Seminar on Fixed Point Theory Cluj-Napoca, Volume 3, 2002, 261-270
http://www.math.ubbcluj.ro/~nodeacj/journal.htm

ON THE NUMERICAL APPROACH OF KORTEWEG - DE VRIES - BURGER EQUATIONS BY SPLINE FINITE ELEMENT AND COLLOCATION METHODS

G. MICULA AND MARIA MICULA

Department of Applied Mathematics
Babeş-Bolyai University, Cluj-Napoca, Romania

Abstract. The detailed study of nonlinear partial differential equations arising in physical applications is not enough developed and not exhaustive investigated subject. For many such equations little quantitative and detailed qualitative information is known about their solutions, about a basic questions of existence, uniqueness and stability as well. Moreover, the general procedures for the numerical and approximative treatment of such equations are worthily very desired. In this paper the spline finite element and collocation methods for some special nonlinear Korteweg-de Vries-Burger equations are presented.

AMS Subject Classification: 35B35, 65M12, 65N30, 65N35, 65D07, 65M50.

1. Introduction

The Korteweg - de Vries equation introduced in 1895 by D.J. Korteweg and G. de Vries was originally derived in order to describe the behaviour of one-dimensional shallow water waves with small but finite amplitudes.

Investigating turbulent fluid motion in 1939 J.M. Burger established a famous equation which is modeling diverse physical phenomena as traffic flow, shock waves, acoustic transmission in fogs, etc., in fact any nonlinear wave propagation problem subject to dissipation. The dissipation may arise from viscosity, heat conduction, chemical reaction, etc.

The important feature of the Burger's equation is that it is a prototype equation for the balance between the nonlinear convection term and the diffusive term. The ability to handle this balance is probably the most difficult aspect of computing fluid dynamic problems.

More recently, such kind of Korteweg - de Vries and Burger equations have been found to describe wave phenomena in **plasma physics** (C.S. Gardner and G.K. Morikawa, 1960, H. Wasmini and T. Taniuti, 1966), **anharmonic crystals** (N.J. Zanuski, 1967, 1981, M.D. Kruskal, 1965), **buble-liquid mixture** (L. Van Wijngarden, 1968), etc.

There has became great interest in these equations because of their special properties.

N.J. Zabuski and M.D. Kruskal (1965) discovered the *concept of solutons localized waves* with special interaction properties, while studying the results of numerical computation on the Korteweg - de Vries equation.

The work of C.S. Gardner (1967) led to the explosion of both of theoretical and numerical aspects which is still growing today.

For these problems no analytical results are known, therefore, numerical studies are essential in order to develop an understanding the phenomena.

In our knowledges, until now to Korteweg - de Vries - Burger equations the following numerical methods are applied:

1. **Finite difference methods**: explicit methods and implicit methods (Zabutski and Kruskal, 1965, 1978; Hopscotch, 1976; Goda, 1975; Taha, 1982, 1984; Kruskal, 1981).

2. **Finite Fourier transform or pseudospectral methods**: splet step Fourier and pseudospectral methods (Tappert, 1974; Fornberg and Whitham, 1978; Nouri and Sloan, 1989).

3. **Collocation (in space and time) method**: spline collocation method (Brunner and Roth, 1997).

4. **Finite element method**: spline approximating subspaces (A.H.A. Ali, R.T.L. Gardner, G.A. Gardner, 1992, 1993, 1995; I. Dag, 1997).

For more details and exhaustive literature on Korteweg - de Vries - Burger equations we mention the comprehensive survey paper [11] of R.A. Miura (1976).

Nonlinear partial differential equations of the form

$$u_t + u^n u_x + \left(\frac{j}{2t} + \alpha\right) u + \left(\beta + \frac{\gamma}{x}\right) u^{n+1} = \delta u_{xx} - \varepsilon u_{xxx}, \quad (x,t) \in D \subset \mathbb{R}^2, \tag{1.1}$$

where $j, \alpha, \beta, \gamma, \delta, \varepsilon$ are nonnegative constants, and n is a positive integer, model many physical phenomena.

