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SOME RESULTS CONCERNING THE MULTIVALUED OPTIMIZATION AND THEIR APPLICATIONS

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1. Introduction

The idea of modelling the optimization problems in terms of the fixed points of the multivalued operator drew the mathematicians' attention to it, determining an intense research activity, in the last three decades (see J.P.Aubin[1]; J.P.Aubin, A.Cellina [2]; R.A.Becker, H.Bercovici, C.Foiaş [3]; K.C.Border [4]; H.W.Corley [5], [6]; Lj.Gajić [7]; W.K.Kim [8]; Z.Liang [9]; J.E.Martinez-Legaz [10]; I.A.Rus [11], [12]; E.Tarafdar [13]; G.X.Z.Yuan, G.Isac, K.K.Tan, J.Yu [14]; J.X.Zhou [15] and others).

The purpose of this paper is to prove some multivalued analysis theorems with econo-
mical uses. The theorems obtained in the main section generalize some results from Corley [6]. We also formulate two mathematical models for consumer's problem, governed by multivalued operators, offering some existence results of these problems.

2. Preliminaries

Let (X, d) be a metric space. Throughout this paper we use the following notations:

$$P(X) := \{A \subset X \mid A \neq \varnothing\};$$

$$P_{cl}(X) := \{A \in P(X) \mid A = \bar{A}\};$$

$$P_{cp}(X) := \{A \in P(X) \mid A \quad \text{a compact set}\}.$$

If $A \in P(X)$ and $\varepsilon > 0$, then we denote:

$$\delta(A) := \sup\{d(a,b) \mid a, b \in A\};$$

$$P_b(X) := \{A \in P(X) \mid \delta(A) < +\infty\};$$

$$V(A;\varepsilon) := \{x \in X \mid d(x,a) < \varepsilon, \quad (\forall)\, a \in A\}.$$

Definition 2.1. Let $T : X \multimap X$ be a multivalued operator.
An element $x \in X$ is a ***strict fixed point of T*** iff $T(x) = \{x\}$.

We denote by $(SF)_T := \{x \in X \mid T(x) = \{x\}\}$ ***the strict fixed points set of T.***

Definition 2.2 ([12]). A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a ***comparison function*** if it satisfies the conditions:

i) φ is monoton increasing;

ii) $\Big(\varphi^n(t)\Big)_{n\in\mathbb{N}}$ converges to 0, for all $t > 0$.

Definition 2.3 ([12]). Let $T : X \to P(X)$ be a multivalued operator and $x_0 \in X$ be an arbitrary point. By definition, a sequence $(x_n)_{n\in\mathbb{N}}$ is called ***sequence of successive approximations of T, starting from*** x_0, if $x_0 \in X$ and $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$.

Definition 2.4. Let (X, d), (Y, d') be two metric spaces and $T : X \to P(Y)$ be a multivalued operator. Then, by definition, T is ***H-upper semicontinuous*** (briefly, H-u.s.c.) in $x_0 \in X$, iff $(\forall)\, \varepsilon > 0$, $(\exists)\, \delta > 0$ such that $(\forall)\, x \in B(x_0; \delta)$ we have $T(x) \subset V(T(x_0); \varepsilon)$.

The multivalued operator T is called ***H-u.s.c. on X***, if it is H-u.s.c. in each point $x_0 \in X$.

Definition 2.5. Let Y be a topological vector space. A set $C \in P(Y)$ is called a ***cone*** iff

$$(\forall)\, (x \in C, \quad 0 < \lambda \in \mathbb{R}) \quad \Rightarrow \quad \lambda \cdot x \in C.$$

Definition 2.6. A cone $C \subset Y$ is:

i) a ***convex cone***, if it is a convex set;

ii) a ***pointed cone***, if $C \cap (-C) = \{\theta\}$;

iii) an ***acute cone***, if $\bar{C}$ is a pointed cone.

Due to the liniar structure of the space Y, the cone C induces a partial order on Y, defined by:

$$x \leqslant_C y \quad \Leftrightarrow \quad y - x \in C, \qquad (\forall)\, x, y \in Y.$$

Let $A \in P(Y)$. We denote by $Max(A, \leqslant_C)$ ***the set of the maximal elements*** of the ordered set $(A, \leqslant_C)$.

