\[
\phi(t) \leq \int_0^T |k(t,s)| L(s) \phi(s)\, ds.
\]

Now suitable integral inequalities (see [9]) yield \(\phi \equiv 0\) and so, by the infinite dimensional version of the Ascoli-Arzela Theorem, \(N(C)\) is relatively compact in \(C(I;E)\).
Notice by the above argument we have not proved the complete continuity of \( N \) and in consequence, Schauder’s Fixed Point Theorem and Leray-Schauder Principle do not apply. However, we may use Mönch’s extensions of these two theorems.

3. Mönch’s fixed point theorems

**Theorem 3.1.** ([5]) Let \( X \) be a Banach space, \( D \subset X \) be closed convex and \( N : D \to D \) be continuous with the further property that for some \( x_0 \in D \) one has
\[
C \subset D, \ C \text{ countable, } \quad \overline{C} = \overline{\{x_0\} \cup N(C)} \implies \overline{C} \text{ compact.}
\]

Then \( N \) has at least one fixed point.

**Theorem 3.2.** ([5]) Let \( X \) be a Banach space, \( K \subset X \) closed convex, \( U \subset K \) open in \( K \) and \( N : U \to K \) continuous, with the further property that for some \( x_0 \in U \) one has
\[
C \subset U, \ C \text{ countable, } \quad C \subset \overline{\{x_0\} \cup N(C)} \implies \overline{C} \text{ compact.}
\]

In addition, assume that
\[
x \neq (1 - \lambda)x_0 + \lambda N(x) \quad \text{for all } x \in U \setminus U, \lambda \in (0,1).
\]

Then \( N \) has at least one fixed point in \( U \).

**STEP III:** Consider the (IVP) and the (BVP) for a differential inclusion in the Banach space \( E \), i.e.
\[
\begin{align*}
\begin{cases}
u' \in f(t,u), \quad t \in I \\
u(0) = 0;
\end{cases} & \quad \begin{cases}
u'' \in f(t,u), \quad t \in I \\
u \in B.
\end{cases}
\end{align*}
\]

If we wish to discuss the inclusions (4) in a similar way like the equations (1), we need to give multivalued analogs to Mönch’s Theorems. This was achieved in [6] replacing (2)-(3) by some slightly more general conditions expressed in terms of a pair \((M,C)\) instead of a single set \(C\):

4. Mönch type theorems for inclusions

**Theorem 4.1.** ([6]) Let \( D \) be a closed, convex subset of a Banach space \( X \) and \( N : D \to 2^D \setminus \{\emptyset\} \) a mapping with convex values. Assume \( \text{graph}(N) \) is closed, \( N \) maps compact sets into relatively compact sets and that for some \( x_0 \in D \) one has
\[
M \subset D, \ M = \text{conv}(\{x_0\} \cup N(M)), \quad \overline{M} = \overline{C} \text{ with } C \subset M, \ C \text{ countable} \implies \overline{M} \text{ compact.}
\]

Then there exists \( x \in D \) with \( x \in N(x) \).

**Theorem 4.2.** ([6]) Let \( K \) be a closed, convex subset of a Banach space \( X \), \( U \) a relatively open subset of \( K \) and \( N : U \to 2^K \setminus \{\emptyset\} \) a mapping with convex values.
Assume graph(N) is closed, N maps compact sets into relatively compact sets and that for some \( x_0 \in U \), the following two conditions are satisfied:

\[
M \subset U, \ M \subset \text{conv} \{ \{x_0\} \cup N(M)\}, \quad \frac{d}{dt} \text{with } C \subset M, \ C \text{ countable} \quad \Rightarrow \quad \frac{\partial}{\partial t} M \text{ compact;}
\]

\[
x \notin (1 - \lambda)x_0 + \lambda N(x) \quad \text{for all } x \in U \setminus U, \ \lambda \in (0, 1).
\]

Then there exists \( x \in \overline{U} \) with \( x \in N(x) \).