For $n = 1$, $j = \alpha = \beta = \gamma = \varepsilon = 0$, the equation (1.1) becomes the well-known **Burger's equation**

$$u_t + uu_x = \delta u_{xx}, \quad (x,t) \in D \subset \mathbb{R}^2. \tag{1.2}$$

When $n = 2$, $j = \alpha = \beta = \gamma = \varepsilon = 0$, (1.1) becomes

$$u_t + u^2 u_x = \delta u_{xx}, \quad (x,t) \in D \subset \mathbb{R}^2 \tag{1.3}$$

and it is called the **modified Burger's equation**.

For $n = 1$, $\alpha = \beta = \gamma = \varepsilon = 0$, the equation (1.1) takes the form:

$$u_t + uu_x + \frac{j}{2t} u = \delta u_{xx}, \quad (x,t) \in D \subset \mathbb{R}^2 \tag{1.4}$$

and it is called the **nonplanar Burger's equation**.

It describe the propagation of weakly nonlinear longitudinal waves in a gas or liquid, subject not only to diffusion effects associated with the viscosity and thermal conductivity, but also to the geometrical effects of change of ray tub area.

When $j = \alpha = \beta = \gamma = \delta = 0$, the equation (1.1) becomes

$$u_t + u^n u_x + \varepsilon u_{xxx} = 0, \quad x \in [0,1],\ t \geq 0. \tag{1.5}$$

This equation is called the **Kortweg - de Vries equation** (1895). Here the function $u = u(x,t)$ is a 1-periodic in the spatial variable and the temporal variable $t \geq 0$ is prescribed at $t = 0$ by $u(x,0) = u_0(x)$, $0 \leq x \leq 1$.

The Korteweg - de Vries equation arise in modelling the propagation of small-amplitude long waves in nonlinear dispersive media. The case $n = 1$ is the classical Korteweg - de Vries equation, about which much has been written in the last decades, and which arises in a number of interesting physical problems.

In real physical situations, dissipative effects are often as important as nonlinear and dispersive effects and this fact has given currency to study the **Korteweg - de Vries - Burger equation**

$$u_t + uu_x - \delta u_{xx} + \varepsilon u_{xxx} = 0 \tag{1.6}$$

(obtained from (1.1) for $n = 1$, $j = \alpha = \beta = \gamma = 0$) as a model that incorporates all three effects: dissipative, dispersive and blow-up one.

In (1.6) the parameter ε is fixed and positive that is related to a generalization of the classical Stokes number of surface water-wave theory, and $\delta > 0$ is another parameter expressing the relative strenght of dissipative to nonlinear effects.

The equation (1.6) is one of the few nonlinear partial differential equations which has been extremly much investigated in the last time.

The most difficulties arise in the numerical solution of Korteweg - de Vries - Burger equation (1.6) with given initial and boundary conditions. In many cases, numerical methods are likely to produce results which include non-physical oscillations unless the size of the elements is unrealistic small.

Recently, finite element methods have been used successfully to obtain accurate numerical solution even for small value of parameters.

Following the idea of A.H.A. Ali, L.R.T. Gardner and G.A. Gardner [1], [2] we shall apply the spline finite element method using quadratic polynomial spline for the numerical solution of the Korteweg - de Vries - Burger equation (1.6).

