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A GENERALIZATION OF JENSEN EQUATION FOR SET-VALUED MAPS

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Abstract. It is given a representation of the solutions of a generalization of Jensen equation for set-valued maps with closed and convex values.
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1. Introduction

Let X be a real vector space, Y a real topological vector space. We denote by $0_X, 0_Y$ the origin of X and Y, by $\mathcal{P}_0(Y)$ the collection of all nonempty subsets of Y and by $CCl(Y)$ the collection of all nonempty, closed and convex subsets of Y.

For two nonempty subsets A and B of X (or Y) and a real number t we define the sets

$$\begin{aligned} A+B &= \{x \mid x = a+b,\ a \in A,\ b \in B\} \\ t \cdot A &= \{x \mid x = ta,\ a \in A\}. \end{aligned} \tag{1}$$

For any nonempty sets $A, B \subseteq X$ and any real numbers s, t the following relations holds

$$\begin{aligned} s(A+B) &= sA+sB \\ (s+t)A &\subseteq sA+tA. \end{aligned} \tag{2}$$

If A is a convex set and $st \geq 0$ then

$$(s+t)A = sA+tA. \tag{3}$$

Let K be a convex cone in X with $0_X \in K$ and p a real number $0 < p < 1$. We are looking for set-valued solutions $F : K \to \mathcal{P}_0(Y)$ of the equation

$$F((1-p)x+py) = (1-p)F(x)+pF(y)+M \tag{4}$$

where $M \in \mathcal{P}_0(Y)$.

For $p = \dfrac{1}{2}$ and $M = \{0_Y\}$ the equation (4) becomes the classical Jensen equation. It is well known that the solutions of classical Jensen equation for single valued maps are of the form $F(x) = a(x)+k$, where $a : K \to Y$ is an additive function and $k \in Y$

is a constant [4]. Classical Jensen equation for set-valued maps was studied by Z. Fifer [4], K. Nikodem [6], [7]. They give a characterization of the solutions of Jensen equation for set-valued maps with compact and convex values in a topological vector space. A generalization of Jensen equation was considered and studied by the author in [9]. The goal of this paper is to give a representation theorem of the solutions of the equation (4) if F has close and convex values.

2. Main results

We start by proving an auxiliary lemma.

Lemma 2.1. *Let X, Y be real vector spaces and K a convex cone in X containing the origin of X. If the set-valued map $F : K \to \mathcal{P}_0(Y)$ satisfies the equation (4) then*

$$F(x+y) + F(0_X) = F(x) + F(y) \tag{5}$$

for every $x, y \in K$.

Proof. For $x = y = 0_X$ in (4) we get

$$F(0_X) = (1-p)F(0_X) + pF(0_X) + M \tag{6}$$

and for $x = 0_X$, respectively $y = 0_X$ in (4) we obtain

$$F((1-p)x) = (1-p)F(x) + pF(0_X) + M, \quad x \in K, \tag{7}$$

$$F(py) = (1-p)F(0_X) + pF(y) + M, \quad y \in K. \tag{8}$$

Now let $u, v \in K$. By (4) we have

$$F(u+v) = F\left((1-p)\frac{u}{1-p} + p\frac{v}{p}\right) = (1-p)F\left(\frac{u}{1-p}\right) + pF\left(\frac{v}{p}\right) + M \tag{9}$$

and taking account of (6), (7), (8) we get

$$F(u+v)+F(0_X) = (1-p)F\left(\frac{u}{1-p}\right)+pF\left(\frac{v}{p}\right)+M+(1-p)F(0_X)+pF(0_X)+M =$$
$$= \left[(1-p)F\left(\frac{u}{1-p}\right) + pF(0_X) + M\right] + \left[(1-p)F(0_X) + pF\left(\frac{v}{p}\right) + M\right] =$$
$$= F(u) + F(v)$$

and the lemma is proved.

Consider in what follows the equation

$$F(x+y) + C = F(x) + F(y) \tag{10}$$

where $F : K \to \mathcal{P}_0(Y)$, K is a convex cone in X with $0_X \in K$, $C \in \mathcal{P}_0(Y)$, X is a real vector space and Y is a real topological vector space.

Lemma 2.2. *Let C be a convex and sequentially compact subset of Y with $0_Y \in C$. A set-valued map $F : K \to CCl(Y)$ satisfies the equation (10) if and only if there exists an additive set-valued map $A : K \to CCl(Y)$ such that*

$$F(x) = A(x) + C$$

for every $x \in X$.

Proof. Suppose that $F : K \to CCl(Y)$ satisfies the equation (10). It can be easily proved by induction that

$$F(nx) + (n-1)C = nF(x) \tag{11}$$

for every $n \in \mathbb{N}$ and every $x \in X$.