Notice any upper semicontinuous mapping \( N \) with compact convex nonempty values, has closed graph and maps compact sets into relatively compact sets.

5. Hammerstein Integral Inclusions

Let us present an application of Theorem 4 to the Hammerstein integral inclusion

\[
u(t) \in \int_0^T k(t, s) f(s, u(s)) \, ds \quad \text{a.e. } t \in I.
\]

**Theorem 5.1.** ([8]) Let \( p \in [1, \infty], \ q \in [1, \infty) \) and let \( r \in (1, \infty] \) be the conjugate of \( q \), i.e., \( \frac{1}{q} + 1/r = 1 \). Assume \( k : \mathbb{I}^2 \to \mathbb{R} \) is measurable and

\[
\begin{align*}
\quad \text{(a) if } p < \infty : \text{ the map } t &\mapsto k(t, .) \text{ belongs to } L^p(I; L^r(I)); \\
\quad \text{(b) if } p = \infty : \text{ the map } t &\mapsto k(t, .) \text{ belongs to } C(I; L^r(I)).
\end{align*}
\]

In addition suppose:

\begin{enumerate}
\item \( f : I \times E \to 2^E \setminus \{\emptyset\} \) is a Carathéodory function with compact convex values;
\item there exists \( a \in L^q(I; \mathbb{R}_+) \), \( b \in \mathbb{R}_+ \) and \( R > 0 \) such that

\[
\begin{align*}
\text{(a) if } p < \infty : \ |f(t, x)| &\leq a(t) + b|x|^{p/q}, \ x \in E \\
\text{(b) if } p = \infty : \ |f(t, x)| &\leq a(t) \quad \text{for } |x| \leq R
\end{align*}
\]

(i.e., \( f \) is a \((q, p/q)-\text{Carathéodory function})
\item there exists a \((q, p/q)-\text{Carathéodory function } \omega : I \times \mathbb{R}_+ \to \mathbb{R}_+ \) with

\[
\alpha(f(t, M)) \leq \omega(t, \alpha(M))
\]

a.e. \( t \in I \), for every bounded \( M \subset E \);
\item \( \varphi \equiv 0 \) is the unique solution in \( L^p(I; \mathbb{R}_+) \) to the inequality

\[
\varphi(t) \leq 2 \int_0^T |k(t, s)| \omega(s, \varphi(s)) \, ds, \ a.e. \ t \in I;
\]
\item \( |u|_p < R \) for any solution \( u \in L^p(I; E) \) with \( |u|_p \leq R \) of

\[
u(t) \in \lambda \int_0^T k(t, s) f(s, u(s)) \, ds, \ a.e. \ t \in I,
\]

for \( \lambda \in (0, 1) \).

Then (7) has at least one solution \( u \in L^p(I; E) \) (respectively, in \( C(I; E) \) if \( p = \infty \)) with \( |u|_p \leq R \).
6. Fixed point results for acyclic mappings

STEP IV: Let us now discuss the problems

\[
\begin{cases}
u' \in Au + f(t, u), & t \in I \\ u(0) = 0;
\end{cases}
\]

\[
\begin{cases}
u'' \in Au + f(t, u), & t \in I \\ u \in \mathcal{B}.
\end{cases}
\]

Notice semilinear parabolic, respectively hyperbolic and elliptic inclusions can be put under the abstract form \( u' \in Au + f(t, u) \), respectively \( u'' \in Au + f(t, u) \).