2. Spline finite element solution for the Korteweg - de Vries - Burger equation

Consider the Korteweg - de Vries - Burger equation:

$$u_t + uu_x - \delta u_{xx} + \varepsilon u_{xxx} = 0, \quad t \geq 0,\ a \leq x \leq b, \tag{2.1}$$

where δ, ε are positive parameters with the following initial and boundary conditions:

$$u(x,0) = u_0(x), \quad a \leq x \leq b \tag{2.2}$$

$$\left.\begin{array}{ll} u(a,t) = \beta_1, & u(b,t) = \beta_2 \\ u_x(a,t) = 0, & u_x(b,t) = 0 \end{array}\right\} \tag{2.3}$$

Apply Galerkin technique to the problem (2.1)-(2.3) using quadratic spline interpolation functions over finite elements with weight function $v(x)$, integrate by parts and use the boundary conditions (2.3) we obtain:

$$\int_a^b v(u_t + uu_x)dx + \int_a^b (\delta u_x v_x - \varepsilon v_x u_{xx})dx = 0. \tag{2.4}$$

The presence of the second spatial derivative within the integrand means that the interpolating functions and their first derivatives must be continuous throughout the region. Quadratic B-spline finite elements satisfy this requirement.

Now, the interval $[a,b]$ is partitioned into uniformly size finite elements by knots $\{x_i\}$ such that:

$$a = x_0 < x_1 < \cdots < x_N = b.$$

Each quadratic B-spline basis denoted by $\{Q_m\}_{m=-1}^{N}$ has the form:

(2.5)

$$h^2 Q_m(x) := \begin{cases} (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2 + 3(x_m-x)^2, & x_{m-1} \le x < x_m \\ (x_{m+2}-x)^2 - 3(x_{m+1}-x)^2, & x_m \le x < x_{m+1} \\ (x_{m+2}-x)^2, & x_{m+1} \le x < x_{m+2} \\ 0, & \text{otherwise} \end{cases}$$

$m = -1, 0, \ldots, N$ where $h := x_{m+1} - x_m$ for all m, and covers three intervals, $x_{m-1} \le x < x_{m+2}$, so that three splines Q_{m-1}, Q_m and Q_{m+1} cover each finite element $[x_m, x_{m+1}]$, all other basis splines are zero in this region.

Use a local coordinate system for the finite subintervals $[x_m, x_{m+1}]$, $h\xi = x - x_m$, $0 \le \xi < 1$, to obtain for the trial functions expressions that are independent of the element position:

$$Q^e := (Q_{m-1}, Q_m, Q_{m+1}) = (1 - 2\xi + \xi^2, 1 + \varepsilon\xi - 2\xi^2, \xi^2). \tag{2.6}$$

The variation of a function u over the element $[x_m, x_{m+1}]$ is

$$u = Q_{m-1}\delta_{m-1} + Q_m\delta_m + Q_{m+1}\delta_{m+1} = Q^e \cdot d^e = (1-2\xi+\xi^2, 1+2\xi-\xi^2, \xi^2) \cdot d^e. \tag{2.7}$$

The modal values u_m, u'_m at the knot x_m are given in terms of the parameters δ_i by:

$$u_m = \delta_m + \delta_{m-1}, \quad hu'_m = 2(\delta_m - \delta_{m-1}).$$

As known, the splines $(Q_{-1}, Q_0, \ldots, Q_N)$ form a basis for the quadratic spline function space defined over $[a,b]$.

The global approximation $u_n(x,t)$ for the exact solution $u(x,t)$ of the problem (2.1)-(2.3) which use these splines as trial functions is:

$$u_N(x,t) := \sum_{j=-1}^{N} \delta_j(t) Q_j(x) \tag{2.8}$$

where δ_i are time dependent functions to be found.