For $n = 1$ the relation (11) is obvious. Suppose that the relation (11) holds for $n \in \mathbb{N}$ and we have to prove that

$$F((n+1)x) + nC = (n+1)F(x).$$

We have:

$$F((n+1)x) + nC = F(nx + x) + C + (n-1)C =$$
$$= F(nx) + F(x) + (n-1)C = nF(x) + F(x) = (n+1)F(x).$$

Now let $x \in K$. From (11) we get:

$$\frac{1}{2^n}F(2^n x) + \frac{2^n - 1}{2^n}C = F(x), \quad n \in \mathbb{N}. \tag{12}$$

Denote $A_n(x) = \dfrac{1}{2^n}F(2^n x)$, $n \geq 0$. The sequence of sets $(A_n(x))_{n\geq 0}$ is decreasing. Indeed:

$$A_{n+1}(x) = \frac{1}{2^{n+1}}F(2^n x + 2^n x) \subseteq \frac{1}{2^{n+1}}(F(2^n x + 2^n x) + C) =$$
$$= \frac{1}{2^{n+1}}(F(2^n x) + F(2^n x)) = \frac{1}{2^{n+1}} \cdot 2F(2^n x) = \frac{1}{2^n}F(2^n x) = A_n(x).$$

Put $A(x) = \bigcap\limits_{n\geq 0} A_n(x) \in CCl(Y)$. Prove that $A(x) \neq \emptyset$.

Let $u \in F(x)$ fixed. From (12) it results that for every $n \in \mathbb{N}$ there exists $a_n \in A_n(x)$ and $c_n \in C$ such that $u = a_n + \dfrac{2^n - 1}{2^n}c_n$. The set C is sequentially compact, so there exists a subsequence $(c_{n_k})_{k\geq 0}$ of $(c_n)_{n\geq 0}$ convergent to $c \in C$ and

$$u = a_{n_k} + \frac{2^{n_k - 1} - 1}{2^{n_k}}c_{n_k}, \quad k \geq 0.$$

It results that $a_{n_k} \to u - c$ as $k \to \infty$. We show that $u - c \in \bigcap\limits_{n\geq 0} A_n(x)$. Suppose that $u - c \notin \bigcap\limits_{n\geq 0} A_n(x)$. Then there exists $p \in \mathbb{N}$ such that $u - c \notin A_{n_p}$. We have $a_{n_k} \in A_{n_p}$ for $k \geq p$ and $\lim\limits_{k\to\infty} a_{n_k} \in A_{n_p}$, because A_{n_p} is closed, contradiction with $u - c \notin A_{n_p}$.

We prove that

$$A(x) + C = F(x), \quad x \in X. \tag{13}$$

Let $u \in A(x) + C$, $u = a + c$, $a \in A(x)$, $c \in C$. It results that $a \in A_n(x)$ for every $n \geq 0$ and let $c_n = \dfrac{2^n - 1}{2^n}c \in \dfrac{2^n - 1}{2^n}C$. From the relation (12) it results that there exists $b_n \in F(x)$ such that $a + c_n = b_n$, $n \geq 0$ and $\lim\limits_{n\to\infty} b_n = a + c \in F(x)$, because $F(x)$ is closed. Hence $A(x) + C \subseteq F(x)$.

Now let $b \in F(x)$. From (12) it results that for every $n \in \mathbb{N}$ there exists $a_n \in A_n(x)$ and $c_n \in C$ such that $b = a_n + \frac{2^n - 1}{2^n} c_n$. The sequence $(c_n)_{n \geq 0}$ has a subsequence $(c_{n_k})_{k \geq 0}$ convergent to $c \in C$, taking account of the sequential compactity of C. Hence the sequence $(a_n)_{n \geq 0}$ is convergent to $b - c \in A(x)$. Then $b = (b - c) + c \in A(x) + C$. The relation $F(x) \subseteq A(x) + C$ is proved. It follows that the relation (13) is true.

We prove that A is an additive set-valued map.

By the relation (10) and (13) we obtain

$$A(x + y) + C + C = A(x) + A(y) + C + C$$

and taking account of the cancellation law of Radström [2] it results that $A(x + y) = A(x) + A(y)$ for every $x, y \in X$, hence A is an additive set-valued map.

If $F(x) = A(x) + C$, $x \in K$, where $A : K \to CCl(Y)$ is an additive set-valued map, then we get:

$$F(x + y) + C = A(x + y) + C + C = A(x) + A(y) + C + C =$$
$$= (A(x) + C) + (A(y) + C) = F(x) + F(y)$$

for every $x, y \in X$.

The lemma is proved.

Theorem 2.1. *If a set-valued map $F : K \to CCl(Y)$, with $F(0_X)$ sequentially compact set, satisfies the equation (4) then there exists an additive set-valued map $A : K \to ccl(Y)$ and a compact convex set $B \in \mathcal{P}_0(Y)$ such that*

$$F(x) = A(x) + B \tag{14}$$

for every $x \in X$.

Proof. Suppose that F satisfies the equation (1) and let $\alpha \in F(0_X)$. The set-valued map $G : K \to CCl(Y)$, given by the relation

$$G(x) = F(x) - \alpha, \quad x \in X \tag{15}$$

satisfies the equation

$$G((1 - p)x + py) = (1 - p)G(x) + pG(y) + M \tag{16}$$

and $0_Y \in G(0_X)$.

By Lemma 2.1 it results that G satisfies also the relation

$$G(x + y) + G(0_X) = G(x) + G(y) \tag{17}$$

for every $x, y \in X$ and $G(0_X)$ is sequentially compact set. Then in view of Lemma 2.2 it results that there exists an additive set-valued map $A : K \to CCl(Y)$ such that $G(x) = A(x) + G(0_X)$ for every $x \in X$. It follows that $F(x) = A(x) + F(0_X)$ for every $x \in X$. Denoting $B = F(0_X)$ the theorem is proved.

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