Here we suppose that \( A \) is a multivalued map from \( E \) into \( 2^E \) such that for each \( v \) in a given space of functions, there exists a unique solution \( S(v) := u \) to the initial value problem, respectively boundary value problem:

\[
\begin{cases}
u' \in Au + v, & t \in I \\ u(0) = 0;
\end{cases}
\]

\[
\begin{cases}
u'' \in Au + v, & t \in I \\ u \in \mathcal{B}.
\end{cases}
\]

We note that the solution operator \( S \) is not linear, so even \( f \) has convex values, the mapping \( N = SF \) may have non convex values. Thus, a natural problem was to give extensions of Mönch’s Theorems for multivalued operators with non convex values. As a result we obtained a Mönch type generalization of the Eilenberg-Montgomery Theorem [2] (see also [4]):

**Theorem 6.1.** ([9]) Let \( D \) be a closed convex subset of a Banach space \( X, Y \) a metric space, \( N : D \to 2^Y \setminus \{\emptyset\} \) a map with acyclic values, and \( r : Y \to D \) continuous. Assume \( \text{graph}(N) \) is closed, \( N \) maps compact sets into relatively compact sets and that for some \( x_0 \in D \) one has

\[
M \subset D, \quad M = \text{conv} \left( \{x_0\} \cup rN(M) \right), \quad \overline{M} \text{ compact.}
\]

Then there exists \( x \in D \) with \( x \in rN(x) \).

The next result is the continuation type version of Theorem 6.

**Theorem 6.2.** ([9]) Let \( K \) be a closed convex subset of a Banach space \( X, U \) a convex, relatively open subset of \( K, Y \) a metric space, \( N : U \to 2^Y \setminus \{\emptyset\} \) with acyclic values and \( r : Y \to K \) continuous. Assume \( \text{graph}(N) \) is closed, \( N \) maps compact sets into relatively compact sets and that for some \( x_0 \in U \), the following two conditions are satisfied:

\[
M \subset U, \quad M \subset \text{conv} \left( \{x_0\} \cup rN(M) \right) \quad \Rightarrow \quad \overline{M} \text{ compact;}
\]

\[
x \notin (1 - \lambda)x_0 + \lambda rT(x) \quad \text{for all} \quad x \in U \setminus U, \quad \lambda \in (0, 1).
\]

Then there exists \( x \in U \) with \( x \in rN(x) \).
7. Abstract Hammerstein inclusions

STEP V: Here we discuss the abstract inclusion

\[ u \in SF (u), \quad u \in L^p (I; E), \]

where

\[ S : L^q (I; E) \to L^p (I; E) \]

is a given single valued operator and \( F : L^p (I; E) \to 2^{L^q (I; E)} \) is the Nemytskii multivalued operator associated to a function \( f : I \times E \to 2^E \), given by

\[ F (u) = \{ w \in L^q (I; E) : w (t) \in f (t, u (t)) \text{ a.e. } t \in I \}. \]

As a direct consequence of Theorem 7, we have the following existence principle for (10).

**Theorem 7.1.** ([1]) Let \( K \) be a closed convex subset of \( L^p (I; E) \) \((1 \leq p \leq \infty)\), \( U \) a relatively open subset of \( K \) and \( u_0 \in U \). Assume

1. \( SF : U \to 2^K \setminus \{ \emptyset \} \) has acyclic values, closed graph and maps compact sets into relatively compact sets;
2. \( M \subset U, M \subset \text{conv } (\{0\} \cup SF (M)) \implies M \text{ compact}; \)
3. \( u \notin (1 - \lambda)u_0 + \lambda SF (u) \) for all \( u \in U \setminus K, \lambda \in (0, 1) \).

Then (10) has at least one solution in \( U \).

In what follows: \( u_0 = 0, \quad U = B_R = \{ u \in K : |u|_p < R \} \). We shall give sufficient conditions for (H1)-(H2):

(S1) There exists a function \( k : I^2 \to R_+ \) such that \( k(t, \cdot) \in L^r (I) \)

\( (1/r + 1/q = 1) \), the function \( t \mapsto |k(t, \cdot)|_r \) belongs to \( L^p (I) \) and

\[ |S (w_1) (t) - S (w_2) (t)| \leq \int_I k (t, s) |w_1 (s) - w_2 (s)| \, ds \]

a.e. \( t \in I \), for all \( w_1, w_2 \in L^q (I; E) \).