An element contributs to equation (2.4) is the following integral:

$$\int_{x_m}^{x_{m+1}} [v(u_t + uu_x) + \delta v_x u_x - \varepsilon v_x u_{xx}] dx. \tag{2.9}$$

Use (2.8) in (2.9) and identify the weight function v with a quadratic B-spline to produce the element contribution. Combine together the contributions from each element we obtain a couple set of the following nonlinear ordinary differential equations:

$$A\dot{d} + B(d)d + \delta Cd - \varepsilon Dd = 0 \tag{2.10}$$

where the vector of unknown functions is

$$d := (\delta_{-1}(t), \delta_0(t), \ldots, \delta_N(t))^T. \tag{2.11}$$

Here $A, B(d), C, D$ are pentagonal matrices and row m of each has the following form:

$$A : \frac{h}{30}(1, 26, 66, 26, 1), \quad B : \frac{2}{3h}(-1, -2, 6, -2, -1), \quad C : \frac{2}{h^2}(1, -2, 0, 2, -1)$$

$$D : \frac{h}{30}[(-1, 0, 1, 0, 0)d_m, (-7, -31, 31, 7, 0)d_m, (-2, -62, 0, 62, 2)d_m,$$
$$(0, -7, -31, 31, 7)d_m, (0, 0, -1, 0, 1)d_m]$$

where $d_m := (\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2})^T$.

Let d be linearly interpolated between two levels n and $n+1$; then d and its time derivative $\dot{d}$ are given by

$$d := \frac{1}{2}(d^n + d^{n+1}), \quad \dot{d} := \frac{1}{\Delta t}(d^{n+1} - d^n). \tag{2.12}$$

Use the definitions (2.12) to write the equation (2.1) in the form:

$$\left[A + \frac{\Delta t}{2}(B(d) + \delta C - \varepsilon D)\right] d^{n+1} = \left[A - \frac{\Delta t}{2}(B(d) + \delta C - \varepsilon D)\right] d^n. \tag{2.13}$$

This is the recurrence relationship for d which, since the matrix $B(d)$ depends on d, is nonlinear.

Before solving this system apply the boundary conditions (2.3) to produce from (2.13) a recurrence relationship for $(\delta_1^n, \dots, \delta_{N_2}^n)^T$. The system (2.13) contains $(N-2) \times (N-2)$ pentadiagonal matrices, and it can be solved by iteration, but an inner iteration is also needed at each step to cope with nonlinear term.

The boundary parameters $\delta_{-1}, \delta_0, \delta_{N-1}, \delta_N$ can be computed at each time step from the boundary conditions.

To use the iterative procedure (2.13) a starting vector d^0 must first be determined from the initial condition (2.2).

The starting value d^0 is determined from the initial condition $u(x, 0) = u_0(x)$ by interpolating using quadratic splines. For $t = 0$, (2.8) becomes:

$$u_N(x, 0) = \sum_{i=-1}^{N} \delta_i^0 Q_i(x) \tag{2.14}$$

where δ_i^0 are to be determined. To find d^0 require $u_N(x, 0)$ to satisfy the following two conditions:

(a) It shall agree with the initial condition $u(x, 0)$ at the knots x_i, $i = 0, 1, \dots, N$, leading $N + 1$ conditions.

(b) Its first derivative should agree with that of the exact solution at either x_0 and x_N.

The initial vector d^0 is thus determined as the solution of a matrix equation of the form $Md^0 = b$.

In conclusion the Galerkin method with quadratic spline finite elements is capable to produce accurate and stable numerical solutions for the Korteweg - de Vries - Burger equation even when the values of viscosity coefficient are small.

Remark. The similar method can be applied to the numerical approximation of solutions to the initial and periodic value problem for the generalized Korteweg - de Vries - Burger equation:

$$u_t + u^p u_x - \delta u_{xx} + \varepsilon u_{xxx} = 0, \quad (x,t) \in [0,1] \times [0,T]$$
$$u(x,0) = u_0(x), \quad 0 \le x \le 1$$

where $p \ge 2$, u_0 is a reasonably smooth periodic, real-valued function on $\mathbb{R}$, and ε and δ are positive constants as previously.