Definition 2.7 ([5]). Let Y be a topological vector space, C a cone in Y and $B \in P(Y)$. B is said to be ***C-semicompact*** if every open cover $\{\mathcal{C}(y_i - \bar{C}) \mid y_i \in B,\ i \in I\}$ of B

has a finite subcover (where the symbol $\mathcal{C}(y_i - \bar{C})$ means the complement of a set $y_i - \bar{C}$).

If X is a topological vector space, then we denote by $P_{C-scp}(X)$ ***the space of all nonempty and C-semicompact subsets of X***.

In the sequel is necessary the following known results:

Theorem 2.8. (Corley, 1980). *Let Y be a topological vector space and $C \subset Y$ be an acute convex cone. If $B \in P(Y)$ is a C-semicompact set, then $Max(B, \leqslant_C) \neq \varnothing$.*

Theorem 2.9. (Corley, 1986). *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$ and $T : Y \to P_b(Y)$. We suppose that:*

i) there exist $0 \leqslant k < 1$, $y_0 \in Y$ and $y_{n+1} \in T(y_n)$, $(\forall)\, n \in \mathbb{N}$, for which $\delta(T(y_{n+1})) \leqslant k \cdot \delta(T(y_n))$;

ii) $y \in T(y), \quad (\forall)\, y \in Y.$
Then, $y_n \to y^* \in Y$ *as* $n \to \infty$ *and* $\{y^*\} = T(y^*).$

3. Main results

First result of this section is the following topological strict fixed point theorem, that generalizes Theorem 2, in Corley [6].

Theorem 3.1. *Let X be a topological vector space,* $Y \in P_{C-scp}(X)$*, C be a acute convex cone in X and* $T : Y \to P_{C-scp}(Y)$ *be a multivalued operator. We suppose that* $y \in T(y)$*, for each* $y \in Y$*. Then* $(SF)_T \neq \varnothing$*.*

Proof. Applying Theorem 2.8, we obtain $B := Max(Y, \leqslant_C) \neq \varnothing$. Let $x \in B$. Clearly, $x \in Y$. From Theorem 2.8 it results that $F(x) := Max(T(x), \leqslant_C) \neq \varnothing$. Let $y \in F(x)$. Hence, $y \in T(x) \subset Y$. Because, $x, y \in Y$ and x is a maximal element in Y, it follows that $y \leqslant_C x$.

On the other hand, by the hypothesis iii) we have $x \in T(x)$. Since $y \in T(x)$ and y is a maximal element in $T(x)$, we deduce that $x \leqslant_C y$.

As, $\leqslant_C$ is an ordered relation on Y, we obtain $x = y$. Consequently, there exists $y \in Y$ such that $T(y) = \{y\}$. So, $(SF)_T \neq \varnothing$. $\square$

We'll establish now a continuity property for a multivalued operator, from the optimization theory.

Theorem 3.2. *Let X and Y be two normed spaces,* $C \subset Y$ *be an acute convex cone and* $T : X \to P_{b,C-scp}(Y)$ *be a H-u.s.c. multivalued operator.*

In these conditions, $F(x) := Max(T(x), \leqslant_C)$ *is H-u.s.c. multivalued operator.*

Proof. From Theorem 2.8, we have $F(x) \neq \varnothing$, for each $x \in X$. Let $x_0 \in X$, be an arbitrary point. Because T is H-u.s.c. in x_0, we have that for each $\varepsilon > 0$, there exists $\varepsilon_1 = \dfrac{\varepsilon}{2M} > 0$, for this ε_1 there exists $\delta > 0$ such that for all $x \in B(x_0; \delta)$ we have $T(x) \subset V(T(x_0); \varepsilon_1)$. In fact, we get:

$$(\exists)\ F(x) = Max(T(x), \leqslant_C) \subset T(x) \subset V(T(x_0); \varepsilon_1) \subset V(Max(T(x_0)); \varepsilon) = V(F(x_0); \varepsilon).$$

Indeed, let us denote by: $\mu := \sup\{\|u - v\| \ \mid\ u, v \in T(x_0)\} > 0$. Since T has bounded values, it follows that $\mu < +\infty$. Hence, we can take $M := \mu + 1 > 1$. Also, it is known that:

$$V(T(x_0); \varepsilon_1) := \{y \in Y \ \mid\ \|y - y_0\| < \frac{\varepsilon}{2M}, \quad (\forall)\, y_0 \in T(x_0)\};$$

$$V(Max(T(x_0)); \varepsilon) := \{y \in Y \ \mid\ \|y - y_0^m\| < \varepsilon, \quad (\forall)\, y_0^m \in Max(T(x_0))\}.$$