(S2) \( S : L^q (I; E) \to K \) and for every compact convex subset \( C \) of \( E \), \( S \) is sequentially continuous from \( L^q_w (I; C) \) to \( L^p (I; E) \) (Here \( L^q_w (I; C) \) stands for the set \( L^1 (I; C) \) endowed with the weak topology of \( L^1 (I; E) \)).

1. \( f : I \times E \to 2^E \setminus \{ \emptyset \} \) has compact convex values.
2. \( f (\cdot, x) \) has a strongly measurable selection on \( I \), for each \( x \in E \).
3. \( f (t, \cdot) \) is upper semicontinuous, for a.e. \( t \in I \).
4. There exists \( a \in L^q (I; R_+) \), \( b \in R_+ \) and \( R > 0 \) such that

\[ \begin{cases} \text{if } p < \infty : |f (t, x)| \leq a (t) + b |x|^{p/q}, \text{ for all } x \in E; \\ \text{if } p = \infty : |f (t, x)| \leq a (t), \text{ for } |x| \leq R. \end{cases} \]

5. For every separable closed subspace \( E_0 \) of \( E \), there exists a \((q, p/q)\)-Carathédory function \( \omega : I \times R_+ \to R_+ \) such that

\[ \beta_{E_0} (f (t, M) \cap E_0) \leq \omega (t, \beta_{E_0} (M)) \]
a.e. \( t \in I \), for every set \( M \subset E_0 \) satisfying
\[
|M| \leq |S(0)(t)| + \left(|a|_q + bR^{p/q}\right)|k(t,.)|_r
\]
if \( p < \infty \), respectively
\[
|M| \leq |S(0)(t)| + |a|_q |k(t,.)|_r
\]
if \( p = \infty \). In addition \( \varphi \equiv 0 \) is the unique solution in \( L^p(I;R_+) \) to
\[
\varphi (t) \leq \int_I k(t,s) \omega (s, \varphi (s)) \, ds, \quad \text{a.e. } t \in I.
\]
(15) Here \( \beta_{E_0} \) is the ball measure of noncompactness in \( E_0 \).

Theorem 7.2. ([1]) Assume (S1)-(S2), (f1)-(f5) and (SF) hold. In addition suppose (H3). Then (10) has at least one solution \( u \) in \( K \subset L^p(I;E) \) with \( |u|_p \leq R \).

If \( q \leq p \), then a sufficient condition for (f5) is
(15*) For every separable closed subspace \( E_0 \) of \( E \), there exists a \( \delta \in L^{pq/(p-q)}(I) \) such that
\[
\beta_{E_0}(f(t,M) \cap E_0) \leq \delta(t) \beta_{E_0}(M)
\]
a.e. \( t \in I \), for every subset \( M \subset E_0 \) satisfying
\[
|M| \leq |S(0)(t)| + \left(|a|_q + bR^{p/q}\right)|k(t,.)|_r,
\]
if \( p < \infty \), respectively
\[
|M| \leq |S(0)(t)| + |a|_q |k(t,.)|_r.
\]
if \( p = \infty \), and
(16) \[
|\delta|_{pq/(p-q)} ||k(t,.)||_r < 1.
\]
Here \( pq/(p-q) = q \) if \( p = \infty \) and \( pq/(p-q) = \infty \) if \( p = q \).

Notice in the Volterra case, i.e. when \( k(t,s) = 0 \) for \( s > t \), condition (16) can be dropped.

Example 7.1. Let \( f(t,x) = a|x|^{p-2}x \), where \( a > 0 \), \( p > 2 \). Then, if \( |M| \leq \eta(t) \),
one has
\[
\beta(f(t,M)) \leq a(p-1) \eta(t)^{p-2} \beta(M).
\]
Here \( \delta(t) = a(p-1) \eta(t)^{p-2} \) and (16) holds for a sufficiently small \( a \).

We note that the technique we use to verify compactness conditions like (5), (6) equally applies to check the Palais-Smale condition in critical point theory (see [10]).
References