3. The spline collocation method for Korteweg - de Vries - Burger equation

Following the idea of H. Brunner and H. Roth [4] we shall describe the spline collocation method in both space and time to solve the Korteweg - de Vries - Burger equation (2.1) written in the form:

$$\begin{aligned} & u_t + A(u) = 0, \quad 0 \le t < T, \; x \in \mathbb{R} \\ & u(x,0) = u_0(x), \quad x \in \mathbb{R} \\ & \frac{\partial^j u(x,t)}{\partial x^j} \to 0 \text{ as } |x| \to +\infty, \; t \in [0,T]. \end{aligned} \tag{3.1}$$

Here $A(u)$ is a spatial differential operator, so that $A(u)$ is a function of u and its special derivatives containing in particular all equation (1.1)-(1.6).

One assumes that the solution u of (3.1) exists, it is unique and sufficiently smooth.

We seek an approximation function $U(x,t) \approx u(x,t)$ on $[a,b] \times [0,T]$ from a tensor product of polynomial spline spaces $S(p, p-q, \Delta_x) \otimes S(l, 0, \Delta_t)$, where q is a positive integer not exceeding $p+1$ and $S(p, r, \Delta_v)$ consists of splines of degree p and continuity class C^r on a partition Δ_v on the domain of variables v.

The special interval $[a,b]$ is chosen sufficiently large so that during the time interval of interest $[0,T]$, the solution u and its special derivatives remain small enough at a and b, that is

$$\frac{\partial^j u(x,t)}{\partial x^j} \to 0 \text{ as } x \to a_+, \; x \mapsto b_-, \; t \in [0,T], \; j = 0, 1, \ldots$$

Following a method of lines approach, one begins with discretization of space, thereby obtaining a system of nonlinear differential equations, and then discretize time to obtain a system of nonlinear algebraic equations.

Collocation in space. Let

$$\Delta_x = \Delta_x^N : \; a = x_0 < x_1 < \cdots < x_N = b$$

be a partition of spatial interval $[a,b]$ and set

$$I_k := [x_k, x_{k+1}], \quad h_k := x_{k+1} - x_k, \quad h := \max\{h_k, \; k = 0, 1, \ldots, N-1\}.$$

If $\mathcal{P}_p$ denoted the set of real polynomials of degree at most p for $q \le p+1$ let

$$S(p, p-q, \Delta_x) := \{v | \; v \in C^{p-q}, \; v|_{I_k} \in \mathcal{P}_p\}$$

be the polynomial spline space consisting of real polynomials of degree p and having $p-q$ continuous derivatives on $[a,b]$. The linear space $S(p, p-q, \Delta_x)$ has the dimension

$d = Nq + p + 1 - q$. A computationally convenient basis for this space is the B-spline basis $\mathcal{B} := \{B_j(x) : \ 1 \le j \le d\}$ constructed from a knot sequence $\{\widetilde{x}\}$ consisting of q knots placed at each interior meshpoints $x_1, x_2, \dots, x_{N_1}$ (establishing the meshpoints as the breakpoints and confering the desired smoothness) and an additional $p+1$ knots at (or outside) each end of the interval $[a, b]$. We shall place $p + 1$ knots at a and at b and we shall refer to the resulting knot sequence as $\{\overline{x}\}_A$.

The approximation $U(x, t)$ of the solution of (3.1) will take the form of a time dependent linear combination of basis functions:

$$U(x,t) := \sum_{j=1}^{d} B_j(x) v_j(t) \tag{3.2}$$

where $v_j, \ j = 1, 2, \dots, d$ are functions to be determined in $C[0, T]$.

Now introduce the collocation parameters

$$0 \le c_1 < c_2 < \cdots < c_q \le 1.$$

On each subinterval $I_k, \ k = 0, 1, \dots, N-1$ set

$$\widehat{x}_{kq+i} := x_k + c_i h_k, \quad i = 1, 2, \dots, q$$

and let

$$X_M := \{\widehat{x}_j : \ 1 \le j \le Nq\}$$

be the special collocation points.

Note that we have q collocation points on each subinterval of Δ_x yielding Nq conditions on U.

