A GENERALIZATION OF JENSEN EQUATION FOR SET-VALUED MAPS

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Abstract. It is given a representation of the solutions of a generalization of Jensen equation for set-valued maps with closed and convex values.

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1. Introduction

Let $X$ be a real vector space, $Y$ a real topological vector space. We denote by $0_X, 0_Y$ the origin of $X$ and $Y$, by $\mathcal{P}_0(Y)$ the collection of all nonempty subsets of $Y$ and by $\mathcal{CCl}(Y)$ the collection of all nonempty, closed and convex subsets of $Y$.

For two nonempty subsets $A$ and $B$ of $X$ (or $Y$) and a real number $t$ we define the sets

\begin{align*}
A + B &= \{ x \mid x = a + b, \ a \in A, \ b \in B \} \\
t \cdot A &= \{ x \mid x = ta, \ a \in A \}.
\end{align*}

For any nonempty sets $A, B \subseteq X$ and any real numbers $s, t$ the following relations holds

\begin{align*}
s(A + B) &= sA + sB \\
(s + t)A &\subseteq sA + tA.
\end{align*}

If $A$ is a convex set and $st \geq 0$ then

\begin{align*}
(s + t)A &= sA + tA.
\end{align*}

Let $K$ be a convex cone in $X$ with $0_X \in K$ and $p$ a real number $0 < p < 1$. We are looking for set-valued solutions $F : K \rightarrow \mathcal{P}_0(Y)$ of the equation

\begin{align*}
F((1 - p)x + py) &= (1 - p)F(x) + pF(y) + M
\end{align*}

where $M \in \mathcal{P}_0(Y)$.

For $p = \frac{1}{2}$ and $M = \{0_Y\}$ the equation (4) becomes the classical Jensen equation. It is well known that the solutions of classical Jensen equation for single valued maps are of the form $F(x) = a(x) + k$, where $a : K \rightarrow Y$ is an additive function and $k \in Y$. 
is a constant \([4]\). Classical Jensen equation for set-valued maps was studied by Z. Fifer \([4]\), K. Nikodem \([6]\), \([7]\). They give a characterization of the solutions of Jensen equation for set-valued maps with compact and convex values in a topological vector space. A generalization of Jensen equation was considered and studied by the author in \([9]\). The goal of this paper is to give a representation theorem of the solutions of the equation \((4)\) if \(F\) has close and convex values.

2. Main results

We start by proving an auxiliary lemma.

\textbf{Lemma 2.1.} Let \(X, Y\) be real vector spaces and \(K\) a convex cone in \(X\) containing the origin of \(X\). If the set-valued map \(F : K \rightarrow \mathcal{P}_0(Y)\) satisfies the equation \((4)\) then
\[
F(x + y) + F(0_X) = F(x) + F(y)
\]
for every \(x, y \in K\).

\textbf{Proof.} For \(x = y = 0_X\) in \((4)\) we get
\[
F(0_X) = (1 - p)F(0_X) + pF(0_X) + M
\]
and for \(x = 0_X\), respectively \(y = 0_X\) in \((4)\) we obtain
\[
F((1 - p)x) = (1 - p)F(x) + pF(0_X) + M, \quad x \in K,
\]
\[
F(py) = (1 - p)F(0_X) + pF(y) + M, \quad y \in K.
\]

Now let \(u, v \in K\). By \((4)\) we have
\[
F(u + v) = F\left(\left(1 - \frac{u}{1 - p}\right) + \frac{v}{p}\right) = (1 - p)F\left(\frac{u}{1 - p}\right) + pF\left(\frac{v}{p}\right) + M
\]
and taking account of \((6)\), \((7)\), \((8)\) we get
\[
F(u + v) + F(0_X) = (1 - p)F\left(\frac{u}{1 - p}\right) + pF\left(\frac{v}{p}\right) + M + (1 - p)F(0_X) + pF(0_X) + M =
\]
\[
= \left[(1 - p)F\left(\frac{u}{1 - p}\right) + pF(0_X) + M\right] + \left[(1 - p)F(0_X) + pF\left(\frac{v}{p}\right) + M\right] = F(u) + F(v)
\]
and the lemma is proved.

Consider in what follows the equation
\[
F(x + y) + C = F(x) + F(y)
\]
where \(F : K \rightarrow \mathcal{P}_0(Y)\), \(K\) is a convex cone in \(X\) with \(0_X \in K\), \(C \in \mathcal{P}_0(Y)\), \(X\) is a real vector space and \(Y\) is a real topological vector space.

\textbf{Lemma 2.2.} Let \(C\) be a convex and sequentially compact subset of \(Y\) with \(0_Y \in C\). A set-valued map \(F : K \rightarrow \mathcal{CCl}(Y)\) satisfies the equation \((10)\) if and only if there exists an additive set-valued map \(A : K \rightarrow \mathcal{CCl}(Y)\) such that
\[
F(x) = A(x) + C
\]
for every \(x \in X\).
Proof. Suppose that $F : K \to CCL(Y)$ satisfies the equation (10). It can be easily proved by induction that

$$(11) \quad F(nx) + (n - 1)C = nF(x)$$

for every $n \in \mathbb{N}$ and every $x \in X$.