In set $Max(T(x_0))$ can be more than one element. Next, we want to prove that

$$V(T(x_0); \varepsilon_1) \subset V(Max(T(x_0)); \varepsilon).$$

Let $y \in V(T(x_0); \varepsilon_1)$, then:

$$\|y - y_0^m\| \leqslant \|y - y_0\| + \|y_0 - y_0^m\| < \frac{\varepsilon}{2M} + \frac{\varepsilon}{2M} = \frac{\varepsilon}{M} < \varepsilon.$$

Finally, we obtained that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(x_0; \delta)$, we have $F(x) \subset V(F(x_0); \varepsilon)$. $\square$

The following result is a strict fixed point metrical theorem, that generalizes Theorem 2.9 belonging to H.W.Corley.

Theorem 3.3. *Let* (X, d) *be a complete metric space,* $Y \in P_{cl}(X)$ *and* $T : Y \to P_b(Y)$. *We suppose that:*

i) there exist a comparison function φ *and a sequence* $(y_n)_{n\in\mathbb{N}}$ *of succesive approximations of T, starting from* $y_0 \in Y$, *such that* $\delta(T(y_{n+1})) \leqslant \varphi(\delta(T(y_n)))$, *for all* $n \in \mathbb{N}$;

ii) $y \in T(y)$, $(\forall)\, y \in Y$.

In these conditions, $(SF)_T = \{y^*\}$.

Proof. Let $y_0 \in Y$ and $y_{n+1} \in T(y_n)$, $(\forall)\, n \in \mathbb{N}$. From i) we have:

$$\delta(T(y_n)) \leqslant \varphi(\delta(T(y_{n-1}))) \leqslant \ldots \leqslant \varphi^n(\delta(T(y_0))) \to 0, \quad \text{as} \quad n \to \infty.$$

This implies that $\delta(T(y_n)) \to 0$, as $n \to \infty$.

Since $\delta(T(y_n)) \to 0$, it result $d(y_m, y_n) \to 0$, as $m, n \to \infty$. It follows that the sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy in the complete metric space (X, d). Hence, there exists $y^* \in X$ such that $\lim\limits_{n\to\infty} y_n = y^*$. Moreover, Y being closed we have $y^* \in Y$.

Because, by the hypothesis ii), $y^* \in T(y^*)$ and on the other side $\delta(T(y^*)) = 0$, we conclude that $\{y^*\} = T(y^*)$. $\square$

Remark 3.4. If $\varphi(t) = k \cdot t$, $0 \leqslant k < 1$, then from Theorem 3.3 we obtain Theorem 2.9.

4. Applications to consumer's problem

In the equilibrium-point theory for abstract economies, appear the preference operators. In our context, preference or indifference will be taken as primary notions, meaning that, if x and x' are two variants of commodity consumption, then a consumer may express a preference for one of them or he may say they are equivalent (indifferent) to him to choose one or the other.

Let an economy, there are n commodities and m consumers. Considering that the commodity space is $\mathbb{R}^n$, then for the consumption vectors $x, x' \in \mathbb{R}^n$, we will define ***the preference-indifference relation*** (***the weak preference relation***) $\succeq$ on $\mathbb{R}^n$, as a logical combination, in this way:

$x \succeq x' \quad \Leftrightarrow \quad x$ is rather prefered to x' or the consumption vectors x and x' are indifferent to me.

We assume that the consumption set of the consumer i, $i \in \{1, ..., m\}$, is given by a subset $Y_i \subset \mathbb{R}^n$. The preference relation of consumer i is a binary relation $\succeq_i$ defined on Y_i. For $x \in Y_i$ define the ***upper contour set*** $U_i(x) := \{x' \in Y_i \mid x' \succeq_i x\}$. We'll omit the index i, when the meaning is clear.

The preference relation $\succeq$ being given, ***the preference multivalued operator*** will be defined as follows: $U : Y \multimap Y$, $\quad U(x) := \{y \in Y \mid y \succeq x\}$.

By definition, a consumption vector $x^* \in Y$ is a ***optimal preference for U***, if $U(x^*) = \{x^*\}$. It this way, the consumer's problem can be modelled in terms of strict fixed points of a multivalued operator. For example, we will prove the following result.