There remain $p+1-q$ condition to impose and we obtain these from our boundary conditions.

We shall require that the solution and its first n_0 derivatives to be zero directly at the endpoints of $[a, b]$. This requires that $p + 1 - q$ be an even number.

Now let $B \in \mathbb{R}^{d\times d}$ be the matrix $B := (B_j(\widehat{x}_i))_{i,j}$ in row i and column j.

Putting the approximation $U(x, t)$ in the equation (3.1) one obtains the following system of d ordinary differential equations

$$Bv' + A(Bv) = 0 \tag{3.3}$$

with initial conditions

$$Bv = U^0, \text{ where } U^0 := (u_0(\widehat{x}_1), \dots, u_0(\widehat{x}_{Nq}))^T$$

is the restriction of $u_0(x)$ to the set of collocation points $\widehat{x}_N$. Due to the minimal compact support of B-splines, there are at most $p + 1$ nonzeros on any row of B, so that B has a special block row structure.

In addition, it is easy to show that the matrix $B := (B_j(\widehat{x}_i))_{i,j}$ is strictly positive if and only if $x_i < \widehat{x}_i < x_{i+p+1}$.

It follows that $\det B > 0$ whenever the collocation points are distinct from meshpoints.

Collocation in time. Continuing with the discretization, we partition the time interval $[0,T]$ with the temporal mesh:

$$\Delta_t := \Delta_t^M : \ 0 = t_0 < t_1 < \cdots < t_M = T$$

and take our functions $v_j(t), \ j = 1, 2, \ldots, Mq$ to be elements of a polynomial spline space. In particular, we will consider only the case where $v_j \in C[0,T]$ are piecewise linear functions. In [4] Brunner and Roth considered also the piecewise quadratic case.

Let denote $\tau_n := t_{n+1} - t_n, \ n = 0, 1, \ldots, M-1$.

We represent the functions $v(t)$ on $[t_n, t_{n+1}]$ by a piecewise linear interpolant

$$v(t) = a^{(n)} L_0^{(n)}(t) + b^{(n)} L_1(t) \tag{3.4}$$

where $a^{(n)}$ and $b^{(n)}$ are vectors to be determined in $\mathbb{R}^d$ and $L_0^{(n)}, L_1^{(n)}$ are the fundamental Lagrange polynomials defined by:

$$L_0^{(n)}(t) := \begin{cases} \dfrac{1}{\tau_n}(t_{n+1} - t), & t \in [t_n, t_{n+1}] \\ 0, & \text{elsewhere} \end{cases}$$

$$L_1^{(n)}(t) := \begin{cases} \dfrac{1}{\tau_n}(t - t_n), & t \in [t_n, t_{n+1}] \\ 0, & \text{elsewhere.} \end{cases}$$

Thus on $]t_n, t_{n+1}[$ we have

$$\frac{dv}{dt} = -\frac{1}{\tau_n} a^{(n)} + \frac{1}{\tau_n} b^{(n)}.$$

Introducing the set of time collocation points:

$$\widehat{T}_M := \{\widehat{t}_n := t_n + \gamma_1 \tau_n | \ \gamma_1 \in [0,1], \ 0 \le n \le M-1\}$$

we find

$$L_0^{(n)}(\widehat{t}_n) = 1 - \gamma_1, \quad L_1^{(n)}(\widehat{t}_n) = \gamma_1. \tag{3.5}$$

If we now collocate at $\widehat{t}_n$ in (3.3) using (3.4) and (3.5) the collocation equation becomes:

$$\frac{1}{\tau_n} B(b^{(n)} - a^{(n)}) + A\{B[(1-\gamma_1)a^{(n)} + \gamma_1 b^{(n)}]\} = 0.$$