For $n = 1$ the relation (11) is obvious. Suppose that the relation (11) holds for $n \in \mathbb{N}$ and we have to prove that

$$F((n + 1)x) + nC = (n + 1)F(x).$$

We have:

$$F((n + 1)x) + nC = F(nx + x) + C + (n - 1)C = F(nx) + F(x) + (n - 1)C = nF(x) + F(x) = (n + 1)F(x).$$

Now let $x \in K$. From (11) we get:

$$(12) \quad \frac{1}{2^n} F(2^n x) + \frac{2^n - 1}{2^n} C = F(x), \quad n \in \mathbb{N}.$$

Denote $A_n(x) = \frac{1}{2^n} F(2^n x), \, n \geq 0$. The sequence of sets $(A_n(x))_{n \geq 0}$ is decreasing. Indeed:

$$A_{n+1}(x) = \frac{1}{2^{n+1}} F(2^{n+1} x + 2^n x) \subseteq \frac{1}{2^{n+1}} (F(2^n x + 2^n x) + C) =$$

$$= \frac{1}{2^{n+1}} (F(2^n x) + F(2^n x)) = \frac{1}{2^{n+1}} \cdot 2F(2^n x) = \frac{1}{2} F(2^n x) = A_n(x).$$

Put $A(x) = \bigcap_{n \geq 0} A_n(x) \in CCL(Y)$. Prove that $A(x) \neq \emptyset$.

Let $u \in F(x)$ fixed. From (12) it results that for every $n \in \mathbb{N}$ there exists $a_n \in A_n(x)$ and $c_n \in C$ such that $u = a_n + \frac{2^n - 1}{2^n} c_n$. The set $C$ is sequentially compact, so there exists a subsequence $(c_{n_k})_{k \geq 0}$ of $(c_n)_{n \geq 0}$ convergent to $c \in C$ and

$$u = a_{n_k} + \frac{2^{n_k} - 1}{2^{n_k}} c_{n_k}, \quad k \geq 0.$$ 

It results that $a_{n_k} \to u - c$ as $k \to \infty$. We show that $u - c \in \bigcap_{n \geq 0} A_n(x)$. Suppose that $u - c \not\in \bigcap_{n \geq 0} A_n(x)$. Then there exists $p \in \mathbb{N}$ such that $u - c \not\in A_{n_p}$. We have $a_{n_k} \in A_{n_k}$ for $k \geq p$ and $\lim_{k \to \infty} a_{n_k} \in A_{n_p}$, because $A_{n_p}$ is closed, contradiction with $u - c \not\in A_{n_p}$.

We prove that

$$(13) \quad A(x) + C = F(x), \quad x \in X.$$

Let $u \in A(x) + C, \, u = a + c, \, a \in A(x), \, c \in C$. It results that $a \in A_{n_0}(x)$ for every $n \geq 0$ and let $c_n = \frac{2^n - 1}{2^n} c \in \frac{2^n - 1}{2^n} C$. From the relation (12) it results that there exists $b_n \in F(x)$ such that $a + c_n = b_n, \, n \geq 0$ and $\lim_{n \to \infty} b_n = a + c \in F(x)$, because $F(x)$ is closed. Hence $A(x) + C \subseteq F(x)$. 
Now let \( b \in F(x) \). From (12) it results that for every \( n \in \mathbb{N} \) there exists \( a_n \in A_n(x) \) and \( c_n \in C \) such that \( b = a_n + \frac{2^n - 1}{2^n} c_n \). The sequence \((c_n)_{n \geq 0}\) has a subsequence \((c_{n_k})_{k \geq 0}\) convergent to \( c \in C \), taking account of the sequential compactity of \( C \). Hence the sequence \((a_n)_{n \geq 0}\) is convergent to \( b - c \in A(x) \). Then \( b = (b - c) + c \in A(x) + C \).

The relation \( F(x) \subseteq A(x) + C \) is proved. It follows that the relation (13) is true.

We prove that \( A \) is an additive set-valued map.

By the relation (10) and (13) we obtain

\[
A(x + y) + C + C = A(x) + A(y) + C + C
\]

and taking account of the cancellation law of Radström [2] it results that \( A(x + y) = A(x) + A(y) \) for every \( x, y \in X \), hence \( A \) is an additive set-valued map.

If \( F(x) = A(x) + C, x \in K \), where \( A : K \to CCl(Y) \) is an additive set-valued map, then we get:

\[
F(x + y) + C = A(x + y) + C = A(x) + A(y) + C + C = (A(x) + C) + (A(y) + C) = F(x) + F(y)
\]

for every \( x, y \in X \).

The lemma is proved.

**Theorem 2.1.** If a set-valued map \( F : K \to CCl(Y) \), with \( F(0_X) \) sequentially compact set, satisfies the equation (4) then there exists an additive set-valued map \( A : K \to ccl(Y) \) and a compact convex set \( B \in \mathcal{P}_0(Y) \) such that

\[
(14) \quad F(x) = A(x) + B
\]

for every \( x \in X \).

**Proof.** Suppose that \( F \) satisfies the equation (1) and let \( \alpha \in F(0_X) \). The set-valued map \( G : K \to CCl(Y) \), given by the relation

\[
(15) \quad G(x) = F(x) - \alpha, \quad x \in X
\]

satisfies the equation

\[
(16) \quad G((1 - p)x + py) = (1 - p)G(x) + pG(y) + M
\]

and \( 0_Y \in G(0_X) \).

By Lemma 2.1 it results that \( G \) satisfies also the relation

\[
(17) \quad G(x + y) + G(0_X) = G(x) + G(y)
\]

for every \( x, y \in X \) and \( G(0_X) \) is sequentially compact set. Then in view of Lemma 2.2 it results that there exists an additive set-valued map \( A : K \to CCl(Y) \) such that \( G(x) = A(x) + G(0_X) \) for every \( x \in X \). It follows that \( F(x) = A(x) + F(0_X) \) for every \( x \in X \). Denoting \( B = F(0_X) \) the theorem is proved.

**References**


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