Theorem 4.1. *Consider* $Y \in P(\mathbb{R}^n)$, $C \subset \mathbb{R}^n$ *be an acute convex cone and* $U : Y \to P(Y)$ *be the preference operator. We suppose that:*

i) Y *is a C-semicompact set;*

ii) For all $x \in Y$ *, the upper contour set* $U(x)$ *is a C-semicompact set.*

Then, the consumer's problem has at least one solution.

Proof. Because all vectors of consumer admit the relation $x \succeq x$, we have $x \in U(x)$, $(\forall)\, x \in Y$. Thus, the conclusion follows immediately by virtue of theorem 3.1. $\square$

Remark 4.2. It is not necessary for the validity of our result to assume that the relation $\succeq$ is either complete or transitive.

In the economic models, there is also the possibility for another more subtle approach to the fundamental concepts from the consumer's theory, using the numeric reprezentation of consumer's preferences by an utility operator.

In order to create a proper background for an optimization model, the following sets are necessary, being requested by the economic use.

The ***consumption set*** will be $K \subset \mathbb{R}^n$. It is a closed convex cone in $\mathbb{R}^n$.

The ***price space*** is defined by $Y = \{p \in \mathbb{R}^n \mid p \cdot x \geqslant 0, \quad x \in K\}$. Y is a closed convex cone. It is well known that $K = \{x \in \mathbb{R}^n \mid p \cdot x \geqslant 0 \quad \text{for all} \quad p \in Y\}$.

Note that, without loss of generality, spendable income is normalized to unity. We may define the ***budget set***, $B(p) := \{x \in K \mid p \cdot x \leqslant 1\}$.

Let $\succeq$ be a preference ordering on K. Assume that $\succeq$ can be reprezented by a ***utility function*** $u : K \to \mathbb{R}$, i.e. a real valued function such that for all vectors x, x' in K, $x \succeq x'$ if only if $u(x) \geqslant u(x')$.

The consumer's problem supposes the solving of the following optimization problem: $\sup\limits_{x \in B(p)} u(x)$. The solution of the consumer's problem, regarded as a price function, leads to a new operator $v : Y \to \mathbb{R}$, $v(p) := \sup\{u(x) \mid p \cdot x \leqslant 1\}$, whom we'll call the ***indirect utility function associated with*** u.

Theorem 4.3. *Let* $K, Y \in P(\mathbb{R}^n)$ *and let* $T : K \to P_{cp}(Y)$ *be continuous.*

Consider an utility function $u : K \to \mathbb{R}$ *such that* $v : Y \to \mathbb{R}$ *is the indirect utility function of* u*. Define* $F : K \multimap Y$, $F(x) := \{p \in T(x) \mid v(p) = \sup\limits_{x' \in B(p)} u(x')\}$.

In these conditions, we have:

a) If v *is continuous, then* F *is closed and u.s.c.;*

b) If u *is continuous, then* $F(x) \in P_{cp}(Y)$.

Proof. a) Let $x_n \to x_0$, $p_n \in F(x_n)$, $p_n \to p_0$. We will prove that $p_0 \in F(x_0)$. Since T is u.s.c. and compact-valued it follows that T is closed, so $p_0 \in T(x_0)$. Suppose $p_0 \notin F(x_0)$. Then there is $q_0 \in T(x_0)$ with $v(q_0) > v(p_0)$. Because T is l.s.c. at x, there is a sequence $q_n \to q_0$, $q_n \in T(x_n)$. Since $q_n \to q_0$, $p_n \to p_0$ and $v(q_0) > v(p_0)$, the continuity of v implies that eventually $v(q_n) > v(p_n)$, contradicting $p_n \in F(x_n)$.

Now, $F = T \cap F$ implies that F is u.s.c. at x.

b) Because the sets $B(p) = \{x \in K \mid p \cdot x \leqslant 1\}$ are compact and u is continuous on $B(p)$, u takes its maximum value at some $\bar{x} \in B(p)$, i.e. $\sup\limits_{x' \in B(p)} u(x') = u(\bar{x}) =$

$v(p)$. So, $F(x) \neq \varnothing$. Now, from the assumption $T(x) \in P_{cp}(Y)$ we have that F is compact-valued. $\square$

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