Because $v(t_{n+1}) = b^{(n)}$ on $[t_n, t_{n+1}]$ and on $[t_{n+1}, t_{n+2}]$ we get $v(t_{n+1}) = a^{(n+1)}$, it follows

$$a^{(n+1)} = b^{(n)}.$$

Meanwhile, the initial condition $Bv = U^0$ becomes

$$B[a^{(0)} L_0^{(0)}(t_0) + b^{(0)} L_1^{(0)}(t_0)] = Ba^{(0)} = U^0.$$

We can now recursively compute the approximating solution $U(x,t)$ on the stripts $[0,b] \times [t_n, t_{n+1}]$, $n = 0, 1, \ldots, M-1$ and it is clear that $U \in S(p, p-q, \Delta_x) \otimes S(1, 0, \Delta_t)$.

For the convergence analysis of the present collocation method and the estimation of error we refer to the paper [4] of Brunner and Roth.

We conclude that the spline collocation method both in space and time is capable of delivering global, accurate and efficient approximating solution. It can be extended to more general boundary conditions.

References

[1] Ali A.H.A., Gardner R.T.L., Gardner G.A., *Numerical studies of the Korteweg - de Vries - Burger equation using B-spline finite elements*, J. Math. Phys. Sci. 27(1993), No.1, 37-53.

[2] Ali A.H.A., Gardner R.T.L., Gardner G.A., *A collocation solution for Burgers equation using cubic B-spline finite elements*, Comput. Meth. Appl. Mech. Engrg. 100(1992), 325-337.

[3] Bona J.L., Dougalis V.A., Karakashian O.A., McKinney W.R., *The effect of dissipation on solutions of the generalized Korteweg - de Vries equation*, J. Comput. Appl. Maths. 74(1996), 127-154.

[4] Brunner H., Roth H.K., *Collocation in space and time: experience with Korteweg - de Vries equation*, Appl. Numer. Math. 25(1997), 369-390.

[5] Dag I., *A cubic B-spline collocation method for the regularized long-wave equation*, OGU Maths. Preprints 14(1997), Turkey.

[6] Gardner R.T.L., Gardner G.A., *Solitary waves on the regularized long-wave equation*, J. Comput. Physics 91(1990), No.2, 441-459.

[7] Gardner R.T.L., Gardner G.A., Geyikli T., *Solitary wave solutions of the MKdV equation*, Comput. Meth. Appl. Mech. Engrg. 124(1995), 321-333.

[8] Gardner C.S., Green J.M., Kruskal M.D., Miura R.M., *Korteweg - de Vries equation and generalizations*, Communications on Pure and Applied Mathematics 27(1974), 97-133.

[9] Kato T., *On the Korteweg - de Vries equation*, Manuscripta Math. 28(1979), 89-99.

[10] Micula G., Micula Sanda, *Handbook of Splines*, Kluwer Acad. Publishers, Dordrecht - London - Boston, 1999.

[11] Miura R.M., *The Korteweg - de Vries equation. A survey of results*, SIAM Review 18(1976), No.3, 412-459.

[12] Prenter M., *Splines and variational methods*, J. Wiley, 1975.

[13] Rubin J., Graves R.A., *Viscous flow solutions with a cubic spline approximation*, Computers and Fluids 3(1975), 1-36.

[14] Sachdev P.L., Joseph K.T., Vaganan Mayil B., *Exact N-wave solutions of generalized Burgers equations*, Studies in Applied Mathematics 97(1996), No.4, 349-367.

[15] Schoombie S.W., *Spline Petrov-Galerkin methods for the numerical solution of the Korteweg - de Vries equation*, IMA J. of Numer. Anal. 2(1982), 95-109.

[16] Zabusky N.J., *A synergetic approach to problem of nonlinear dispersive wave propagation*, Interaction Proc. Symp. Nonlinear Partial Diff. Equations, W. Ames ed., Academic Press, 1978, 223-258.

[17] Zabusky N.J., Kruskal M.D., *Interaction of solitons in a collitionless plasma and the recurrence of initial states*, Phys. Rev. Lett. 15(1965), 240-243.